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GAUSSIAN INTEGER SOLUTIONS FOR AN ELLIPTIC CURVE $34J^2 + 751K^2 - 304JK = 1296$

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Abstract

A Diophantine equation is a polynomial equation $P(x_1, x_2, \ldots, x_n) = 0$ where the polynomial P has integral coefficients such that the only solutions of interest are the integer ones. But in another direction, we seek Gaussian integer solution. On account of that, we are considering a elliptic Diophantine equation $34J^2 + 751K^2 - 304JK = 1296$. To do so, we entail on linear transformation, Pell equation.

1. Introduction

Number theory is a branch of mathematics that deals mainly with the zero, counting numbers and negative of them. It has so many interesting things which has to be used in other areas. Apart from theories, there is an engrossing content which motivates many researchers to carry number theory in their hands is solving a Diophantine equation. A Diophantine equation is a polynomial equation over $\mathbb{Z}$ with one or more variables and has to be over integers alone [5]. Many researchers contributed themselves in

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finding the integer solutions of Diophantine equations. [7-11] discusses the solutions of certain Diophantine Equations.

As days pass by, there is some development in the concept of Diophantine equations. One such development is extension of solutions from integers to Gaussian integers. [13] is an example for this kind. A Gaussian integer is one having the form $a + ib$ where a, b are integers. It is denoted by $\mathbb{Z}[i]$. For example, the Diophantine equation $x^4 \pm y^4 = z^2$ where x, y and z being Gaussian integers was examined by Hilbert. It was proved that there is only inconsequential solutions in $\mathbb{Z}[i]$. Elliptic curves are also used to show that the Diophantine equation $x^3 + y^3 = z^3$ has only inconsequential solutions in Gaussian integers. These results have motivated us to find non-zero distinct Gaussian integer solutions to a homogeneous quadratic Diophantine equation in three variables.

In this paper, we investigate Gaussian integer solutions of the Diophantine equation $34J^2 + 751K^2 - 304JK = 1296$ which is transformed into a Pell's equation by suitable transformation. To solve the reduced Pell Equation over integers, we make use of [3] and [14]. Using those solutions, we derive the Gaussian integer solutions.

2. Reduction to Pell's Equation

Consider the Diophantine equation

$$D : 34J^2 + 751K^2 - 304JK = 1296 \tag{1}$$

to be solved over $\mathbb{Z}[i]$. It is complicated to solve and find the nature and properties of the solutions of (1). So we apply a linear transformation T to (1) to transfer it to a simpler form for which we can determine the integral solutions.

Let

$$T : \begin{cases} J = hx + iy \\ K = x + iky \end{cases} \tag{2}$$

be the transformation where $x, y, h, k \in \mathbb{Z}[i]$.

Applying T to D, we get

$$T(D) = \widetilde{D} : 34(hx + iy)^2 + 751(x + iky)^2 - 304((hx + iy)(x + iky)) = 1296 \quad (3)$$

Equating the imaginary part to zero and coefficient of x^2 and y^2 to least integer, we get $h = 5$ and $k = 2$. Hence for $J = 5x + iy$ and $K = x + 2iy$, we have the Diophantine equation

$$\widetilde{D} : x^2 - 30y^2 = 16 \quad (4)$$

This is a Pell equation. Now we try to find all integer solutions (x_n, y_n) of $\widetilde{D}$ and then we can reverse all results from $\widetilde{D}$ to D by using the inverse process of T.

3. Solutions of Pell Equation $x^2 - 30y^2 = 16$

Theorem 1. *Let D be the DE in* (4). *Then*

1. *The CFE of* $\sqrt{30}$ *is* $\sqrt{30} = [5, \overline{2, 10}]$

2. *The fundamental solution of* $x^2 - 30y^2 = 1$ *is* $(u_1, v_1) = (11, 2)$

3. *For* $n \geq 2$,

$$u_n = 11u_{n-1} + 60v_{n-1}$$

$$v_n = 2u_{n-1} + 11v_{n-1}$$

4. *For* $n \geq 4$,

$$u_n = 23(u_{n-1} + u_{n-2}) + u_{n-3}$$

$$v_n = 23(v_{n-1} + v_{n-2}) + v_{n-3}$$

Proof.

