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AVERAGE DOMINATION ON ANTI FUZZY GRAPH

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Abstract

In this paper, we introduce the average domination on anti fuzzy graph. Further, we investigate the exact value of average domination number for some standard anti fuzzy graphs. Also, establish the relationship between the average domination number and independent domination number on anti fuzzy graph with suitable example.

1. Introduction

A. Rosenfeld introduced the concept of Fuzzy graphs in 1975. R. Seethalakshmi and R. B. Gnanajothi [7] introduced the concept of Anti fuzzy graphs. Further, R. Muthuraj and A. Sasirekha [5, 6] developed the concept on anti fuzzy graph and also introduced the domination on anti fuzzy graph. Henning [1] introduced the concept of average domination and independent domination numbers. In this paper, average domination number is studied for some standard anti fuzzy graphs and the relationship between the average domination number and independence number on anti fuzzy graph is established.

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2. Preliminaries

Definition 2.1. A pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ with $\mu(a, b) \geq \sigma(a) V \sigma(b)$ for all a, b in V is called an *anti fuzzy graph* where V is a finite non empty set and V denote maximum. It is denoted by $\mathcal{A}$.

Definition 2.2. If the underlying graph $\mathcal{A}^*$ is complete and $\mu(a, b) = \sigma(a) V \sigma(b)$ for every (a, b) in E then the anti fuzzy graph $\mathcal{A}$ is called the *complete anti fuzzy graph*.

Definition 2.3. Every vertices and edges in an anti fuzzy graph $\mathcal{A}$ have same membership value then $\mathcal{A}$ is called *uninodal* anti fuzzy graph.

Definition 2.4. If for every vertex $b \in V(\mathcal{A}) \setminus D$ then there exists a in D such that a is a strong neighborhood of b otherwise it dominates itself. Then the set $D \subseteq V(\mathcal{A})$ is said to be a *dominating set of an anti fuzzy graph* $\mathcal{A}$. Among all minimal dominating set in $\mathcal{A}$ the maximum fuzzy cardinality is called a *domination number of an anti fuzzy graph* $\mathcal{A}$. It is denoted by $\gamma_{\mathcal{A}}$.

Definition 2.5. An anti fuzzy graph $\mathcal{A} = (\sigma, \mu)$ is said to be *strong* if $\mu(a, b) = \sigma(a) V \sigma(b)$ for all (a, b) in E

3. Main Results

Definition 3.1. The maximum fuzzy cardinality taken over all minimal dominating set of $\mathcal{A}$ that contains the vertex a is called the *lower domination number* on an anti fuzzy graph. It is denoted by $\gamma_a(\mathcal{A})$. A dominating set of cardinality $\gamma_a(\mathcal{A})$ is called the γ_a-set of anti fuzzy graph. The γ_a-set is denoted by D_a.

Definition 3.2. The *average domination number* on anti fuzzy graph is defined as $\frac{\sum_{a \in V} \gamma_a(\mathcal{A})}{|V|}$ where $\gamma_a(\mathcal{A})$ is the lower domination number. It is denoted by $\gamma_{Ag}(\mathcal{A})$.

Example 3.3.

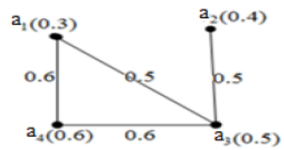

Figure 1. Anti Fuzzy graph $\mathcal{A}$.

$\{a_3\}$ forms a dominating set of $\mathcal{A} \cdot \gamma_{\mathcal{A}} = 1$.

Table 1. Lower Domination number on Anti Fuzzy graph.

$a \in V(\mathcal{A})$	γ_a - set	$\gamma_a(\mathcal{A})$
a_1	$\{a_1, a_3\}$	0.8
a_2	$\{a_2, a_4\}$	1
a_3	$\{a_3\}$	0.5
a_4	$\{a_4, a_3\}$	1.1

$$\gamma_{Ag}(\mathcal{A}) = \frac{\sum_{a \in V} \gamma_a(\mathcal{A})}{|V|} = \frac{0.8 + 1 + 0.5 + 1.1}{4} = \frac{3.4}{4} = 0.85.$$

Theorem 3.4. *If* $\mathcal{A} = (\sigma, \mu)$ *is a complete anti fuzzy graph then* $\gamma_{Ag}(\mathcal{A}) = p/|V|$.

