# ABSOLUTELY HARMONIOUS LABELING OF SOME SPECIAL GRAPHS 

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#### Abstract

Absolutely harmonious labeling $f$ is an injection from the vertex set of a graph $G$ with $q$ edges to the set $\{0,1,2, \ldots, q-1\}$, if each edge $u v$ is assigned $f(u)+f(v)$ then the resulting edge labels can be arranged as $a_{0}, a_{1}, a_{2}, \ldots, a_{q-1}$ where $a_{i}=q-i$ or $q+i, 0 \leq i \leq q-1$. however, when $G$ is a tree one of the vertex labels may be assigned to exactly two vertices. A graph which admits absolutely harmonious labeling is called absolutely harmonious graph. In this paper, we study absolutely harmonious labeling of some special graphs.


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## 1. Introduction

In this paper, we consider finite, simple and undirected graphs. M. Seenivasan and A. Lourdusamy [3] introduced another variation of harmonious labeling, namely, absolutely harmonious labeling of graphs. In this paper we investigate the absolutely harmonious labeling of some special graphs such as triangular ladder, globe graph, shadow graph, Sudivision, splitting and duplication of star graph.

Theorem 1. The graph $K_{2}+m K_{1}$ is absolutely harmonious.
Proof. Let $G=K_{2}+m K_{1}$ graph .
The vertex set $V(G)=\left\{x, y, w_{1}, w_{2}, \ldots, w_{m}\right\}$ and the edge set $E(G)=\left\{x y, x w_{i}, y w_{i}: 1 \leq i \leq m\right\}$ here, $G$ is of order $m+2$ and size $2 m+1$.

Now, define $f: V(G) \rightarrow\{0,1,2,3, \ldots, q-1\}$ as follows

$$
\begin{aligned}
& f(x)=0 \\
& f(y)=m+1 \\
& f\left(w_{i}\right)=i, 1 \leq i \leq m
\end{aligned}
$$

The induced edge labels are as follows

$$
f^{*}(x y)=a_{m}
$$

$$
f^{*}\left(x w_{i}\right)=a_{q-j} ; 1 \leq i \leq m \text { and } 1 \leq j \leq m .
$$

$$
f^{*}\left(y w_{i}\right)=a_{k} ; 0 \leq k \leq m-1
$$

From the above, $a_{0}, a_{1}, a_{2}, \ldots, a_{q-1}$ where $a_{i}=q-i \quad$ (or) $q+i: 0 \leq i \leq q-1$ are the arranged edge labels. Therefore $f$ admits absolutely harmonious labeling.
and hence $K_{2}+m K_{1}$ is an absolutely harmonious graph.


Figure 1. $K_{2}+4 K_{1}$.
Definition 1. Let $G$ be a $(p, q)$ graph. The subdivision of each edge of a graph $G$ with a vertex is called the subdivision graph and it is denoted by $S(G)$.

Theorem 2. $S\left(K_{1, n}\right)$ is absolutely harmonious for all $n \geq 1$.
Proof. Let $G=S\left(K_{1, n}\right)$
Let $V(G)=\left\{u, v_{i}, u_{i}: 1 \leq i \leq n\right\}$ and $E(G)=\left\{u v_{i}, v_{i} u_{i}: 1 \leq i \leq n\right\}$
Here $G$ is of order $2 n+1$ and size $2 n$. Now, Define $f: V(G) \rightarrow\{0,1,2, \ldots, q-1\}$ as follows
$f(u)=1$
$f\left(v_{1}\right)=0$
$f\left(v_{i}\right)=2 i-1 ; 2 \leq i \leq n$
$f\left(u_{i}\right)=2 i ; 1 \leq i \leq n-1$
$f\left(v_{n}\right)=2 n-2$
Then the induced edge labels are as follows
$f^{*}\left(u v_{1}\right)=a_{q-1}$

$$
\begin{aligned}
& f^{*}\left(u v_{k}\right)=a_{[q-2 i]} ; 2 \leq k \leq n, 2 \leq i \leq n \\
& f^{*}\left(v_{1} u_{1}\right)=a_{q-2} \text { and } f^{*}\left(v_{n} u_{n}\right)=a_{2 n-3} \\
& f^{*}\left(v_{k} u_{k}\right)=a[q-(4 i-1)] ; 2 \leq k \leq n-1 ; 2 \leq i \leq n-1
\end{aligned}
$$

From the above, $a_{0}, a_{1}, a_{2}, \ldots, a_{q-1}$ where $a_{i}=q-i \quad$ (or) $q+i ; 0 \leq i \leq q-1$ are the arranged edge labels. Therefore $f$ admits absolutely harmonious labeling of $S\left(K_{1, n}\right)$ and hence $S\left(K_{1, n}\right)$ is an absolutely harmonious graph.

