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A NOTE ON (ϕ, A, B) CONTRACTIVE RESULTS USING COMPARISON FUNCTION ON *b*-METRIC SPACES

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Abstract

In this paper, we introduce the pair of functions to satisfy some contractive condition in the setting of b-metric spaces to establish the results regarding to get the unique common fixed point for such pair by using the comparison function.

1. Introduction

In 1989, Bakhtin [1] introduced the concept of b-metric spaces as a generalization of metric spaces. He used it to prove a generalization of the Banach principle in spaces endowed with such kind of metrics. Many authors used this notion to obtain various fixed point theorems. Common fixed point results for single-valued and multi-valued mappings satisfying a weak ϕ-contraction in b-metric spaces was introduced by Aydi et al. in [2]. Starting from the results of Berinde [3], the existence and uniqueness of fixed points of ϕ-contraction on b-metric spaces was proved by Pacurar [4].

"The aim of this paper is to establish the results on b-metric spaces, regarding common fixed points of two mappings, using a cyclic contraction condition defined by means of a comparison function".

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2. Preliminaries

Definition 2.1. Let M be a non-empty set and $b : M \times M \to [0, \infty)$. A function b is called a b-metric with constant $s \geq 1$ if

(i) $b(u, v) = 0$ if and only if $u = v$.

(ii) $b(u, v) = b(v, u)$ for all $u, v \in M$

(iii) $b(u, v) \leq s[b(u, w) + b(w, v)]$ for all $u, v, w \in M$.

The pair (M, b) is called a b-metric space.

Definition 2.2. Let $s \geq 1$ be a constant. A mapping $\phi : [0, \infty) \to [0, \infty)$ is called comparison function with base $s \geq 1$, if the following two axioms are fulfilled:

(i) ϕ is non-decreasing,

(ii) $\lim_{n \to \infty} \phi^n(t) = 0$ for all $t > 0$.

Clearly, if ϕ is a comparison function, then $\phi(t) < t$ for each $t > 0$.

3. Main Results

Definition 3.1. Let A, B be two non-empty closed subsets of a b-metric space (M, b) with coefficient $s \geq .1, \phi : [0, \infty) \to [0, \infty)$ and $f, g : X \to X$ be three mappings, the pair (f, g) is called a cyclic (ϕ, A, B)- contraction if

(i) ϕ is a comparison function

(ii) $A \cup B$ has a cyclic representation with respect to the pair (f, g), that is $f(A) \subseteq B, g(B) \subseteq A, X = A \cup B$,

(iii) There exists $0 < \delta < 1$ such that the following

$$u \in A, v \in B \Rightarrow \phi(sb(fu, gv)) \leq \phi(B(u, v)).$$

Theorem 3.1. *Let (M, b) be a complete b-metric space with constant $s \geq 1$ and A, B be a non-empty closed subsets of M, where $A \cup B$ has a* *cyclic representation with respect to the pair* (f, g), (*i.e.*, $f(A) \subseteq B$, $g(B) \subseteq A$ *and* $M = A \cup B$ *and* $f, g : M \to M$ *be two cyclic mappings on X. If there is a constant* $L < \frac{1}{1+s}$ *and a comparison function* ϕ *such that*

$$\phi(B(u, v)) = \phi\left(\max\left\{sb(u, v), sb(u, fu), sb(v, gv), L\left[\frac{b(u, gv) + b(v, fu)}{2}\right]\right\}\right) \quad (1)$$

holds for each $u, v \in M$. *Suppose that f or g is continuous. Then f and g have a unique common fixed point in* $A \cup B$.

Proof. Choose $u_0 \in A$, let $u_1 = f(u_0)$ since $f(A) \subseteq B$ we have $u_1 \in B$.

Let $u_2 = gu_1$. Since $g(B) \subseteq A$ we have $u_2 \in A$.

Continuing this process, we can construct a sequence $\{u_n\}$ in M such that

$$u_{2n+1} = fu_n \in B, u_{2n+2} = gu_{2n+1} \in A \quad (2)$$

for all $n \in \mathbb{N} \cup \{0\}$.

