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UPWIND SCHEME OF CAPUTO TIME FRACTIONAL ADVECTION DIFFUSION EQUATION

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Abstract

The purpose of the present study is to create an implicit upwind finite difference method to calculate the numerical results of a time fractional advection diffusion equation (TFADE). The time derivative of fractional order is treated by applying the Caputo-based fractional formula of derivative order $\alpha \in (0, 1)$. The finite difference approximations (FDA) take care of the discretization of the differential equation. The study also considers the stability of the fractional equation (TFADE). Finally, a Numerical example study is carried out to illustrate the solutions achieved for various fractional orders of the time derivative.

1. Introduction

The study of the Advection diffusion equation is a very important role due to its application in the field of biological applications, weather forecasting, porous flows, underground water flows, etc. [1-5]. The replacement of the integer-order derivatives to the non-integer order derivatives is known as the fractional differential equation. The improvement in the predictions of the natural phenomenon such as flow process in various conditions such as in aquifer, contaminant transport [6, 7] can be achieved by the fractional order approximations. In this study, we explore the Caputo TDADE in one dimensional with homogeneous Dirichlet boundary conditions.

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} + v\frac{\partial u(x, t)}{\partial x} = D_l \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t), \quad (x, t) \in [a, b] \times (0, T_{\max}] \tag{1}$$

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Initial values,

$$u(x, 0) = u0(x) \tag{2}$$

and boundary values,

$$u(0, t) = ub_1;\ u(1, t) = ub_2. \tag{3}$$

where α time derivative non-integer order, $v \geq 0$ is the coefficient of advection and D_l is the diffusivity. And u is the solute concentration. We define $\frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}$ is the Caputo derivative for fractional order α.

The Caputo fractional derivatives of the function u, respected to the independent variable t is defined as:

$$\frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s},\ 0 < \alpha < 1 \tag{4}$$

The term $\Gamma(.)$ denotes Gamma function.

In this study, we will examine the numerical approach to the TFADE by using upwind scheme of finite difference method (FDM). Assume, the grid sizes in time and space for the FDM of the function $U(x, t)$ as $U_i^n = U(x_i, t_n)$ for some positive integers M and N are defined by $h = \frac{1}{M}$ with $x_i = ih$ and $t_n = n\tau$ respectively, where $i = 0, 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N$. The time fractional derivatives can approximate as follows [5, 9]:

$$\frac{\partial^{\alpha} u(x_i, t_n)}{\partial t^{\alpha}} = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left[\sum_{k=0}^{n} (U_i^{n+1-k} - U_i^{n-k}) \delta_{n,k}^{\alpha} \right] \tag{5}$$

where $\delta_{n,k}^{\alpha} = (k+1)^{(1-\alpha)} - (k)^{(1-\alpha)}$.

There are many studies carried out for TFADE. Liu et al., [8] derived a complete solution for TFADE by using transformations such as Mellin, Laplace and variable transformation. Huang and Liu [9] studied the approach of obtaining the solutions for TFADE and TFDE by using the green function explicitly. Li and Zeng [10] analyzed the FDA for fractional (time) differential equations. The analysis considers the forward and central differences for the time and space terms. The FDA equations are then solved using the matrix approach. Zhuang and Liu [11] and Murio [12] focused on implicit schemes to achieve the solutions for the TFDE using FDA. They suggested that the implicit scheme is stable enough to get the solutions of the TFDE. Liu et al., [13] explored the convergence along with the stability of explicit/implicit based FDA to obtain the solutions for FADE (time and space fractional). They adopted Grunwald and shifted Grunwald formula for Riemman-Liouville space fractional derivative and Caputo for time. It is found that the implicit scheme is stable in all conditions whereas the explicit one is for a particular condition. Also noted that both the schemes are convergent. McLean et al., [14] described the well-posedness of TFADE and their scope of study on finding the uniqueness and existence of the weaker solutions. There are many varieties of the available boundary conditions such as Robin, Dirichlet, and Neumann for the achievement of the solutions for TFADE and these are discussed by Povstenko [15]. Din et al., [16] considered the expanded version of the cubic B spline for the FDA to solve the TFADE implicitly. And other methods such as meshless, operational matrix, and convex duality methods are available for obtaining the numerical solutions of the TFDADE [17-19].

