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TOPOLOGICAL INDICES OF SOME DERIVED GRAPHS THROUGH M-POLYNOMIALS

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Abstract

The M-polynomial of a graph G introduced recently, acts as a single window approach for the computation of more than ten degree based topological indices. In this paper, we have obtained M-polynomials of derived graphs of certain graph families and computed few degree based topological indices. Further, we have obtained some formulas expressing first and second Zagreb indices of derived graphs in terms of $M_1(G)$, $M_2(L(G))$ etc. Finally, we have obtained some bounds and characterizations on $T_1(G)$, $T_2(G)$ and $T(G)$.

1. Introduction

The graphs considered here are finite, undirected without loops and multiple edges. Let $G = (V, E)$ be a connected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The degree $d_G(v)$ of a vertex v is the number of vertices adjacent to v. The edge connecting the vertices u and v will be denoted by uv. Let $d_G(e)$ denote the degree of an edge e in G, which is defined by $d_G(e) = d_G(u) + d_G(v) - 2$ with $e = uv$. For definitions and notions, the reader may refer to [1, 12, 14, 16, 20, 21].

According to the IUPAC definition, a topological index or a connectivity index is a numerical value associated with the molecular graph. There are numerous molecular descriptors, which are also referred to as topological

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descriptors that have found some applications in theoretical chemistry, especially in QSPR/QSAR research. Unfortunately quite many of these indices are inadequate for any structure-property correlations. For details see [5, 8, 9, 13, 23, 24].

The first and second Zagreb indices were introduced to take account of the contributions of pairs of adjacent vertices. The first and second Zagreb indices of a graph G are defined as $M_1(G) = \sum_{v \in V(G)} d_G(v)^2$ or $M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$ and $M_2(G) = \sum_{uv \in E(G)} [d_G(u) d_G(v)]$. The general Randic connectivity index of a graph G is defined as $R_a(G) = \sum_{uv \in E(G)} [d_G(u) d_G(v)]^{\alpha}$. The harmonic index of a graph G is defined on the arithmetic mean as $H(G) = \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)}$. The inverse degree sum index of a graph is defined as $I_n(G) = \sum_{uv \in E(G)} \frac{d_G(u) d_G(v)}{d_G(u) + d_G(v)}$.

Vukicevic and Gasperov posed the symmetric division deg index of a graph G, defined as $SD(G) = \sum_{uv \in E(G)} \frac{\max(d_G(G), d_G(v))}{\min(d_G(G), d_G(v))} + \frac{\min(d_G(G), d_G(v))}{\max(d_G(G), d_G(v))} = \sum_{uv \in E(G)} \frac{d_G(u)}{d_G(v)} + \frac{d_G(v)}{d_G(u)}$. For the details of different degree based topological indices and their applications, we refer the reader to [3, 4, 6, 7, 10, 11, 22, 25]. A degree-based topological index is a graph invariant of the form

$$I(G) = \sum_{e = uv \in E(G)} f(d_G(u), d_G(v))$$

where $f = f(x, y)$ is a function of $d_G(u)$ and $d_G(v)$ choosen appropriately for different topological indices as shown in table 1.

The M-polynomial is an algebraic polynomial with integer coefficients which was introduced in 2015 and useful in computing many degree-based topological indices. Recently, there has been a significant study on M-polynomials [17, 18, 19].

Definition 1.1. For a graph G, the M-polynomial is defined as

$$M(G, x, y) = \sum_{i>j} m_{ij}(G)x^i y^j$$

where $m_{ij}(G), i, j \geq 1$, is the number of edges $e = uv$ of G such that $\{d_G(G), d_G(v)\} = \{i, j\}$.

Table 1. Some degree based topological indices and formulas how to compute them from the M-polynomial.