Suppose for some $n \in \mathbb{N}$ such that $u_n = u_{n+1}$. If $n = 2k$, then $u_{2k} = u_{2k+1}$ and from the contraction condition (1) with $u = u_{2k}$ and $v = u_{2k+1}$, we have $\phi(B(u, v)) = \phi(\max\{sb(u_{2k}, u_{2k+1}), sb(u_{2k}, gu_{2k}), sb(u_{2k+1}, gu_{2k+1}), L\left[\frac{b(u_{2k}, gu_{2k+1})b(u_{2k+1}, fu_{2k})}{2}\right]\}\bigg)$

$$= \phi(\max\{sb(u_{2k}, u_{2k+1}), sb(u_{2k}, u_{2k+1}), sb(u_{2k+1}, u_{2k+2}),$$

$$L\left[\frac{b(u_{2k}, u_{2k+2})b(u_{2k+1}, u_{2k+1})}{2}\right]\}\bigg)$$

$$= \phi\left(\max\left\{sb(u_{2k}, u_{2k+1}), sb(u_{2k+1}, u_{2k+2}), L\left[\frac{b(u_{2k}, u_{2k+2})}{2}\right]\right\}\right).$$

Suppose we have that $u_{2k} = u_{2k+1}$ as a comparison function and ϕ is non-decreasing,

$$\phi(B(u, v)) \leq \phi\left(\max\left\{sb(u_{2k+1}, u_{2k+2}), \frac{L}{2}[s(b(u_{2k}, u_{2k+1}) + b(u_{2k+1}, u_{2k+2}))]\right\}\right)$$

$$= \phi\left(\max\left\{sb(u_{2k+1}, u_{2k+2}), \frac{L}{2}[sb(u_{2k+1}, u_{2k+2})]\right\}\right)$$

$$= \phi(sb(u_{2k+1}, u_{2k+2})).$$

If $b(u_{2k+1}, u_{2k+2}) > 0$, then we have

$$\phi(sb(u_{2k+1}, u_{2k+2})) \leq \delta\phi(B(u, v))$$

$$= \delta\phi(sb(u_{2k+1}, u_{2k+2}))$$

$$< \phi(sb(u_{2k+1}, u_{2k+2})).$$

Which is a contradiction. Therefore $b(u_{2k+1}, u_{2k+2}) = 0$. Hence $u_{2k+1} = u_{2k+2}$. Thus we have $u_{2k} = u_{2k+1} = u_{2k+2}$.

By (2) $u_{2k} = u_{2k+1} = fu_{2k}$ and $u_{2k+1} = u_{2k+2} = gu_{2k}$.

$\therefore u_{2k} = fu_{2k} = gu_{2k}$.

This implies that *f* and *g* have u_{2k} as a common fixed point.

If $n = 2k + 1$, then as in the case $u_{2k} = u_{2k+1}$, we can show that u_{2k+1} is a common fixed point of *f* and *g*.

Now we shall prove that

$$\phi(sb(u_n, u_{n+1})) \leq \phi(sb(u_{n-1}, u_n)), \forall n \in \mathbb{N}. \tag{3}$$

Case (a) $n = 2k, k \in \mathbb{N}$.

From (1) with $u = u_{2k}$ and $v = u_{2k-1}$, we have

$$\phi(sb(u_{2k+1}, u_{2k})) \leq \delta\phi(B(u, v))$$

$$\Rightarrow \phi(sb(fu_{2k}, gu_{2k-1})) \leq \phi(B(u_{2k}, u_{2k-1}))$$

$$\phi(B(u, v)) = \phi(\max\{(sb(u_{2k}, u_{2k-1}), sb(u_{2k}, fu_{2k}),$$

$$sb(u_{2k-1}, gu_{2k-1})L\left[\frac{b(u_{2k}, gu_{2k-1}) + b(u_{2k-1}, fu_{2k})}{2}\right]\})$$

$$= \phi(\max\{(sb(u_{2k}, u_{2k-1}), sb(u_{2k}, u_{2k+1}),$$

$$sb(u_{2k-1}, u_{2k})L\left[\frac{b(u_{2k}, u_{2k}) + b(u_{2k-1}, u_{2k+1})}{2}\right]\Big\}\Big)$$

$$= \phi\left(\max\left\{(sb(u_{2k}, u_{2k-1}), sb(u_{2k}, u_{2k+1}), L\left[\frac{b(u_{2k-1}, u_{2k+1})}{2}\right]\right\}\right)$$

$$\leq \phi\left(\max\left\{(sb(u_{2k}, u_{2k-1}), sb(u_{2k}, u_{2k+1}), \frac{L}{2}[sb(u_{2k-1}, u_{2k+1}) + sb(u_{2k}, u_{2k+1})]\right\}\right)$$