Definition 2. Let the graphs $G_{1}$ and $G_{2}$ have disjoint vertex sets $V_{1}$ and $V_{2}$ the edge sets $E_{1}$ and $E_{2}$ respectively. Then their union $G=G_{1} \cup G_{2}$ is a graph with vertex set $V=V_{1} \cup V_{2}$ and the edge set $E=E_{1} \cup E_{2}$. Clearly, $G_{1} \cup G_{2}$ has $p_{1}+p_{2}$ vertices and $q_{1}+q_{2}$ edges.

Theorem 3. $S\left(K_{1, n}\right) \cup K_{1, m}, n, m \succ 1$ is not absolutely harmonious.
Proof. Let $G=S\left(K_{1, n}\right) \cup K_{1, m}$ be a graph with $p$ vertices and $q$ edges.
Let $\left\{u, u_{i}, v_{i}, w, w_{j} ; 1 \leq i \leq n, 1 \leq j \leq m\right\}$ be the vertices of $G$ and $\left\{u u_{i}, u_{i} v_{i}, w w_{j} ; 1 \leq i \leq n, 1 \leq j \leq m\right\}$ be the edges of $G$.

Here, $p=3 n+2$ and $q=3 n$ since $p \succ q$, we cannot give the distinct labels from $\{0,1,2, \ldots, q-1\}$ to the vertices of $G$. Hence $G$ is not an absolutely harmonious graph.

Definition 3. For a graph $G$, the splitting graph $S^{\prime}\left(K_{1, n}\right)$ of $G$ is obtained by adding a new vertex $v^{\prime}$ corresponding to each vertex $v$ of $G$ such that $N(v)=N\left(v^{\prime}\right)$.

Theorem 4. The splitting graph $S^{\prime}\left(K_{1, n}\right)$ is absolutely harmonious.
Proof. Let $v_{1}, v_{2}, \ldots, v_{n}$ be the pendant vertices and $v$ be the apex vertex of $K_{1, n}$ and $u, u_{1}, u_{2}, \ldots, u_{n}$ be added vertices corresponding to $v, v_{1}, v_{2}, \ldots v_{n}$ to obtain $S^{\prime}\left(K_{1, n}\right)$.

Let $G$ be the splitting graph $S^{\prime}\left(K_{1, n}\right)$. Then $G$ is of order $2 n+2$ and size 3n. Now, Define $f: V(G) \rightarrow\{0,1,2, \ldots, q-1\}$ as follows

$$
\begin{aligned}
& f(u)=3 n-1 \\
& f(v)=0 \\
& f\left(v_{i}\right)=i ; 1 \leq i \leq n \\
& f\left(u_{i}\right)=f\left(v_{n}\right)+i ; 1 \leq i \leq n
\end{aligned}
$$

Then the induced edge labels are as follows

$$
\begin{aligned}
& f\left(u v_{i}\right)=a_{i-1} ; 1 \leq i \leq n \\
& f\left(v v_{i}\right)=a_{q-1} ; 1 \leq i \leq n \\
& f\left(v u_{i}\right)=a_{2 n-i} ; 1 \leq i \leq n
\end{aligned}
$$

From the above, $a_{0}, a_{1}, a_{2}, \ldots, a_{q-1}$ where $a_{i}=q-1 \quad$ (or) $q+i ; 0 \leq i \leq q-1$ are the arranged edge labels. Therefore $f$ admits absolutely harmonious labeling of $S^{\prime}\left(K_{1, n}\right)$ and hence $S^{\prime}\left(K_{1, n}\right)$ is an absolutely harmonious graph.

Definition 4. The Bistar graph $B_{n, n}$ is obtained by joining the center (apex) vertices of two copies of $K_{1, n}$ by an edge.