By the above works of literature and from the author's knowledge, the Caputo time fractional based upwind implicit TFADE solutions are not available. The study of the upwind schemes is important where the shock waves generated due to the advective terms. Also, this study carries the stability criteria for an upwind implicit scheme.

2. The Fractional Implicit Upwind Method

The upwind scheme is a numerical discretization method for solving hyperbolic-based PDEs. The upwind FDM uses in the upstream orientation to appropriate the Advection term (assuming $\upsilon \geq 0$) and can be approximated centered difference in second-order spatial derivatives as follows:

$$\left.\frac{\partial u}{\partial x}\right|_{i}^{n+1} = \frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x};\ \left.\frac{\partial^2 u}{\partial x^2}\right|_{i}^{n+1} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \tag{6}$$

The TFADE can be numerically approximated by substituting Equation

(6) and Equation (5) for the time fractional term in Equation (1) by using the fractional implicit upwind method as follows:

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{k=0}^{n}\delta_{n,\,k}^{\alpha}(U_i^{n+1-k}-U_i^{n-k})\right]+v\left(\frac{u_i^{n+1}-u_{i-1}^{n+1}}{\Delta x}\right)$$

$$=D_l\left(\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{\Delta x^2}\right)+f(x,\,t) \tag{7}$$

After rearranging the following equation can be found

$$\left(1-v\,\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x}+2D_l\,\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)u_i^{n+1}$$

$$=\left(v\,\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x}+D_l\,\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)u_{i-1}^{n+1}$$

$$+\left(D_l\,\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)u_{i+1}^{n+1}+u_i^{n}-\left[\sum_{k=1}^{n}\delta_{n,\,k}^{\alpha}(U_i^{n+1-k}-U_i^{n-k})\right]$$

$$+\Delta t^{\alpha}\Gamma(2-\alpha)f(x,\,t). \tag{8}$$

3. Stability Analysis

Let us consider the Von Neumann-based method in preparation for estimating the stability of the upwind implicit scheme for TFADE. Let U_i^n be the approximate solution of fractional schemes (8). For stability, the time along with the Caputo time fractional, term the advective terms are discretized backward FDA (upwind) and the diffusive term with the central FDA.

Lemma. *The coefficients* $\delta_{(n,\,k)}^{\alpha}$, $i=1,2,\ldots,$ *satisfy*

1. $\delta_{n,\,k}^{\alpha}>0$, $i=1,2,\ldots,$;

2. $\delta_{n,\,k}^{\alpha}>\delta_{n,\,k+1}^{\alpha}$

We defined error as, $\epsilon_i^n = u_i^n - U_i^n$; $n = 0, 1, \ldots, N$ and $i = 1, 2, \ldots, M$. By using the induction method terms are developed for finite difference implicit upwind numerical scheme in Equation (8). From (8), when $n = 0$

$$\left(1 + v\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x} + 2D_l\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)\epsilon_i^1 = \left(v\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x} + D_l\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)\epsilon_{i-1}^1$$

$$+\left(D_l\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)\epsilon_{i+1}^1 + \epsilon_i^0 \tag{9}$$

when $n \geq 1$,

$$\left(1 + v\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x} + 2D_l\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)\epsilon_i^{n+1}$$

$$= \left(v\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x} + D_l\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)\epsilon_{i-1}^{n+1}$$

$$+\left(D_l\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)\epsilon_{i+1}^{n+1} + \epsilon_i^n - \left[\sum_{k=1}^{n}\delta_{n,\,k}^{\alpha}(\epsilon_i^{n+1-k} - \epsilon_i^{n-k})\right] \tag{10}$$

define $\epsilon_i^m = \zeta_m e^{lq(i\Delta x)}$, where q is a real spatial wave number and $I = \sqrt{-1}$. Inserting this equation into (8) one gets and boundary values,

$$\zeta_{m+1} = \frac{1}{(h - le^{-lq\Delta x} - \rho e^{lq\Delta x})}\zeta_m - \sum_{k=0}^{n}\delta_{n,\,k}^{\alpha}(\zeta_{n+1-k} - \zeta_{n-k}) \tag{11}$$

where

$$h = \left(1 + v\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x} + 2D_l\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right); \quad l = \left(v\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x} + D_l\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)$$

and $\rho = \left(D_l\frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\right)$.