$$= \phi(\max\{sb(u_{2k}, u_{2k-1}), sb(u_{2k}, u_{2k+1})\}).$$

If we suppose that $\phi(\max\{sb(u_{2k}, u_{2k-1}), sb(u_{2k}, u_{2k+1})\}) = sb(u_{2k}, u_{2k+1})$ then by the property of ϕ, we get

$$\begin{aligned}\phi(\max\{sb(u_{2k+1}, u_{2k})\}) &= \phi(sb(fu_{2k}, gu_{2k-1}))\\ &\leq \delta\phi(B(u, v))\\ &= \delta\phi(sb(u_{2k}, u_{2k+1}))\\ &< \phi(sb(u_{2k+1}, u_{2k}))\end{aligned}$$

which contradicts the fact.

Therefore the maximum is $sb(u_{2k}, u_{2k-1})$. We have

$$\begin{aligned}\phi(sb(u_{2k+1}, u_{2k})) &= \phi(fu_{2k}, gu_{2k-1})\\ &\leq \delta\phi(B(u, v))\\ &= \delta\phi(sb(u_{2k}, u_{2k-1}))\end{aligned}$$

$$\therefore \phi(sb(u_{2k}, u_{2k+1})) \leq \delta\phi(sb(u_{2k-1}, u_{2k})). \tag{4}$$

Case (b): $n = 2k + 1$, $k \in \mathbb{N}$.

As in the case (a), it can be proved that equation (3) holds for $n = 2k + 1$, that is

$$\phi(sb(u_{2k+1}, u_{2k+2})) \leq \delta\phi(sb(u_{2k}, u_{2k+1})). \tag{5}$$

From equations (4) and (5), we conclude that the inequality (3) holds $\forall n \in \mathbb{N}$. From (3), by the induction it is easy to prove that

$$\therefore \phi^n(sb(u_n, u_{n+1})) \leq \delta^n\phi^n(sb(u_0, u_1))\forall n \in \mathbb{N}. \tag{6}$$

Since $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$, from (6), we obtain

$$\lim_{n\to\infty} b(u_n, u_{n+1}) = 0. \tag{7}$$

We now have to prove that $\{u_n\}$ is a Cauchy sequence.

Let $\epsilon > 0$ since $L < \frac{1}{1+s}$. This implies $s - 2L > 0$ and $1 - L(1+s) > 0$. From (7), we conclude that there exists $n_0 \in \mathbb{N}$ such that

$$b(u_n, u_{n-1}) < \frac{1 - L - L_s}{2s}, \forall n > n_0. \tag{8}$$

Let $m, n \in \mathbb{N}$ with $m > n$. By induction on m, we will prove that

$$b(u_n, u_m) < \varepsilon \, \forall m > n \geq n_0. \tag{9}$$

Let $n \geq n_0$ and $m = n + 1$. Then from (3) and (8), we get

$$b(u_n, u_m) = b(u_n, u_{n+1}) \leq b(u_n, u_{n-1}) < \frac{1 - L - Ls}{2s} < \epsilon.$$

Thus $\{u_n\}$ is the Cauchy sequence for $m = n + 1$.

We assume that (9) holds for some $m \geq n + 1$

Now we have to prove that (9) holds for $m + 1$.

Case (i): n is odd and $m + 1$ is even.

$$\begin{aligned} \phi(sb(u_n, u_{m+1})) &= \phi(sb(fu_{n-1}, gu_m)) \\ &\leq \delta\phi(B(u_{n-1}, u_m)) \end{aligned}$$

$$\phi(B(u, v)) = \phi\left(\max\left\{sb(u_{n-1}, u_n), sb(u_m, u_{m+1}), \left[\frac{b(u_n, u_{m+1}) + b(u_n, u_m)}{2}\right]\right\}\right).$$

Since $b(u_m, u_{m+1}) < b(u_{n-1}, u_n)$ and for all $t > 0$, we have

$$\begin{aligned} \phi(B(u, v)) &< \phi(\max\{sb(u_{n-1}, u_n), L[b(u_{n-1}, u_{m+1}) + b(u_n, u_m)]\}) \\ &= \phi(sb(u_{n-1}, u_n)) \end{aligned} \tag{10}$$

$$\begin{aligned}\phi(B(u, v)) &< \phi(sb(u_{n-1}, u_n)) \\ &< \phi\left(s\left[\frac{(1 - L - Ls)}{2s}\varepsilon\right]\right) \\ &= \phi\left(\frac{1 - L - Ls}{2}\varepsilon\right) \\ &< \phi(\varepsilon)\end{aligned}$$

$$\begin{aligned}\phi(sb(u_n, u_{m+1})) &= \phi(sb(fu_{n-1}, gu_m)) \\ &\le \delta\phi(B(u_{n-1}, u_m)) \\ &< \delta\phi(\varepsilon) \\ &< \phi(\varepsilon)\end{aligned}$$