Notation	Topological Index	$f(x, y)$	Derivation from $M(G, x, y)$
$M_1(G)$	First Zagreb	$x+y$	$(D_x + D_y)(M(G, x, y))\mid_{x=y=1}$
$M_2(G)$	Second Zagreb	xy	$(D_x D_y)(M(G, x, y))\mid_{x=y=1}$
$R_\alpha(G)$	General Randic	$(xy)^\alpha$	$(D_x^\alpha D_y^\alpha)(M(G, x, y))\mid_{x=y=1}$
$H(G)$	Harmonic	$\frac{2}{x+y}$	$2S_x J(M(G, x, y))\mid_{x=y=1}$
$I_n(G)$	Inverse Sum Index	$\frac{xy}{x+y}$	$S_x J D_x D_y(M(G, x, y))\mid_{x=y=1}$
$S_D(G)$	Symmetric Division	$\frac{x^2+y^2}{xy}$	$(D_x S_y + S_x D_y)(M(G, x, y))\mid_{x=y=1}$

where $D_x = x\frac{\partial f(x, y)}{\partial x}$, $D_y = y\frac{\partial f(x, y)}{\partial y}$, $S_x = \int_0^x \frac{f(t, y)}{t}dt$, $S_x = \int_0^y \frac{f(x, t)}{t}dt$, $J(f(x, y)) = f(x, x)$ and $\alpha \in \mathbb{N}$.

2. M-polynomials of some derived graphs

In this section, to compute the M-polynomials and some standard topological indices of derived graphs of certain families of graphs, we need the following definitions.

Definition 2.1. The line graph $L(G)$ is the graph with vertex set $V(L(G)) = E(G)$ and whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common.

Definition 2.2. The subdivision graph $S(G)$ is the graph obtained from G by replacing each of its edges by a path of length two, or equivalently, by inserting an additional vertex into each edge of a graph G.

Definition 2.3. The vertex-Semitotal graph $T_1(G)$ with vertex set $V(G) \cup E(G)$ and edge set $E(S(G)) \cup E(G)$ is the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it.

Definition 2.4. The edge-Semitotal graph $T(G)$ with vertex set $V(G) \cup E(G)$ and edge set $E(S(G)) \cup E(L(G))$ is the graph obtained by inserting a new vertex into each edge of G and by joining with edges those pairs of these new vertices which lie on adjacent edges of G.

Definition 2.5. The total graph of a graph G denoted by $T(G)$ with vertex set $V(G) \cup E(G)$ and any two vertices of $T(G)$ are adjacent if and only if they are either incident or adjacent in G.

Throught this paper, we take G^* as derived graph of a r-regular graph G with $n \geq 2$ vertices or a complete bipartite graph $G = K_{r,s}$ with $1 \leq r \leq s$ vertices.

Observation 2.1. The aforesaid derived graphs possess vertices corresponding to the vertices of the parent graph G, and vertices corresponding to the edges of the parent graph. We call the former as γ-vertex set of the derived graph whereas the latter its λ-vertex set.

Proposition 2.1 [15]**.** *Edge partitions of derived graphs of regular graph and complete bipartite graph are as follows*.

Table 2. Edge Partitions of Derived Graphs.

G^*	Regular Graph	$K_{r,s}$
$L(G)$	E_{2r-2}	E_{r+s-2}

$S(G)$	$E_{2,r}$	$E_{2,r}, E_{2,s}$
$T_1(G)$	$E_{2,2r}, E_{2r}$	$E_{2r,2}, E_{2s,2}, E_{2r,2s}$
$T_2(G)$	$E_{2,2r}, E_{2r}$	$E_{r,r+s}, E_{s,r+s}, E_{r+s}$
$T(G)$	E_{2r}	$E_{2s,2r}, E_{2s,r+s}, E_{2r,r+s}, E_{r+s}$

where $E_i = \{e = uv \in E(G) : d_G(u) = d_G(v) = i\} = (i, i)$

$E_{i,j} = \{e = uv \in E(G) : d_G(u) = i, d_G(v) = j\} = (i, j)$.

