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PRIME LABELING OF PAPPUS GRAPH

V. ANNAMMA and HAMEEDA BEGUM N

L. N. Government College
Ponneri, 601204, Tamil Nadu, India

Department of Mathematics
Justice Basheer Ahmed Sayeed
College for Women (Autonomous)
Teynampet, Chennai, 600018, Tamil Nadu, India
E-mail: hameedarafi1106@gmail.com

Abstract

A graph $G = (V(G), E(G))$ is observed to admit prime labeling when the vertices of the graph are labeled with unique integral values from $[1, |V|]$ in a way that for every edge uv the labels designated to u and v share no common positive factors except 1. A graph that receives a prime labeling is called a prime graph. In this research article we investigate that the pappus graph admits prime labeling. We construct the mirror graph and shadow graph of the pappus graph. We also discuss and establish prime labeling in the framework of some graph operations such as duplication, switching and fusion with few of their application ideas.

1. Introduction

In this research article we examine only finite, simple, connected and undirected graphs. We denote the vertex set and the edge set by $V(G)$ and $E(G)$ of the graph G and the corresponding cardinality by $|V(G)|$ and $|E(G)|$ respectively. A graph labeling is an assignment of integers to the vertices or edges, or both subject to certain conditions. One of the important areas in graph theory is graph labeling used in many applications like coding theory, X-ray, crystallography, radar, astronomy, circuit design, communication network addressing, data base management [7]. The notion of

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prime labeling was introduced by R. Entringer. A graph $G = (V(G), E(G))$ is observed to admit prime labeling when the vertices of the graph are labeled with unique integral values from $[1, |V|]$ in a way that for every edge uv the labels designated to u and v share no common positive factors except 1. A graph that receives a prime labeling is called a prime graph [3]. In this research article we investigate that the pappus graph admits prime labeling. We construct the mirror graph and shadow graph of the pappus graph. We also discuss and establish prime labeling in the framework of some graph operations such as duplication, switching and fusion with few of their application ideas.

2. Preliminaries

Definition 2.1[1]**.** The pappus graph is a bipartite 3-regular undirected graph with 18 vertices and 27 edges. It is a graph with girth 6. It is named after pappus of Alexandria, an ancient Greek mathematician.

Definition 2.2[4]**.** Let G be a bipartite graph with partite sets V_1 and V_2 and G' be the copy of G with corresponding partite sets V_1' and V_2'. The mirror graph $M(G)$ of G is obtained from G and G' by joining each vertex of V_2 to its corresponding vertex in V_2' by an edge. The concept of Mirror graphs was introduced by Bresar et al. in 2004 as an intriguing class of graphs.

Definition 2.3[6]**.** The open neighbourhood set $N(v)$ for every vertex $v \in V(G)$ is defined as the set of all vertices of G that are adjacent to the vertex v in G.

Definition 2.4[5]**.** The shadow graph $D_2(G)$ of a connected graph G is obtained by taking two copies of G, say G' and G'' and joining each vertex u' in G' to the neighbours of corresponding vertex u'' in G''.

Definition 2.5[2]**.** Duplication of a vertex v_i of a graph G constructs a new graph G_1 by adding a vertex v_i' with $N(v_i) = N(v_i')$. In other words, a vertex v_i' is said to be a duplication of the vertex v_i if all the vertices which are adjacent to v_i in G are now adjacent to v_i' in G_1.

Definition 2.6[2]. A vertex switching G_v of a graph G is obtained by taking a vertex v of G and by removing the entire edges incident with v and adding edges joining v to every vertex which are not adjacent to v in G.

Definition 2.7[2]. Let u and v be two distinct vertices of a graph G. A new graph G_1 is constructed by fusing (identifying) two vertices u and v by a single vertex x in G_1 such that every edge which was incident with either u or v in G now incident with x in G_1.

3. Prime Labeling of Pappus Graph

Theorem 3.1. *The pappus graph is a prime graph.*

Proof. Let G be the pappus graph with 18 vertices and 27 edges. The vertex set $(G) = \{v_1, v_2, \cdots, v_{18}\}$. In general $V(G) = \{v_i/1 \leq i \leq 18\}$ and $|V(G)| = 18$.

The edge set $E(G) = \{v_i v_{i+1}/1 \leq i \leq 17\} \cup \{v_i v_{i+7}/i = 1, 3, 7, 9\} \cup \{v_i v_{i+5}/ i = 6, 12\} \cup \{v_i v_{i+11}/i = 2, 4\} \cup \{v_1 v_{18}\} \cup \{v_5 v_{18}\}$ and $|E(G)| = 27$. Let us define a labeling $f : V(G) \to \{1, 2, 3, \cdots, 18\}$ by $f(v_i) = i, 1 \leq i \leq 13$ and $i = 15, 17, 18, f(v_{14}) = 16$ and $f(v_{16}) = 14$. Then for the edges,

$v_i v_{i+1}$, gcd $(f(v_i), f(v_{i+1})) = 1$ for $1 \leq$ i ≤ 17

$v_i v_{i+7}$, gcd $(f(v_i), f(v_{i+7})) = 1$ for i $= 1, 3, 7, 9$

$v_i v_{i+5}$, gcd $(f(v_i), f(v_{i+5})) = 1$ for i $= 6, 12$

$v_i v_{i+11}$, gcd $(f(v_i), f(v_{i+5})) = 1$ for i $= 2, 4$

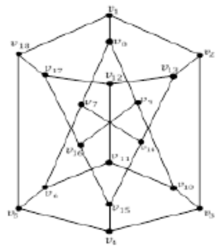
$v_i v_{i+18}$, gcd $(f(v_1), f(v_{18})) = 1$

$v_i v_{i+18}$, gcd $(f(v_5), f(v_{18})) = 1$.