$\therefore\ sb(u_n, u_{m+1}) < \varepsilon$

$$\Rightarrow b(u_n, u_{m+1}) < \frac{\varepsilon}{s} < \varepsilon.$$

Thus we proved that (9) holds for $m + 1$ in this case.

Therefore, by induction (9) holds for all $m > n$ in case (i).

Case (ii): When n is even, $m + 1$ is odd.

This case can be proved in a similar way as we proved for case (i).

Case (iii): When n is even, $m + 1$ is even

$$\begin{aligned}\phi(b(u_n, u_{m+1})) &\le \phi(sb(u_n, u_{n+1}) + sb(u_{n+1}, u_{m+1})) \\ &\le \phi(sb(u_n, u_{n+1})) + \phi(sb(u_{n+1}, u_{m+1})) \\ &= \phi(sb(u_n, u_{n+1})) + \phi(sb(fu_n, gu_m)) \\ &\le \phi(sb(u_n, u_{n+1})) + \delta\phi(b(u_n, u_m))\end{aligned}$$

$$\le \phi(sb(u_n, u_{n+1}))$$

$$+ \phi\left(\max\left\{sb(u_n, u_m), sb(u_n, u_{n+1}) + sb(u_m, u_{m+1}), L\left[\frac{b(u_n, u_{m+1}) + b(u_m, u_{n+1})}{2}\right]\right\}\right)$$

$$\phi(b(u_n, u_{m+1})) \leq \phi(sb(u_n, u_{n+1}))$$

$$+\phi\left(\max\left\{s\varepsilon, sb(u_n, u_{n+1}) + sb(u_n, u_{n+1}), L\left[\frac{b(u_n, u_{m+1}) + b(u_m, u_{n+1})}{2}\right]\right\}\right) \quad (11)$$

Equality (11) implies

$$\phi(b(u_n, u_{m+1})) < \phi(sb(u_n, u_{n+1})) + \varphi(sb(u_n, u_{n+1}))$$

$$= 2\phi(sb(u_n, u_{n+1}))$$

$$< 2\phi\left(s\left[\frac{1 - L - Ls}{2s}\varepsilon\right]\right)$$

$$= \phi\left(2s\left[\frac{1 - L - Ls}{2s}\right]\varepsilon\right)$$

$$= \phi(\varepsilon)$$

$$\Rightarrow b(u_n, u_{m+1}) < \varepsilon.$$

If the inequality (11) implies

$$\phi(b(u_n, u_{m+1})) \leq \phi(sb(u_n, u_{n+1})) + \phi\left[\frac{L(b(u_n, u_{m+1}) + b(u_{n+1}, u_m)}{2}\right].$$

By triangle inequality, we have

$$\phi(b(u_n, u_{m+1})) < \phi(sb(u_n, u_{n+1})) + \frac{1}{2}\phi[Lb(u_n, u_{m+1})$$

$$+Lsb(u_{n+1}, u_n) + Lsb(u_n, u_m)]$$

$$= \left(\frac{L}{2} + 1\right)\phi(sb(u_n, u_{n+1})) + L\varphi\frac{b(u_n, u_{m+1})}{2}$$

$$+ L\varphi\frac{sb(u_n, u_m)}{2}$$

$$\left(1 - \frac{L}{2}\right)(b(u_n, u_{m+1})) = \left(1 + \frac{L}{2}\right)\phi(sb(u_n, u_{n+1})) + \frac{L}{2}\phi(b(u_n, u_m)).$$