Proof. Let $\mathcal{A}$ be a complete anti fuzzy graph with n vertices $\{a_1, a_2, a_3, \ldots, a_n\}$. Then, a_i is adjacent to all the other vertices $\{a_1, a_2, a_3, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$.

By the definition of complete anti fuzzy graph, each a_i dominates all the other vertices. Moreover, by the definition of lower domination number $\sum_{a \in V} \gamma_a(\mathcal{A}) = \sigma(a_1) + \sigma(a_2) + \ldots + \sigma(a_n) = \sum_{i=1}^{n} \sigma(a_i) = p$. Clearly $\gamma_{Ag}(\mathcal{A}) = p/|V|$.

Theorem 3.5. *If an anti fuzzy graph* $\mathcal{A} = (\sigma, \mu)$ *is a uninodal path with* *effective edges then*

$$\gamma_{Ag}(\mathcal{A}) = \begin{cases} \left[\dfrac{3k^2 + 2k}{|V|}\right]\sigma(a) & \text{for } n = 3k \\ (k+1)\sigma(a) & \text{for } n = 3k+1 \\ \left[\dfrac{3k^2 + 6k + 2}{|V|}\right]\sigma(a) & \text{for } n = 3k+2 \end{cases}$$

where $k = 1, 2, 3, \ldots,$ and $|V| = n.$

Proof. Let $\mathcal{A}$ be a uninodal path anti fuzzy graph with effective edges.

Case 1: For $n = 3k,$ Let $\mathcal{A}$ be a uninodal path anti fuzzy graph with n vertices $\{a_1, a_2, a_3, \ldots, a_{3k}\}$ where $k = 1, 2, 3, \ldots$ Then a_1 is adjacent to a_2 and a_{n-1} is adjacent to a_n, a_i is adjacent to a_{i-1} and a_{i+1}. It is obvious that a_1 dominates a_2 and a_{n-1} dominates a_n, a_i is dominated by a_{i-1} and a_{i+1}, $2 \le i \le n-1$. By the definition of lower domination number, In P_{3k}, $\sum_{a\in V}\gamma_a(\mathcal{A})$- set having the fuzzy cardinality of $(k+1)\sigma(a)$ in 2k times and $k\ \sigma(a)$ in k times.

Therefore, we get $\sum_{a\in V}\gamma_a(\mathcal{A}) = [(k+1)2k + k(k)]\sigma(a) = [2k^2 + 2k + k^2]$ $\sigma(a) = [3k^2 + 2k]\sigma(a)$. Hence, $\gamma_{Ag}(\mathcal{A}) = \left[\dfrac{3k^2 + 2k}{|V|}\right]\sigma(a)$, $k = 1, 2, \ldots$

Case 2: For $n = 3k+1,$ Let $\mathcal{A}$ be a uninodal path anti fuzzy graph with n vertices $\{a_1, a_2, a_3, \ldots, a_{3k+1}\}$ where $k = 1, 2, 3, \ldots$ From the definition of lower domination number, $\sum_{a\in V}\gamma_a(\mathcal{A})$- set having the fuzzy cardinality of $(k+1)\sigma(a)$ in $3k+1$ times. Therefore, we get $\sum_{a\in V}\gamma_a(\mathcal{A})$ $= (k+1)\ (3k+1)\sigma(a)$. Clearly, $\gamma_{Ag}(\mathcal{A}) = \dfrac{(k+1)\ (3k+1)\sigma(a)}{|V| = 3k+1} = (k+1)\ \sigma(a)$, $k = 1, 2, \ldots.$

Case 3: For $n = 3k+2,$ Let $\mathcal{A}$ be a uninodal path anti fuzzy graph with n vertices $\{a_1, a_2, a_3, \ldots, a_{3k+2}\}$ where $k = 1, 2, 3, \ldots$ By the definition of lower domination number, in P_{3k+2}, $\sum_{a \in V} \gamma_a(\mathcal{A})$- set having the fuzzy cardinality of $(k+1)\sigma(a)$ in $(2k+2)$ times and $(k+2)\sigma(a)$ in k times.