1. The CFE of $\sqrt{30}$ is given as

$$\sqrt{30} = 5(\sqrt{30} - 5)$$

$$= 5 + \cfrac{1}{\cfrac{1}{\sqrt{30} - 5}}$$

$$= 5 + \cfrac{1}{\cfrac{\sqrt{30} - 5}{5}}$$

$$= 5 + \cfrac{1}{2 + \cfrac{\sqrt{30} - 5}{5}}$$

$$= 5 + \cfrac{1}{2 + \cfrac{\frac{1}{5}}{\sqrt{30} - 5}}$$

$$= 5 + \cfrac{1}{2 + \cfrac{1}{\sqrt{30} - 5}}$$

$$= 5 + \cfrac{1}{2 + \cfrac{1}{10 + \sqrt{30} - 5}}$$

Therefore, the CFE of $\sqrt{30}$ is $\sqrt{30} = [5, \overline{2, 10}]$

2. Clearly $(11, 2)$ is a solution of $x^2 - 30y^2 = 1$.

3. If $(u_1, v_1) = (11, 2)$ is the minimal solution of $x^2 - 30y^2 = 1$, then the further solutions (u_n, v_n) of $x^2 - 30y^2 = 1$ can be derived by using the equalities

$$(u_n + v_n\sqrt{30}) = (u_1 + v_1\sqrt{30})^n \text{ for } n \geq 2.$$

In other words,

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & 30v_1 \\ 2 & u_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4. We prove that u_n satisfy the recurrence relation. For $n = 4$, we get

$u_n = 116161$. Now

$$u_4 = 23(u_3 - u_2) + u_1$$

$$u_4 = 23(529 - 241) + 11$$

$$u_4 = 116161.$$

So, $u_4 = 23(u_3 - u_2) + u_1$ is satisfied for $n = 4$. Let us suppose that this relation is satisfied for $n - 1$, that is $u_{n-1} = 23(u_{n-2} - u_{n-3}) + u_{n-4}$.

Now to prove the outcome is true for n. Using the earlier statement, we terminate that

$$u_n = 23(u_{n-1} - u_{n-2}) + u_{n-3} \text{ for } n \geq 4$$

Likewise,

$$v_n = 23(v_{n-1} - v_{n-2}) + v_{n-3} \text{ for } n \geq 4$$

Now we think about the problem $x^2 - 30y^2 = 16$

Note that we indicate the integral solutions of $x^2 - 30y^2 = 16$ by (x_n, y_n) and indicate the integral solutions of $x^2 - 30y^2 = 1$ by (u_n, v_n). Then we have the following theorem.

Theorem 2. *Define a sequence* (x_n, y_n) *of positive integers by* $(x_1, y_1) = (44, 8)$ *and*

$$x_n = 44u_{n-1} + 240v_{n-1}$$

$$y_n = 44u_{n-1} + 240v_{n-1}$$

where (u_n, v_n) *is a sequence of positive integer solutions of* $x^2 - 30y^2 = 1$. *Followed by*

1. (x_n, y_n) *is a solution of* $x^2 - 30y^2 = 16$ *for any integer* $n \geq 1$.

2. *For* $n \geq 2$,

$$x_{n+1} = 11x_n + 60y_n$$

$$y_{n+1} = 2x_n + 11y_n$$

3. *For* $n \geq 4$,

$$x_n = 23(x_{n-1} - x_{n-2}) + x_{n-3}$$

$$y_n = 23(y_{n-1} - y_{n-2}) + y_{n-3}$$

Proof.

1. It is easily seen that $(x_1, y_1) = (44, 8)$ is a solution of $x^2 - 30y^2 = 16$, since

$$x_1^2 - 30y_1^2 = 44^2 - 30(8)^2$$

$$= 16(11^2) - 30(2^2)$$

$$= 16(1) = 16$$

Note that by description (u_{n-1}, v_{n-1}) is a solution of $x^2 - 30y^2 = 1$, that is,

$$u_{n-1}^2 - 30v_{n-1}^2 = 1 \tag{5}$$

Also we see as above that (x_1, y_1) is a solution of $x^2 - 30y^2 = 16$, that is,

$$x_1^2 - 30y_1^2 = 16 \tag{6}$$

Applying (5) and (6), we get

$$x_n^2 - 30y_n^2 = (44u_{n-1} + 240v_{n-1})^2 - 30(8u_{n-1} + 44v_{n-1})^2$$

$$= u_{n-1}^2(2)^4 - v_{n-1}^2(2)^4(30)$$

$$= (2)^4(u_{n-1}^2 - 30v_{n-1}^2)$$

$$= (2)^4$$

Therefore (x_n, y_n) is a solution of $x^2 - 30y^2 = 2^4$.