Theorem 5. The Bistar graph $B_{n, n}$ is absolutely harmonious.
Proof. Let $G=B_{n, n}$ Let $V(G)=\left\{u, v, u_{i}, v_{i}: 1 \leq i \leq n\right\}$, $E(G)=\left\{u v, v v_{i}, u u_{i}: 1 \leq i \leq n\right\}$. Then $G$ is of order $2 n+2$ and size $2 n+1$. Now, define $f: V(G) \rightarrow\{0,1,2, \ldots, q-1\}$ as follows

$$
\begin{aligned}
& f(u)=0 \\
& f(v)=q-1 \\
& f\left(v_{i}\right)=2 i-1 ; 1 \leq i \leq n \\
& f\left(u_{1}\right)=1 f\left(u_{k}\right)=2 i ; 1 \leq i \leq n, 2 \leq k \leq n
\end{aligned}
$$

Then the induced edge labels are as follows

$$
\begin{aligned}
& f^{*}(u v)=a_{1} \\
& f^{*}\left(v v_{1}\right)=a_{0} \\
& f^{*}\left(v v_{k}\right)=a_{2 i} ; 1 \leq i \leq n, 2 \leq k \leq n \\
& f^{*}\left(u u_{1}\right)=a_{q-1} \\
& f^{*}\left(u u_{k}\right)=a_{2 i-1} ; n \leq i \leq 2 \leq k \leq n
\end{aligned}
$$

From the above, $a_{0}, a_{1}, a_{2}, \ldots, a_{q-1} \quad$ where $\quad a_{i}=q-1 \quad$ (or) $q+i ; 0 \leq i \leq q-1$ are the arranged edge labels.

Therefore $f$ is an absolutely harmonious labeling of Bistar graph $B_{n, n}$ and hence the Bistar graph $B_{n, n}$ is an absolutely harmonious graph.

Theorem 6. $S\left(K_{1, n}\right) \cup B_{r, s}$ is not Absolutely harmonious for all $n, r, s \succ 1$.

Proof. Let $G=S\left(K_{1, n}\right) \cup B_{r, s}$ since the number of vertices of $G$ is greater than the number of edges of $G$. We cannot give the distinct labels from $\{0,1,2, \ldots, q-1\}$ to be the vertices of $G$.

Hence $G$ is not an absolutely harmonious graph.
Definition 5. The triangular ladder $T L_{n}$ is graph obtained from $L_{n}$ by adding edges $u_{i} v_{i+1}, 1 \leq i \leq n$, where $u_{i}$ and $v_{i}, 1 \leq i \leq n$ are the vertices of $L_{n}$ such that $u_{1}, u_{2}, \ldots, u_{n}$ and $v_{1}, v_{2}, \ldots, v_{n}$ are the two paths of the length $n$ in the graph $L_{n}$.

Theorem 7. The Triangular ladder $T L_{n}$ is an absolutely harmonious graph.

Proof. Let $G$ be a $T L_{n}$ graph. Let $V(G)=\left\{u_{i} v_{i}: 1 \leq i \leq n\right\}$ and $E(G)=\left\{u_{i} u_{i+1}, v_{i} v_{i+1}, u_{i} v_{i+1}: 1 \leq i \leq n-1\right\} \cup\left\{u_{i} v_{i}: 1 \leq i \leq n\right\}$.

Then $G$ is of order $2 n$ and size $4 n-3$. Now, Define
$f: V(G) \rightarrow\{0,1,2, \ldots, q-1\}$ as follows

$$
\begin{aligned}
& f\left(u_{1}\right)=0 \\
& f\left(v_{i}\right)=2 i-1 ; 1 \leq i \leq n-1 \\
& f\left(u_{k}\right)=2 i ; 1 \leq i \leq n-1,2 \leq k \leq n
\end{aligned}
$$

Then the induced edge labels are as follows

$$
\begin{aligned}
& f^{*}\left(u_{1} v_{1}\right)=a_{q-1} \\
& f^{*}\left(u_{i} v_{i+1}\right)=a_{q-2 k}, 1 \leq i \leq n-1 ; 1 \leq k \leq n+2 \text { and } k \text { is odd } \\
& f^{*}\left(v_{i} v_{i+1}\right)=a_{q-4 i}, 1 \leq i \leq n-1 \\
& f^{*}\left(u_{k} v_{k}\right)=a_{q-(4 i+1)}, 1 \leq i \leq n-1,2 \leq k \leq n \\
& f^{*}\left(u_{k} v_{k+1}\right)=a_{q-(4 i+3)} ; 0 \leq i \leq n-1,1 \leq k \leq n-1
\end{aligned}
$$

From the above, $a_{0}, a_{1}, a_{2}, \ldots, a_{q-1}$ where $a_{i}=q-i \quad$ (or) $q+i, 0 \leq i \leq q-1$ are the arranged edge labels.