Therefore, we get $\sum_{a \in V} \gamma_a(\mathcal{A}) = [(k+1)(2k+2) + (k+2)(k)]\sigma(a)$ $= [2k^2 + 2k + 2k + 2 + k^2 + 2k]\ \sigma(a) = [3k^2 + 6k + 2]\ \sigma(a)$. Hence, $\gamma_{Ag}(\mathcal{A})$ $= \left[\frac{3k^2 + 6k + 2}{|V|}\right]\sigma(a),\ k = 1, 2, \ldots.$

Theorem 3.6. *If $\mathcal{A}$ is an anti fuzzy path then $\gamma_{Ag}(\mathcal{A}) \leq p - \delta$ where p is the order and δ is the minimum degree of $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be an anti fuzzy path with order p and δ be the minimum degree of $\mathcal{A}$.

In an anti fuzzy path, except starting and ending vertices have maximum two neighbors. Clearly, from the definition average domination number of an anti fuzzy graph is less than $p - \delta$. Hence, $\gamma_{Ag}(\mathcal{A}) \leq p - \delta$.

Theorem 3.7. *If an anti fuzzy graph $\mathcal{A} = (\sigma, \mu)$ is a cycle C_n with all effective edges then*

$$\gamma_{Ag}(C_n) = \frac{(\frac{n}{3})p}{|V|} \text{ for } n = 3k \text{ where } k = 1, 2, \ldots \text{ and } |V| = n.$$

Proof. Let C_n be the anti fuzzy cycle with n vertices $\{a_1, a_2, a_3, \ldots, a_{n=3k} = a_0\}$ where $k = 1, 2, 3, \ldots$ such that a_i is adjacent with a_{i-1} and a_{i+1}. Also, a_i is dominated by a_{i-1} and a_{i+1}, $2 \leq i \leq n = 3k$. By the definition of lower domination number, In C_n, $\sum_{a \in V} \gamma_a(\mathcal{A})$- set having the fuzzy cardinality p in $(n/3)$ times. Therefore, $\sum_{a \in V} \gamma_a(\mathcal{A}) = (n/3)p$.

Hence, $\gamma_{Ag}(C_n) = \dfrac{\left(\frac{n}{3}\right)p}{|V|}$ for $n = 3k$ where $k = 1, 2, \ldots$ and $|V| = n$

Theorem 3.8. *If an anti fuzzy graph $\mathcal{A} = (\sigma, \mu)$ is a wheel with all* *effective edges then* $\gamma_{Ag}(W_{n+1}) = \frac{p + n(\sigma(b))}{|V|}$ *for* $n \geq 7$ *where b is the center vertex of the wheel.*

Proof. Let W_{n+1} be the anti fuzzy wheel with vertex set $\{b, a_1, a_2, a_3, \ldots, a_{n=3k} = a_0\}$ such that a_i is adjacent with a_{i-1} and a_{i+1}. Then, a_i is dominated by a_{i-1} and a_{i+1}. Also b dominates a_i, $i = 1, 2, \ldots, n$. By the definition of lower domination number, each vertex must be included. Hence, $\sum_{a \in V} \gamma_a(\mathcal{A})$- set having the fuzzy cardinality $\sigma(b), \sigma(a_1), \sigma(b), \sigma(a_2), \sigma(b), \ldots, \sigma(a_n), \sigma(b)$. (i.e.) It has $\sigma(b)$ in n times and p.

$$\sum_{a \in V} \gamma_a(W_{n+1}) = \sigma(b) + \sigma(a_1) + \sigma(a_2) + \ldots + \sigma(a_n) + \sigma(b) \ldots (n \text{ times})$$

$= p + n\sigma(b)$ Hence, $\gamma_{Ag}(W_{n+1}) = \frac{p + n(\sigma(b))}{|V|}$.