2. Recall that $x_{n+1} + y_{n+1}\sqrt{d} = (u_1 + v_1\sqrt{d})(x_n + y_n\sqrt{d})$. Therefore

$$x_{n+1} = u_1 x_n + v_1 d y_n$$

$$y_{n+1} = v_1 x_n + u_1 y_n$$

So $x_{n+1} = 11x_n + 60y_n$ and $y_{n+1} = 2x_n + 11y_n$ (7)

3. Applying the equalities

$$x_n = 2^2(11)u_{n-1} + 2^3(30v_{n-1}) \text{ and } x_{n+1} = 11x_n + 60y_n$$

we find by induction on n that $x_n = 23(x_{n-1} - x_{n-2}) + x_{n-3}$ for $n \geq 4$. Similarly, it can be shown that $y_n = 23(y_{n-1} - y_{n-2}) + y_{n-3}$. From the above, we can see that the DE D might be altered into the DE $\widetilde{D}$ via the conversion T. Also we proved that $J = 5x + iy$ and $K = x + i2y$. So we know how to re transfer all outcomes from $\widetilde{D}$ to D by using the inverse of T. Thus we know to give the following main theorem.

Theorem 3. *Let D be the DE in* (1). *Then*

1. *The primary solution of D is* $(J_1, K_1) = (220 + 8i, 44 + 16i)$.

2. *Define the sequence* $(J_n, K_n)_{n \geq 1} = (5x_n + iy_n, x_n + i2y_n)$ *where* (x_n, y_n) *defined in* (7). *In that case* (J_n, K_n) *is solution of D.*

3. *The solution* (J_n, K_n) *satisfy*

$$J_n = 5(11x_{n-1} + 60y_{n-1}) + i(2x_{n-1} + 11y_{n-1}),\ n \geq 2$$

$$K_n = (11x_{n-1} + 60y_{n-1}) + i2(x_{n-1} + 11y_{n-1}),\ n \geq 2.$$

4. *The solutions* (J_n, K_n) *convince the recurrence associations*

$$J_n = (115(x_{n-1} - x_{n-1}) + 5x_{n-3}) + i(23(y_{n-1} - y_{n-2}) + y_{n-3}),\ n \geq 4$$

$$K_n = (23(x_{n-1} - x_{n-2}) + x_{n-3}) + i(46(y_{n-1} - y_{n-2}) + 2y_{n-3}),\ n \geq 4.$$

Proof.

1. It is without difficulty seen that $(J_1, K_1) = (220 + 8i, 44 + 16i)$ is the primary solution of D since

$$34(220+8i)^2+751(44+16i)^2-304(220+8i)(44+16i)-1296=0.$$

2. We verify it by induction. Let $n=1$. Then $(J_1, K_1)=(5x_1+iy_1, x_1+i2y_1)=(220+8i, 44+16i)$ which is the primary solution and so is a solution of D. Let us suppose that the DE in (1) is fulfilled for $n-1$, that is, $34(5x_{n-1}+iy_{n-1})^2+751(x_{n-1}+i2y_{n-1})^2$ $-304(5x_{n-1}+iy_{n-1})(x_{n-1}+i2y_{n-1})-1296=0$. Now, we desire to prove that this equation is also fulfilled for n.

$$34J^2+751K^2-304JK=1296$$

$$=34(5x_n+iy_n)^2+751(x_n+i2y_n)^2-304(5x_n+iy_n)(x_n+i2y_n)-1296=0$$

$$=81(x_n^2-30y_n^2-16)$$

$$=0$$

So $(J_n, K_n)=(5x_n+iy_n, x_n+i2y_n)$ is also a solution D.

3. From (7), we have $x_n=11x_{n-1}+60y_{n-1}$ and $y_n=2x_{n-1}+11y_{n-1}$. Substitute these in $J_n=5x_n+iy_n$, we get

$$J_n=5(11x_{n-1}+60y_{n-1})+i(2x_{n-1}+11y_{n-1}) \quad (8)$$

Similarly,

$$K_n=(11x_{n-1}+60y_{n-1})+i2(x_{n-1}+11y_{n-1}). \quad (9)$$

4. We show that J_n fulfill the recurrence relation. We have $J_n=220+8i$, $J_2=4820+176i$, $J_3=105820+3864i$, $J_4=2323220$ $+84832i$.