The above stability condition is unconditionally stable.

4. Numerical Results

Example 1. It is important to enhance the accuracy of the upwind implicit scheme, as a result, numerical results obtained are compared with the exact solution. Consider a TFADE with an initial/boundary condition (1-3) with the following initial condition,

$$u0 = x - x^2 \tag{12}$$

and the boundaries are,

$$ub_1 = ub_2 = 0 \tag{13}$$

The respective source term is given as,

$$f(x,\, t) = \frac{(2x - 2x^2)\, t^{2-\alpha}}{\Gamma(3-\alpha)} + v(1 - 2x)(t^2 + 1) + 2(t^2 + 1) \tag{14}$$

The exact/analytical solution is $u(x,\, t) = (x - x^2)(t^2 + 1)$.

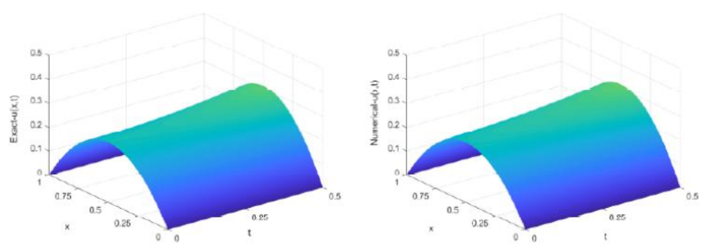

Figure 1. Validating the (left) exact and (right) numerical solutions for example 1.

The exact/numerical results are calculated for the time $t = 0.5$ using the proposed upwind implicit scheme for $v = 1$ and $D_l = 1$. The maximum space length of $x = 1$ is considered and the results are compared with the exact results as depicted in figure 1.

Example 2. Also, for the TFADE without the source term as,

$$\frac{\partial^\alpha U}{\partial t^\alpha} + V\frac{\partial U}{\partial X} = \frac{\partial^2 U}{\partial X^2} \quad 0 \le U \le 1;\ 0 \le t \le T_{\max} \tag{15}$$

$$U(X,\, 0) = 0 \tag{16}$$

$$U(0, t) = 1; U(1, t) = \frac{\partial U}{\partial t} = 0 \tag{17}$$

Here the characteristics of the fractional-order term for various α are discussed with the left boundary of Dirichlet and right side with the Neumann boundary (Equation 17) and also with zero initial values (Equation 16).

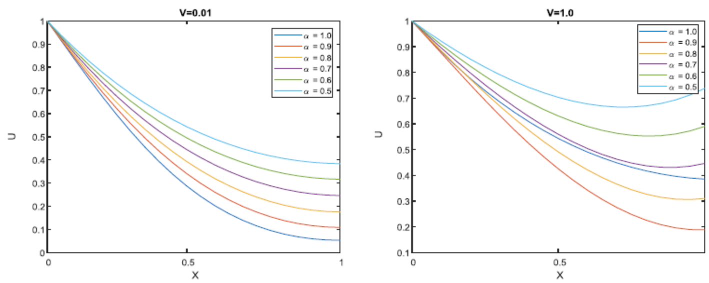

Figure 2. Change of U for various α at $t = 0.1$.

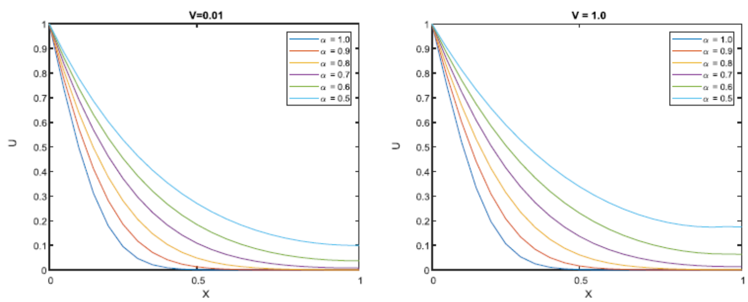

Figure 3. Change of U for various α at $t = 0.1$.

The variation of the parameter U concerning the time and space under specific initial and boundary conditions, as shown in Figures 2 and 3. By increasing the fractional order (α), the parameter U decreases and also U increased with increasing the advective term (V).

3. Conclusions

The TFADE is studied numerically using the upwind implicit method. The upwind implicit method is stable at all conditions and the scheme is a good comparison with the exact solution. The numerical finding shows that the change of parameter U for the time-fractional order term. Mainly U increases with the decrease of fractional order (α), also with the increase in V.