$|E_{2r-2}| = \dfrac{nr(r-1)}{2}$, $|E_{r+s-2}| = \dfrac{rs(r+s-2)}{2}$, $|E_{2,r}| = nr$, $|E_{2,r}| =$

$= rs(in\ S(K_{r,s}))$, $|E_{2,s}| = rs$, $|E_{2,2r}| = nr$, $|E_{2r}| = \dfrac{nr}{2}$, $|E_{2r,2}| = rs$,

$|E_{2s,2}| = rs$, $|E_{2r,2s}| = rs$, $|E_{r,2r}| = nr$, $|E_{2r}| = \dfrac{nr(r-1)}{2}$ $(\text{in}\, T_2(G))$,

$|E_{r,r+s}| = rs$, $|E_{s,r+s}| = rs$, $|E_{r+s}| = \dfrac{rs(r+s-2)}{2}$, $|E_{2r}| = \dfrac{nr^2}{2} + nr$

$(\text{in}\, T(G))$, $|E_{2s,2r}| = rs$, $|E_{2s,r+s}| = rs$, $|E_{2r,r+s}| = rs$, $|E_{r+s}| = \dfrac{rs(r+s-2)}{2}$

$(\text{in}\, T(K_{r,s}))$.

Theorem 2.1. *M-polynomials of some derived graphs of r-regular graph $M(G^*, x, y)$ and complete bipartite graph $M(K^*_{r,s}, x, y)$ are as below.*

Table 3. The M-polynomials of Derived Graphs.

G^*	$M(G^*, x, y)$	$M(K^*_{r,s}, x, y)$
$L(G)$	$\dfrac{nr(r-1)}{2} x^{2r-2} y^{2r-2}$	$\dfrac{rs(r+s-2)}{2} x^{r+s-2} y^{r+s-2}$
$S(G)$	$nrx^2 y^r$	$rsx^2(y^r + y^s)$

$T_1(G)$	$\frac{nr}{2}y^{2r}(2x^2+x^{2r})$	$rs[x^{2r}y^2+x^{2s}y^2+x^{2r}y^{2s}]$
$T_2(G)$	$\frac{nr}{2}[2x^ry^{2r}+(r-1)x^{2r}y^{2r}]$	$\frac{rs}{2}[2x^ry^{r+s}+2x^sy^{r+s}$ $+(r+s-2)x^{r+s}y^{r+s}]$
$T(G)$	$\left(\frac{nr^2}{2}+nr\right)x^{2r}y^{2r}$	$rsx^{2s}y^{2r}+rsx^{2s}y^{r+s}+rsx^{2r}y^{r+s}$ $+\frac{rs(r+s-2)}{2}x^{r+s}y^{r+s}$

Proof. Clearly one can obtain table 3 by using definition 1.1, formulas in table 1 and proposition 2.1. □

Theorem 2.2. *Different topological indices of derived graphs of r-regular graph and complete bipartite graph are shown below.*

Table 4. The topological indices of Derived graphs.