By (8) and by induction, we have

$$\left(1-\frac{L}{2}\right)\phi(b(u_n, u_{m+1})) < \left(1+\frac{L}{2}\right)\phi\left[s\left[\frac{1-L-Ls}{s}\right]\varepsilon\right]+\frac{L}{2}\phi(s\varepsilon)$$

$$< \phi[1-L-Ls]\varepsilon+\frac{L}{2}\phi(s\varepsilon).$$

Taking φ^{-1} on both sides,

$$\left(1-\frac{L}{2}\right)\phi(b(u_n, u_{m+1})) < [1-L-Ls]\varepsilon+\frac{L}{2}s\varepsilon$$

$$= (1-L)\varepsilon - Ls + \frac{L}{s}s\varepsilon$$

$$= (1-L)\varepsilon - \frac{L}{2}s\varepsilon$$

$$< (1-L)\varepsilon$$

$$= \left(1-\frac{L}{2}\right)\varepsilon - \frac{L}{2}\varepsilon$$

$$< \left(1-\frac{L}{2}\right)\varepsilon$$

$$b(u_n, u_{m+1}) < \varepsilon.$$

Thus (9) holds for $m+1$. Hence we conclude that in case (iii) equation (9) holds for all $m > n$.

Similarly, we can prove for the case n is odd, $m+1$ is odd.

From (9), it follows that $\{u_n\}$ is a Cauchy sequence.

Since u is complete, $\{u_n\}$ converges to $u \in M$ as $n \to \infty$ and so

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{2n+1} = \lim_{n\to\infty} u_{2n} = t.$$

Since $\{u_{2n}\}$ is a sequence in A and A is closed and $u_{2n} \to t$, we have $t \in A$.

Also since $\{u_{2n+1}\}$ is a sequence in B and B is closed and $u_{2n+1} \to t$, we have $t \in B$.

Now to prove that t is a fixed point of f and g.

Without loss of generality, we assume that f is continuous.

Since $u_{2n} \to t,$ we have $u_{2n+1} = fu_{2n} \to ft.$

By the uniqueness of the limit, we get $u = fu.$

Similarly, we can prove $t = gt.$

From the contraction (1), we have

$$\begin{aligned}\phi(sb(t, gt)) &= \phi(sb(ft, gt))\\ &\leq \delta\phi(B(u, v))\\ &= \delta\phi\left(\max\{sb(t, t), sb(t, ft), sb(t, gt)\}, \frac{L[b(ft, t) + b(t, gt)]}{2}\right)\\ &= \delta\phi(sb(t, gt))\end{aligned}$$

$$\phi(sb(t, gt)) \leq \delta\phi(sb(t, gt)) < \phi(sb(t, gt))$$

which is a contradiction.

$$\therefore b(t, gt) = 0 \Rightarrow gt = t.$$

Therefore t is common fixed point of f and g.

Suppose that there exist two different fixed points of f and g that is $b(r, t) > 0$

$$\begin{aligned}\phi(sb(r, t)) &= \phi(sb(fr, gt))\\ &\leq \delta\phi(B(r, t))\\ &= \delta\phi\left(\max\left\{sb(r, t), sb(r, fr), sb(t, gt), L\left[\frac{[b(r, gt) + b(t, fr)]}{2}\right]\right\}\right)\\ &= \delta\phi(sb(r, t))\\ &< \phi(sb(r, t))\end{aligned}$$

which is a contradiction.

Finally we have a unique common fixed point for f and g in $A \cap B.$

References

[1] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, In: Functional Analysis, Vol. 30, pp 26-37, Ul'yasnovsk Gos. Ped. Inst., Ul'yanovsk (1989).

[2] H. Aydi, M. F. Bota, E. Karapinar and S. Moradi, A common fixed point for weak φ-contractions on b-metric spaces, Fixed Point Theory 13(2) (2012), 337-346.

[3] V. Berinde, Generalized contractions in quasi metric spaces, In: Seminar on Fixed point Theory. Preprint, vol.3, pp 3-9, "Babes-Bolyai" University, cluj-Napoca,1993

[4] M. Pacurar, A fixed point result for φ-contractions and fixed points on b-metric spaces without the boundness assumption, Fasc. Math. 43(1) (2010), 127-136.

[5] Oratai Yamaod, Wutiphol Sintunavarat and Yeol Je Cho, Common fixed theorems for generalized cyclic contraction pairs in b-metric spaces with applications, Fixed Point Theory Appl. 2015, 164 (2015). https://doi.org/10.1186/s13663-015-0409-z

[6] Wasfi Shatanawi, Ariana Pitea and Rade Lazovic, Contraction conditions using comparison functions on b-metric spaces, Fixed Point Theory Appl. 2014, 135 (2014). https://doi.org/10.1186/1687-1812-2014-135