Therefore $f$ is an absolutely harmonious labeling of triangular ladder $T L_{n}$ and hence the triangular ladder $T L_{n}$ is an absolutely harmonious graph.

Definition 6. Duplication of a vertex $v_{k}$ of a graph $G$ produces a new graph $G^{\prime}$ by adding a vertex $v_{k}^{\prime}$ with $N\left(v_{k}\right)=N\left(v_{k}^{\prime}\right)$.

Theorem 8. The graph obtained by duplication of apex vertex by an edge in $K_{1, n}$ is absolutely harmonious.

Proof. Let $v_{0}$ be the apex vertex of star $K_{1, n}$ and $v_{1}, v_{2}, \ldots, v_{n}$ are pendant vertices of $K_{1, n}$.

Let $G$ denote the graph obtained by duplication of apex vertex $v_{0}$ by an edge $v_{0}^{\prime} v_{0}^{\prime \prime}$. Then $G$ is of order $n+3$ and size $n+3$ Now, define $f: V(G) \rightarrow\{0,1,2, \ldots, q-1\}$ as follows

$$
\begin{aligned}
& f\left(v_{0}\right)=0 \\
& f\left(v_{0}^{\prime}\right)=1 \\
& f\left(v_{0}^{\prime \prime}\right)=n+2 \\
& f\left(v_{k}\right)=i+1 ; 1 \leq k \leq n, 1 \leq i \leq n
\end{aligned}
$$

Then the induced edge labels are as follows

$$
\begin{aligned}
& f^{*}\left(v_{0} v_{0}^{\prime \prime}\right)=a_{1} \\
& f^{*}\left(v_{0}^{\prime} v_{0}^{\prime \prime}\right)=a_{0} \\
& f^{*}\left(v_{0} v_{0}^{\prime}\right)=a_{q-1} \\
& f^{*}\left(v_{0} v_{k}\right)=a[q-(i+1)] ; 1 \leq k \leq n, 1 \leq i \leq n
\end{aligned}
$$

From the above, $a_{0}, a_{1}, a_{2}, \ldots, a_{q-1} \quad$ where $\quad a_{i}=q-i \quad$ (or) $q+i ; 0 \leq i \leq q-1$ are the arranged edge labels.

Therefore $f$ is an absolutely harmonious labeling of the graph obtained by duplication of apex vertex by an edge in $K_{1, n}$ and hence the graph obtained by duplication of apex vertex by an edge in $K_{1, n}$ is an absolutely harmonious graph.

Definition 7. Globe graph is defined as the two isolated vertex are joined by $n$ paths of length 2 . It is denoted by $G l(n)$.

Theorem 9. Globe $G l(n), n \geq 3$ is an absolutely harmonious graph.
Proof. Let $G=G l(n)$.
Let $V(G)=\left\{u, v, w_{i}: 1 \leq i \leq n\right\}$ and $E(G)=\left\{\left[u w_{i}\right] \cup\left[v w_{i}\right]: 1 \leq i \leq n\right\}$.
Here, $|V(G)|=n+1$ and $|E(G)|=2 n$
Define $f: V(G) \rightarrow\{0,1,2, \ldots, q-1\}$ as follows
$f(u)=q-1$
$f(v)=0$

$$
f\left(w_{i}\right)=i ; 1 \leq i \leq q-n
$$

Then the induced edge labels are as follows
$f^{*}\left(u w_{i}\right)=a_{i-1}, 1 \leq i \leq q-n$
$f^{*}\left(w_{i} v\right)=a_{q-i} ; 1 \leq i \leq q-n$
From the above, $a_{0}, a_{1}, a_{2}, \ldots, a_{q-1}$ where $a_{i}=q-i \quad$ (or) $q+i, 0 \leq i \leq q-1$ are the arranged edge labels.

Therefore $f$ is an absolutely harmonious labeling of the globe $G l(n), n \geq 3$ and hence the globe $G l(n), n \geq 3$ is an absolutely harmonious graph.

Definition 8. The shadow graph $D_{2}(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G^{\prime}$ and $G^{\prime \prime}$. Join each vertex $u^{\prime}$ in $G^{\prime}$ to the neighbours of the corresponding vertex $v^{\prime}$ in $G^{\prime \prime}$.