D^*		$M_1(D^*)$	$M_2(D^*)$	$R_\alpha(D^*)$	$H(D^*)$	$I_n(D^*)$	$S_D(D^*)$
$L(G)$	RG	$2nr(r-1)^2$	$2nr(r-1)^3$	$nr2^{2\alpha-1}(r-1)^{2\alpha+1}$	$\frac{nr}{4}$	$\frac{nr(r-1)^2}{2}$	$nr(r-1)$
	$K_{r,s}$	$2rs(r+s-2)^2$	$\frac{rs(r+s-2)^3}{2}$	$\frac{rs}{2}(r+s-2)^{2\alpha+1}$	$\frac{rs}{2}$	$\frac{rs(r+s-2)^2}{4}$	$rs(r+s-2)$
$S(G)$	RG	$nr(r+2)$	$2nr^2$	$n2^{\alpha}r^{\alpha+1}$	$\frac{2nr}{2+r}$	$\frac{2nr^2}{2+r}$	$\frac{nr^2(r+4)}{4}$
	$K_{r,s}$	$rs(r+s+4)$	$2rs(r+s)$	$rs2^{\alpha}(r^{\alpha}+s^{\alpha})$	$\frac{2nr(r+s+4)}{(r+2)(s+2)}$	$\frac{4nr(r+s+4)}{(r+2)(s+2)}$	$\frac{(r+s)(4+rs)}{2}$
$T_1(G)$	RG	$2nr(2r+1)$	$2nr^2(2+r)$	$n2^{\alpha-1}r^{\alpha+1}[2^{\alpha+1}+(2r)^{\alpha}]$	$\frac{n(5r+1)}{4(r+1)}$	$nr^2\left[\frac{2}{r+1}+\frac{1}{2}\right]$	$n(r^2+r+1)$
	$K_{r,s}$	$4rs(r+s+1)$	$4rs(r+s+rs)$	$rs2^{2\alpha}[r^{\alpha}+s^{\alpha}+(rs)^{\alpha}]$	$rs\left[\frac{1}{r+1}+\frac{1}{s+1}+\frac{1}{r+s}\right]$	$2rs\left[\frac{r}{r+1}+\frac{s}{s+1}+\frac{rs}{r+s}\right]$	$(r+s)(rs+1)+r^2+s^2$
$T_2(G)$	RG	$nr^2(2r+1)$	$2nr^4$	$nr^{2\alpha+1}[2^{\alpha+1}+(r-1)2^{2\alpha-1}]$	$\frac{n(5+3r)}{12}$	$2nr^3\left[\frac{1}{3}+\frac{r-1}{4}\right]$	$\frac{nr}{2}(2r+3)$
	$K_{r,s}$	$rs(r+s)(r+s+1)$	$\frac{rs}{2}(r+s)^3$	$\frac{rs}{2}(r+s)^{\alpha}[2r^{\alpha}+2s^{\alpha}+(r+s-2)(r+s)^{\alpha}]$	$rs\left[\frac{2}{2r+s}+\frac{2}{2s+r}+\frac{r+s-2}{2r+2s}\right]$	$\frac{rs}{2}(r+s)\left[\frac{2r}{2r+s}+\frac{2s}{r+2s}+\frac{(r+s-2)^2}{2}\right]$	$rs\left[r+s-1+\frac{(r+s)^2}{rs}\right]$
$T(G)$	RG	$2nr^2(r-2)$	$2nr^3(r+2)$	$\frac{nr(r+2)}{2}(2r)^{2\alpha}$	$\frac{n(r+2)}{4}$	$\frac{nr^2(r+2)}{4}$	$nr^2(r+2)$
	$K_{r,s}$	$rs(r+s)(r+s+4)$	$rs\left[(r+s)^2+4rs+\frac{(r+s)^3}{2}\right]$	$rs\left[(r+s)^{\alpha}((2s)^{\alpha}+(2r)^{\alpha})+(4rs)^{\alpha}+\frac{(r+s-2)(r+s)^{2\alpha}}{2}\right]$	$2rs\left[\frac{1}{r+3s}+\frac{1}{3r+3s}+\frac{1}{3r+s}+\frac{r+s-2}{4(r+s)}\right]$	$rs\left[\frac{2rs}{r+s}+\frac{3s(r+s)}{r+3s}+\frac{3r(r+s)}{3r+s}+\frac{(r+s-2)(r+s)}{4}\right]$	$rs(r+s+1)+\frac{3}{2}(r^2+s^2)$

where RG stands for r-regular graph.

Proof. Applying the formulas tabulated in table 1 for respective topological indices on the M-polynomials in table 3, table 4 can be obtained. □

Theorem 2.3. *Let G be any connected graph with* $n \geq 2$ *vertices and m edges, then*

(i) $M_1(S(G)) = M_1(G) + 4m$

(ii) $M_1(T_1(G)) = 4M_1(G) + 4m$

(iii) $M_1(T_2(G)) = 5M_1(G) + M_1(L(G)) - 4m$

(iv) $M_1(T(G)) = 8M_1(G) + M_1(L(G)) - 4m$

(v) $M_1(T(G)) = M_1(T_1(G)) + M_1(T_2(G)) - M_1(S(G))$

(vi) $M_2(T(G)) = 2M_2(T_1(G)) + M_2(T_2(G)) - 4M_2(S(G))$ *if G is regular*.