Now,

$$23(J_3-J_2)+J_1=23((5x_3+iy_3)-(5x_2+iy_2))+(5x_1+iy_1)$$

$$=(115(21164-964)+220)+i(23(3864-176)+8)$$

$$=2323220+84832i.$$

So the recurrence relation is satisfied for $n=4$. Let us suppose that this relation is fulfilled for $n-1$, that is,

$$J_{n-1} = 23((5x_{n-2} + iy_{n-2}) - (5x_{n-3} + iy_{n-3})) + (5x_{n-4} + iy_{n-4}) \tag{10}$$

Then using the preceding statement, (8) and (10), we terminate that J_n fulfill the recurrence relation. Now, we show that K_n gratify the recurrence relation. For $n = 4$, we get $K_1 = 44 + 16i$, $K_2 = 964 + 352i$, $K_3 = 21164 +7728i$, $K_4 = 464644 + 169664i$.

Hence

$$K_4 = 23(K_3 - K_2) + K_1 = (23(21164 - 964) + 44) + i(46(3864 - 176) + 8)$$

$$= 464644 + i1696644i.$$

So

$$K_n = (23(x_{n-1} - x_{n-1}) + x_{n-3}) + i(46(y_{n-1} - y_{n-2}) + 2y_{n-3}) \text{ for } n \geq 4.$$

Let us suppose that this relation is fulfilled for $n - 1$, that is

$$K_{n-1} = 23((x_n - 2 + 2iy_n - 2) - (x_n - 3 + 2iy_n - 3)) + (x_n - 4 + 2iy_n - 4) \tag{11}$$

Then using the preceding declarations (9) and (11), we conclude that

$K_n = (23(x_n - 1 + x_n - 2) + x_n - 3) + i(46(y_n - 1 - y_n - 2) + 2y_n - 3),$ for $n \geq 4$.

4. Conclusion

Diophantine equations are wealthy in diversity. There is no common technique to discover all feasible Gaussian integer solutions (if it exists) for DEs. This method looks to be trouble-free but actually it is extremely complex for attainment of solutions. One may search for other choices of Diophantine equations to find their corresponding Gaussian integer solutions.

References

[1] L. E. Dickson, History of the theory of numbers, Volume II-Diophantine Analysis, Dover Publications, (2015).

[2] M. A Gopalan, Sumathi and Vidhyalakshmi, Gaussian Integer Solutions of Homogeneous Quadratic Equation with Four Unknowns $x^2 + y^2 = 3z^2 + w^2$, IAAST, 04(03) (2013), 58-61.

[3] Hensley Doug, Continued fractions, World Scientific, 2006.

[4] J. Kannan, Manju Somanath and K. Raja, Solutions of Negative Pell Equation Involving Twin Prime, JP Journal of Algebra, Number Theory and Applications 40(5) (2018), 869-874.

[5] Kenneth H. Rosen, Elementary Number Theory and its Applications, Addison-Wesley Publishing Company, (1984).

[6] H. W. Lenstra Jr, Solving the Pell equation, Notices of the AMS 49(2) (2002), 182-192.

[7] Manju Somanath and J. Kannan, Congruum Problem, International Journal of Pure and Applied Mathematical Sciences 9(2) (2016), 123-131.

[8] Manju Somanath, K. Raja, J. Kannan and A. Akila, Integral Solutions of an Infinite Elliptic Cone $x^2 + 9y^2 + 11z^2$, Advances and Applications in Mathematical Sciences, 19(11) (2020), 1119-1124.

[9] Manju Somanath, K. Raja, J. Kannan and B. Jeyashree, Non Trivial Integral Solutions of Ternary Quadratic Diophantine Equation, Advances and Applications in Mathematical Sciences 19(11) (2020), 1105-1112.

[10] Manju Somanath, K. Raja, J. Kannan and K. Kaleeswari, Solutions of Negative Pell Equation Involving Chen Prime, Advances and Applications in Mathematical Sciences, 19(11) (2020), 1089-1095.

[11] Manju Somanath, K. Raja, J. Kannan and M. Mahalakshmi, On a Class of Solutions for a Quadratic Diophantine Equation, Advances and Applications in Mathematical Sciences 19(11) (2020), 1097-1103.

[12] Manju Somanath, K. Raja, J. Kannan and S. Nivetha, Exponential Diophantine Equation in Three Unknowns, Advances and Applications in Mathematical Sciences 19(11) (2020), 1113-1118.

[13] Manju Somanath, K. Raja, J. Kannan and V. Sangeetha, On the Gaussian Integer Solutions for an Elliptic Diophantine Equation, Advances and Applications in Mathematical Sciences 20(05) (2021), 815-822.

[14] A. Tekcan, Continued fractions expansion of $\sqrt{D}$ and Pell equation $x^2 - Dy^2 = 1$, Mathematica Moravica 15(2) (2011), 19-27.