Theorem 3.9. *If an anti fuzzy graph* $\mathcal{A} = (\sigma, \mu)$ *is a star* $K_{1, n}$ *with all effective edges then* $\gamma_{Ag}(K_{1, n}) = \frac{p + n(\sigma(b))}{|V|}$ *where b is the root vertex.*

Proof. Let $K_{1, n}$ be the star anti fuzzy graph with the vertex set $\{b, a_1, a_2, a_3, \ldots, a_n\}$ such that b is adjacent with a_i, $i = 1, 2, \ldots, n$. Clearly, b dominates a_i. Hence, $\sum_{a \in V} \gamma_a(\mathcal{A})$- set having the fuzzy cardinality $\sigma(b), \sigma(a_1), \sigma(b), \sigma(a_2), \sigma(b), \ldots, \sigma(a_n), \sigma(b)$.

$$\sum_{a \in V} \gamma_a(K_{1, n}) = \sigma(b) + \sigma(a_1) + \sigma(a_2) + \ldots + \sigma(a_n) + \sigma(b) \ldots (n \text{ times})$$

$= p + n\sigma(b)$ Hence, $\gamma_{Ag}(K_{1, n}) = \frac{p + n(\sigma(b))}{|V|}$

Theorem 3.10. *If an anti fuzzy graph* $\mathcal{A} = (\sigma, \mu)$ *is a uninodal cycle* C_n *then*

$$\gamma_{Ag}(C_n) = \begin{cases} \dfrac{\left(\frac{n}{3}\right)p}{|V|} & \text{for } n = 3k \\ \dfrac{\left(\left[\frac{n}{3}\right]+1\right)p}{|V|} & \text{for } n \neq 3k \end{cases}$$

where $\left[\frac{n}{3}\right]$ *is the greatest integer port of* $\frac{n}{3}$ *and* $k = 1, 2, 3, \ldots$.

Proof. Let C_n be the anti fuzzy uninodal cycle with n fuzzy vertices $\{a_1, a_2, a_3, \ldots, a_n\}$.

Case 1: For $n = 3k$, a_i is adjacent with a_{i-1} and a_{i+1}, $2 \leq i \leq n$. Clearly, a_i is dominated by a_{i-1} and a_{i+1}, By the definition of lower domination number, in C_n, $\sum_{a \in V} \gamma_a(\mathcal{A})$- set having the fuzzy cardinality p in $(n/3)$ times. Therefore, $\sum_{a \in V} \gamma_a(C_n) = ([n/3] + 1)p$.

Hence, $\gamma_{Ag}(C_n) = \dfrac{\left(\left[\frac{n}{3}\right]+1\right)p}{|V|}$.

Case 2: For $n \neq 3k$ since $\mathcal{A}$ be the uninodal cycle, one strong arc can dominate two other arcs adjacent to it. Hence, by the definition of lower domination number, $\sum_{a \in V} \gamma_a(\mathcal{A})$- set having the fuzzy cardinality p in $([n/3]+1)$ times.

Therefore, $\sum_{a \in V} \gamma_a(C_n) = ([n/3] + 1)p$. Hence, $\gamma_{Ag}(C_n) = \dfrac{\left(\frac{n}{3}\right)p}{|V|}$.

Theorem 3.11. *For any strong anti fuzzy graph* $\mathcal{A} = (\sigma, \mu)$, $\gamma_{Ag}(\mathcal{A}) \leq i(\mathcal{A}) \leq \alpha(\mathcal{A})$.

Proof. Let $\mathcal{A}$ be a strong anti fuzzy graph. Suppose $\delta = n - 1$ of a graph then $\gamma_a(\mathcal{A}) = p/n$ and $i(\mathcal{A}) = \sigma(a_i)$ where a_i has the maximum fuzzy value.

Then, $\frac{\sigma(a_1) + \sigma(a) + \ldots + \sigma(a_n)}{n} \leq \sigma(a_i)$. It is enough to prove that for $\delta = n - 1$ of a graph G. Because the independence domination number of a graph G with $\delta = n - 1$ is 1. Clearly, $\delta < n - 1$ graphs have the independence domination number greater than 1.

Therefore, $\gamma_{Ag}(\mathcal{A}) \leq i(\mathcal{A})$. Obviously, $i(\mathcal{A}) \leq \alpha(\mathcal{A})$. Hence, $\gamma_{Ag}(\mathcal{A}) \leq i(\mathcal{A}) \leq \alpha(\mathcal{A})$.

Example 3.12.

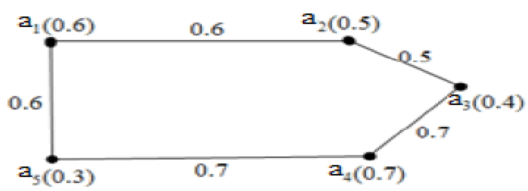

Figure 2. Strong Anti Fuzzy graph.