Proof. (i) As for any v in λ vertex set of $S(G)$, $d_S(v) = 2$ and for any v in γ vertex set of $S(G)$, $d_S(v) = d_G(v)$. Therefore,

$$M_1(S(G)) = \sum_{uv \in E(S)} (d_S(u) + d_S(v)) = \sum_{u \in \gamma,\, v \in \lambda} (d_G(u) + 2)$$

$$= \sum_{uv \in E(S)} [d_S(u) + d_S(w)] + 2(2m)$$

$$= M_1(G) + 4m$$

(ii) Clearly, for any v in λ vertex set of $T_1(G)$, $d_{T_1}(v) = 2$ and for any v in γ vertex set of $T_1(G)$, $d_{T_1}(v) = 2d_G(v)$. From the construction of vertex-semitotal graph, we have two edge partition of $T_1(G)$ as follows.

$$E_1 = \{uv \in E(T_1(G)) : u, v \in \gamma\}, |E_1| = m, \text{ and}$$

$$E_2 = \{uv \in E(T_1(G)) : u \in \gamma, v \in \lambda\}, |E_2| = 2m$$

$$M_1(T_1(G)) = \sum_{un \in E(T_1)} (d_{T_1}(u) + d_{T_1}(v))$$

$$= \sum_{un \in E_1} (d_{T_1}(u) + d_{T_1}(v)) + \sum_{un \in E_2} (d_{T_1}(u) + d_{T_1}(v))$$

$$= \sum_{un \in E(G)} (2d_G(u) + 2d_G(v)) + \sum_{un \in E_2} (d_{T_1}(u) + 2)$$

$$= \sum_{un \in E(G)} (2d_G(u) + 2d_G(v)) + 2 \sum_{un \in E(G)} (d_G(u) + d_G(w))$$

$$+ 2(2m) = 4M_1(G) + 4m.$$

(iii) For any $e' = uv$ in λ vertex set of $T_2(G)$, $d_{T_1}(e') = d_G(u) + d_G(v)$

$-2+2 = d_G(u)+d_G(v)$ and for any v in γ vertex set of $T_2(G)$, $d_{T_2}(e') = d_G(u)$. From the construction of edge-semitotal graph $T_2(G)$, we have two edge partitions.

$$E_1 = \{ue' \in E(T_2(G)) : u \in \gamma,\ e' \in \gamma\},\ |E_1| = 2m, \text{ and}$$

$$E_2 = \{e'f' \in E(T_2(G)) : e',\ f' \in \lambda\},\ |E_2| = E(L(G)) = \frac{1}{2}[M_1(G) - 2m].$$

Note that, for every $uv \in E(G)$, there corresponds two edges ue' and $e'v$ in E_1 and degree of each vertex in λ vertex set is 2 more than the degree of corresponding vertex in $L(G)$. Further, for every edge $e'f' \in E_2$, let ef be the corresponding edge in $L(G)$.

$$\begin{aligned} M_1(T2(G)) &= \sum_{ue' \in E_1}(d_{T_2}(u) + d_{T_2}(e')) \\ &+ \sum_{e'f' \in E_2}(d_{T_2}(e') + d_{T_2}(f')) \\ &= \sum_{uv \in E(G)}(d_G(u) + d_G(v)) + 2(d_G(u) + d_G(v)) \\ &+ \sum_{ef \in E(L)}(d_L(e) + 2 + d_L(f) + 2) \\ &= 3M_1(G) + M_1(L(G)) + 4|E(L(G))| \end{aligned}$$