References

[1] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Elsevier, (1998).

[2] K. Devendra and S. Jagdev, Fractional Calculus in Medical and Health Science, CRC Press, (2020).

[3] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, (2000).

[4] T. M. Atanackovic, S. Pilipovic, B. Stankovic and D. Zorica, Fractional Calculus with Applications in Mechanics: Wave Propagation, Impact and Variational Principles, Wiley Blackwell, (2014).

[5] D. A. Benson, S. W. Wheatcraft and M. M. Meerschaert, Application of a fractional advection-dispersion equation, Water Resour. Res. 36 (2000), 1403-1412. https://doi.org/10.1029/2000WR900031

[6] A. Allwright and A. Atangana, Augmented upwind numerical schemes for a fractional advection-dispersion equation in fractured groundwater systems, Discret. Contin. Dyn. Syst.-Ser. S. 13 (2020), 443-466, available online at https://doi.org/10.3934/dcdss.2020025

[7] Y. E. Aghdam, H. Mesgrani, M. Javidi and O. Nikan, A computational approach for the space-time fractional advection-diffusion equation arising in contaminant transport through porous media, Eng. Comput. (2020), available online at https://doi.org/10.1007/s00366-020-01021-y

[8] F. Liu, V. V. Anh, I. Turner and P. Zhuang, Time fractional advection-dispersion equation, J. Appl. Math. Comput. 13 (2003), 233-245, available online at https://doi.org/10.1007/BF02936089

[9] F. Huang and F. Liu, The time fractional diffusion equation and the advection-dispersion equation, ANZIAM J. 46 (2005), 317-330, available online at https://doi.org/10.1017/s1446181100008282

[10] C. Li and F. Zeng, Finite difference methods for fractional differential equations, Int. J. Bifurc. Chaos. 22 (2012), available online at https://doi.org/10.1142/S0218127412300145

[11] P. Zhuang and F. Liu, Implicit difference approximation for the time fractional diffusion equation, J. Appl. Math. Comput. 22 (2006), 87-99, available online at https://doi.org/10.1007/BF02832039

[12] D. A. Murio, Implicit finite difference approximation for time fractional diffusion equations, Comput. Math. with Appl. 56 (2008), 1138-1145, available online at https://doi.org/10.1016/j.camwa.2008.02.015.

[13] F. Liu, P. Zhuang, V. Anh, I. Turner and K. Burrage, Stability and convergence of the difference methods for the space-time fractional advection-diffusion equation, Appl.

Math. Comput. 191 (2007), 12-20, available online at https://doi.org/10.1016/j.amc.2006.08.162

[14] W. McLean, K. Mustapha, R. Ali and O. Knio, Well-posedness of time-fractional advection-diffusion-reaction equations, Fract. Calc. Appl. Anal. 22 (2019), 918-944, available online at https://doi.org/10.1515/fca-2019-0050

[15] Y. Povstenko, Generalized boundary conditions for the time-fractional advection diffusion equation, Entropy. 17 (2015), 4028-4039, available online at https://doi.org/10.3390/e17064028

[16] S. T. Mohyud-Din, T. Akram, M. Abbas, A.I. Ismail and N.H.M. Ali, A fully implicit finite difference scheme based on extended cubic *B*-splines for time fractional advection-diffusion equation, Adv. Differ. Equations (2018), available online at https://doi.org/10.1186/s13662-018-1537-7.

[17] N. Chen, J. Huang, Y. Wu and Q. Xiao, Operational matrix method for the variable order time fractional diffusion equation using Legendre polynomials approximation, IAENG Int. J. Appl. Math. 47 (2017), 282-286.

[18] A. Mardani, M. R. Hooshmandasl, M. H. Heydari and C. Cattani, A meshless method for solving the time fractional advection-diffusion equation with variable coefficients, Comput. Math. with Appl. 75 (2018), 122-133, available online at https://doi.org/10.1016/j.camwa.2017.08.038.

[19] Q. Tang, On an optimal control problem of time-fractional advection-diffusion equation, Discret. Contin. Dyn. Syst. - Ser. B. 25 (2020), 761-779, available online at https://doi.org/10.3934/dcdsb.2019266.