Table 2. Lower Domination number on Strong Anti Fuzzy graph.

$a \in V(\mathcal{A})$	γ_a - set	$\gamma_a(\mathcal{A})$
a_1	$\{a_1, a_4\}$	1.3
a_2	$\{a_2, a_4\}$	1.2
a_3	$\{a_3, a_1\}$	1
a_4	$\{a_4, a_1\}$	1.3
a_5	$\{a_5, a_2\}$	0.8

$$\gamma_{Ag}(\mathcal{A}) = \frac{\sum_{a \in V} \gamma_a(\mathcal{A})}{|V|} = 1.1$$

Independent dominating sets are $\{a_1, a_3\}$, $\{a_1, a_4\}$, $\{a_2, a_4\}$, $\{a_2, a_5\}$ $\{a_3, a_5\}$ and $i(\mathcal{A}) = 1.3$.

Independent sets are $\{a_1, a_3\}$, $\{a_1, a_4\}$, $\{a_2, a_4\}$, $\{a_2, a_5\}$, $\{a_3, a_5\}\alpha(\mathcal{A}) = 1.3$.

Hence, $\gamma_{Ag}(\mathcal{A}) \leq i(\mathcal{A}) \leq \alpha(\mathcal{A})$.

Theorem 3.12. *If* $\mathcal{A} = (\sigma, \mu)$ *is an anti fuzzy complete bipartite graph* K_{σ_1, σ_2} *then* $\gamma_{Ag}(K_{\sigma_1, \sigma_2}) = \frac{p + m(\max(\sigma(b_j))) + n(\max(\sigma(a_i)))}{|V|}$ *where* $a_i \in V_1$ *and* $b_j \in V_2$ *with* $3 \leq m \leq n$.

Proof. Let K_{σ_1, σ_2} be the anti fuzzy complete bipartite graph. By the definition, V can be partitioned into two sets V_1 and V_2. Let $V_1 = \{a_1, a_2, a_3, \ldots, a_m\}$ and $V_2 = \{b_1, b_2, b_3, \ldots, b_n\}$ with $3 \leq m \leq n$. Now, a_i is adjacent with b_j. $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Let D_a be the lower dominating set that contains a.

Clearly, $D_{a_1} = \{a_1, \max(b_j)\}$ $D_{a_2} = \{a_2, \max(b_j)\}, \ldots, D_{a_m} = \{a_m, \max(b_j)\}$ $D_{b_1} = \{b_1, \max(a_j)\}$ $D_{b_2} = \{b_2, \max(a_j)\}, \ldots, D_{b_m} = \{b_m, \max(a_j)\} \sum_{a \in V} \gamma_a(\mathcal{A})$ $= \sigma(a_1) + \sigma(a_2) + \ldots + \sigma(a_m) + \sigma(b_1) + \sigma(b_2) + \ldots + \sigma(b_n) + m \max(\sigma(b_j))$ $+ n \max(\sigma(a_i)) = p + m \max(\sigma(b_j)) + n \max(\sigma(a_i))$.

Hence, $\gamma_{Ag}(K_{\sigma_1, \sigma_2}) = \frac{p + m(\max(\sigma(b_j))) + n(\max(\sigma(a_i)))}{|V|}$ where $a_i \in V_1$ and $b_j \in V_2$.

Theorem 3.13. *Let* $\mathcal{A} = (\sigma, \mu)$ *be an anti fuzzy Petersen graph with all effective edges then*

$$\gamma_{Ag}(\mathcal{A}) = \frac{\begin{array}{c} p + \sum_{i=1}^{5} \max[(Int\ \sigma(a_{i+4}) + Ext\ \sigma(a_{i+7})), (Int\ \sigma(a_{i+1}) + Ext\ \sigma(a_{i+8}))] \\ + \sum_{i=6}^{10} \max[(Int\ \sigma(a_{i-2}) + Int\ \sigma(a_{i-3})), (Ext\ \sigma(a_{i+2}) + Int\ \sigma(a_{i-4}))] \end{array}}{|V|}$$

where $Int\,\sigma(a_i)$ *denotes the fuzzy value of the interior cycle of* a_i *and Ext* $Ext\ \sigma(a_i)$ *denotes the fuzzy value of the exterior cycle* a_i.

Proof. By routine verification.

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