(iv) For any $e' = uv$ in λ vertex set of $T(G)$, $d_T(e') = d_G(u) + d_G(v)$ $-2+2 = d_G(u)+d_G(v)$ and for any v in γ vertex set of $T(G)$, $d_T(v) = 2d_G(v)$. From the construction of total graph $T(G)$, we have three edge partitions.

$$E_1 = \{uv \in E(T(G)) : uv \in \gamma\},\ |E_1| = m,$$

$$E_2 = \{ue' \in E(T(G)) : u \in \gamma,\ e' \in \lambda\},\ |E_2| = 2m, \text{ and}$$

$$E_3 = \{e'f' \in E(T(G)) : e',\ f' \in \lambda\},\ |E_3| = E(L(G)) = \frac{1}{2}[M_1(G) - 2m].$$

Note that, for every $uv \in E(G)$, there corresponds two edges ue' and $e'v$

in E_2 and degree of each vertex in λ vertex set is 2 more than the degree of corresponding vertex in $L(G)$. Further, for every edge $e'f' \in E_3$, let ef be the corresponding edge in $L(G)$.

$$\begin{aligned} M_1(T(G)) &= \sum_{uv \in E_1} (d_T(u) + d_T(v)) + \sum_{ue' \in E_2} (d_T(u) + d_T(e')) \\ &+ \sum_{e'f' \in E_3} (d_T(e') + d_T(f')) \\ &= \sum_{uv \in E(G)} (2d_G(u) + 2d_G(v)) \\ &+ \sum_{uv \in E(G)} (2d_G(u) + 2d_G(v)) + 2(d_G(u) + d_G(v)) \\ &+ \sum_{ef \in E(L)} (d_L(e) + 2 + d_L(f) + 2) \\ &= 6M_1(G) + M_1(L(G)) + 4|E(L(G))| \end{aligned}$$

(v) The result directly follows from (i)-(iv).

(vi) The required result immediately follows from table 4. □

3. Some Bounds and Characterizations

In this section, we have obtained some upper and lower bounds for different topological indices of some derived graphs using some standard classical inequalities and also in terms of other topological indices. Let Δ and δ be the maximum and minimum vertex degree of G respectively.

Theorem 3.1. *Let G be any connected graph with $n \geq 2$ vertices and m edges. Then*

$$4m(2\delta + 1) \leq M_1(T_1(G)) \leq 4m(2\Delta + 1)$$

Equality on both sides hold if and only if G is regular.

Proof. Two edge partitions of $T_1(G)$ as follows.

$E_1 = \{uv \in E(T_1(G)) : u, v \in \gamma\}, | E_1 | = m$, and

$E_2 = \{uv \in E(T_1(G)) : u \in \gamma, v \in \lambda\}, | E_2 | = 2m$

$$M_1(T_1(G)) = \sum_{uv \in E(T_1)} (d_{T_1}(u) + d_{T_1}(v))$$

$$= \sum_{uv \in E_1} (d_{T_1}(u) + d_{T1}(v)) + \sum_{uv \in E_2} (d_{T_1}(u) + d_{T_1}(v))$$

For any $uv \in E_1$, $4\delta \le d_{T_1}(u) + d_{T_1}(v) \le 4\Delta$ and for any $uv \in E_2$, $2 + 2\delta \le d_{T_1}(u) + d_{T_1}(v) \le 2 + 2\Delta$. Hence the above expression becomes $4m\delta + 4m + 4m\delta \le M_1(T_1(G)) \le 4m\Delta + 4m + 4m\Delta$ and thus the result follows. Second part of the result follows from the definitions of $M_1(T_1(G))$ and the regular graph. □

To prove our next theorem, we make use of the following result.

Theorem 3.2 [2]. *Let G be a* (n, m)*-connected graph with* $n \ge 3$ *vertices. Then*,

$$M_1(L(G)) \ge \frac{(M_1(G) - 2m)^2}{m}$$

Further, equality holds if and only if G is regular.