Theorem 10. The graph $D_{2}\left(K_{1, n}\right), n \geq 2$ is absolutely harmonious.
Proof. Let $\left\{v, v_{i}, 1 \leq i \leq n\right\}$ be the vertices and $\left\{e_{i}, 1 \leq i \leq 4 n\right\}$ be the edges.

Define $f: V(G) \rightarrow\{0,1,2, \ldots, q-1\}$ as follows

$$
\begin{aligned}
& f(u)=q-1, f(v)=0 \\
& f\left(u_{i}\right)=i ; 1 \leq i \leq n \\
& f\left(v_{i}\right)=n+i ; 1 \leq i \leq n
\end{aligned}
$$

Then the induced edge labels are as follows

$$
\begin{aligned}
& f^{*}\left(u u_{i}\right)=a_{i-1} ; 1 \leq i \leq n \\
& f^{*}\left(v u_{i}\right)=a_{q-i} ; 1 \leq i \leq n \\
& f^{*}\left(u v_{i}\right)=a_{n-i} ; 0 \leq i \leq n-1 \\
& f^{*}\left(u u_{i}\right)=a[q-(n+i)] ; 1 \leq i \leq n
\end{aligned}
$$

From the above, $a_{0}, a_{1}, a_{2}, \ldots, a_{q-1} \quad$ where $\quad a_{i}=q-1 \quad$ (or) $q+i ; 0 \leq i \leq q-1$ are the arranged edge labels.

Therefore $f$ is an absolutely harmonious labeling of the graph $D_{2}\left(K_{1, n}\right), n \geq 2$ and hence the graph $D_{2}\left(K_{1, n}\right), n \geq 2$ is an absolutely harmonious graph.

Definition 9. Let $G$ be a graph with set of vertices and edges as $V(G)=\left\{\left(c_{1}, c_{2}, b, w, d\right) \cup\left(x^{1}, x^{2}, x^{3}, \ldots, x^{n}\right) \cup\left(y^{1}, y^{2}, y^{3}, \ldots, y^{n}\right)\right\}$
$E(G)=\left\{\left(c_{1} x^{1}, c_{1} x^{2}, c_{1} x^{3}, \ldots, c_{1} x^{n}\right) \cup\left(c_{2} y^{1}, c_{2} y^{2}, c_{2} y^{3}, \ldots, c_{2} y^{n}\right) \cup\left(c_{1} b, c_{1} w\right)\right.$
$\left.\cup\left(c_{2} w, c_{2} d\right)\right\}$
We shall call it $W$-graph and it shall be denoted by $W(n)$


Figure 2. $w(3)$
Theorem 11. The $W$-graph $W(n), n \geq 1$ admits absolutely harmonious labeling.

Proof. Let $G=W(n)$ with $V=|V(G)|$ and $E=|E(G)|$. Here, $V=2 n+5$ and $E=2 n+4$.

The vertex set and the edge set of $G$ are as follows

$$
\begin{aligned}
& \quad V(G)=\left\{\left(c_{1}, c_{2}, b, w, d\right) \cup\left(x^{1}, x^{2}, x^{3}, \ldots, x^{n}\right) \cup\left(y^{1}, y^{2}, y^{3}, \ldots, y^{n}\right)\right\} \\
& E(G)=\left\{\left(c_{1} x^{1}, c_{1} x^{2}, c_{1} x^{3}, \ldots, c_{1} x^{n}\right) \cup\left(c_{2} y^{1}, c_{2} y^{2}, c_{2} y^{3}, \ldots, c_{2} y^{n}\right) \cup\left(c_{1} b, c_{1} w\right)\right. \\
& \left.\cup\left(c_{2} w, c_{2} d\right)\right\}
\end{aligned}
$$

Now, we define the labeling $f: V(G) \rightarrow\{0,1,2, \ldots, q-1\}$ as follows
$f(b)=1$
$f(w)=n+2$
$f(d)=n+3$
$f\left(c_{1}\right)=0$
$f\left(c_{2}\right)=1$
$f\left(x^{i}\right)=i+1 ; 1 \leq i \leq n$
$f\left(y^{j}\right)=q-j ; 1 \leq j \leq n$
From the above, $a_{0}, a_{1}, a_{2}, \ldots, a_{q-1} \quad$ where $\quad a_{i}=q-i \quad$ (or) $q+i ; 0 \leq i \leq q-1$ are the arranged edge labels.

Therefore $f$ is an absolutely harmonious labeling of the $W$-graph. and hence the $W$-graph is an absolutely harmonious graph.


Figure 3. $W(2)$

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