Theorem 3.3. *Let G be any connected graph with* $n \ge 2$ *vertices and m edges. Then*

(i) $M_1(T_2(G)) \ge \dfrac{M_1(G)(M_1(G) + m)}{m}$

(ii) $M_1(T(G)) \ge \dfrac{M_1(G)(M_1(G) + 4m)}{m}$

Equality holds if and only if G is regular.

Proof. The lower bounds follow from results (iii) and (iv) of theorem 2.3 and theorem 3.2. □

Theorem 3.4. *For any* (n, m)*-connected graph G with* $n \ge 3$ *vertices*, η

pendent vertices and minimal non-pendant vertex degree $\delta_1(G)$, $4\eta + 8m\delta_1 + [4\delta_1^2(m-\eta)] \leq M_2(T_1(G)) \leq 4\eta + 8m\Delta + [4\Delta^2(m-\eta)]$.

Proof. From the definition, we have

$$M_2(T_1(G)) = \sum_{uv\in E(T_1(G))} d_{T_1}(u)d_{T_1}(v)$$

$$= \sum_{uv\in E(G)} (2d_G(u))(2d_G(v)) + \sum_{uv\in E(G)} 2(2d_G(u)) + 2(2d_G(v))$$

$$= 4 \sum_{uv\in E(G),\, d_G(u)\neq 1,\, d_G(v)=1} (d_G(u) + d_G(u) + 1)$$

$$+ 4 \sum_{uv\in E(G),\, d_G(u)\neq 1,\, d_G(v)\neq 1} d_G(u)\, d_G(v) + (d_G(u) + d_G(v))$$

$$\leq 4\eta(2\Delta + 1) + 4(m - n)(\Delta^2 + 2\Delta).$$

Thus the upper bound follows. The lower bound follows on the similar lines. □

Remark 3.5. The bounds of the above theorem is attained if and only if $d_G(u) = d_G(v) = \Delta(G) = \delta_1(G)$ for each $uv \in E(G)$ with $d_G(u) \neq 1$, $d_G(v) \neq 1$ and $d_G(u) = \Delta(G) = \delta_1(G)$ for each $uv \in E(G)$ with $d_G(v) = 1$.

Theorem 3.6. *Let G be any connected graph with* $n \geq 2$ *vertices and m edges. Then*

$$\frac{1}{2}H(G) + \frac{2m}{\Delta + 1} \leq H(T_1(G)) \leq \frac{1}{2}H(G) + \frac{2m}{\delta + 1}$$

Equality on both sides attain if and only if G is regular.

Proof. Edge partitions of $T_1(G)$ are given by

$E_1 = \{uv \in E(T_1(G)) : u, v \in \gamma\}$, $|E_1| = m$, and

$E_2 = \{uv \in E(T_1(G)) : u \in \gamma, v \in \lambda\}$, $|E_2| = 2m$, Therefore,

$$H(T_1(G)) = \sum_{uv \in E(T_1(G))} \frac{2}{d_{T_1}(u) + d_{T_1}(v)}$$

$$= \sum_{uv \in E_1} \frac{2}{d_{T_1}(u) + d_{T_1}(v)} + \sum_{uv \in E_2} \frac{2}{d_{T_1}(u) + d_{T_1}(v)}$$

$$= \sum_{uv \in E(G)} \frac{2}{2d_{T_1}(u) + 2d_{T_1}(v)} + + \sum_{uv \in E_2} \frac{2}{d_{T_1}(u) + 2}$$

$$= \frac{1}{2} H(G) + \sum_{uv \in E_2} \frac{2}{d_{T_1}(u) + 2}$$

Note that, for any $u \in V(T_1(G))$, $2\delta \le d_{T_1}(u) \le 2\Delta$. Hence $\frac{2}{2\Delta + 2} \le \frac{2}{d_{T_1}(u) + 2} \le \frac{2}{2\delta + 2}$. Thus the result follows. □

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