Therefore G is a prime graph.

The pappus graph and its prime labeling are as in the following figures 3.1.1 and 3.1.2 respectively

Illustration 3.1.



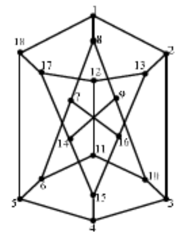


Figure 3.1.1. Pappus graph $G(18, 27)$. **Figure 3.1.2.** Prime labeling of pappus graph $G(18, 27)$.

4. Construction of Mirror Graph $M(G)$ of Pappus Graph

Step 1. Consider the pappus graph G with 18 vertices and 27 edges.

The vertex set $(G) = \{v_1, v_2, \cdots, v_{18}\}$. In general, $V(G) = \{v_i / 1 \leq i \leq 18\}$ and $|V(G)| = 18$.

The edge set $E(G) = \{v_i v_{i+1} / 1 \leq i \leq 17\} \cup \{v_i v_{i+7} / i = 1, 3, 7, 9\} \cup \{v_i v_{i+5} / i = 6, 12\} \cup \{v_i v_{i+11} / i = 2, 4\} \cup \{v_1 v_{18}\} \cup \{v_5 v_{18}\}$ and $|E(G)| = 27$ G is a bipartite graph with partite sets,

$V_1(G) = \{v_i / i = 1, 3, 5, 7, \cdots, 17\}$ and $V_2(G) = \{v_i / i = 2, 4, 6, 8, \cdots, 18\}$.

Step 2. Let G' be the copy of the pappus graph G with 18 vertices and 27 edges. The vertex set $(G') = \{u_1, u_2, \cdots, u_{18}\}$. In general, $V(G') = \{u_i / 1 \leq i \leq 18\}$ and $|V(G')| = 18$.

The edge set $E(G') = \{u_i u_{i+1} / 1 \leq i \leq 17\} \cup \{u_i u_{i+7} / i = 1, 3, 7, 9\} \cup \{u_i u_{i+5} /$ $\{u_i u_{i+5} / i = 6, 12\} \cup \{u_i u_{i+11} / i = 2, 4\} \cup \{u_1 u_{18}\} \cup \{u_5 u_{18}\}$ and $|E(G')| = 27$ G' is a bipartite graph with partite sets,

$$V_1'(G') = \{u_i / i = 1, 3, 5, 7, \cdots, 17\}$$

$$V_2'(G') = \{u_i / i = 2, 4, 6, 8, \cdots, 18\}$$

where V_1' and V_2' are copies of V_1 and V_2 respectively.

Step 3. Let $M(G)$ be the mirror graph of G. The mirror graph $M(G)$ of G is obtained from G and G' by joining each vertex of V_2 to its corresponding vertex in V_2' by additional edges $\{v_iu_i/i = 1, 3, 5, 7, \cdots, 17\}$.

$V[M(G)] = \{v_i, u_i/1 \leq i \leq 18\}$ is the vertex set of $M(G)$.

$E[M(G)] = \{v_iv_{i+1}/1 \leq i \leq 17\} \cup \{v_iv_{i+7}/i = 1, 3, 7, 9\}$

$\cup\{v_iv_{i+5}/i = 6, 12\} \cup \{v_iv_{i+11}/i = 2, 4\} \cup \{v_1v_{18}\} \cup \{v_5v_{18}\}$

$\cup\{u_iu_{i+1}/i \leq i \leq 17\} \cup \{u_iu_{i+7}/i = 1, 3, 7, 9\} \cup \{u_iu_{i+5}/i = 6, 12\}$

$\cup\{u_iu_{i+11}/i = 2, 4\} \cup \{u_1u_{18}\} \cup \{u_5u_{18}\} \cup \{v_iu_i/i = 1, 3, 5, 7, \cdots, 17\}$

is the edge set of $M(G)$.

$|V[M(G)]| = 36$.

The Mirror graph of pappus graph is as in the following figure 4.1.1.

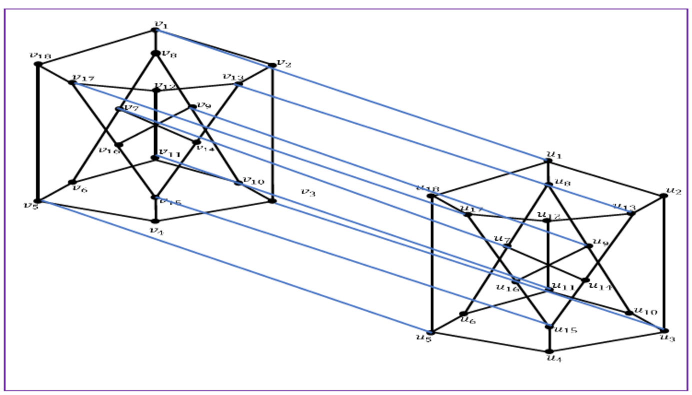


Figure 4.1.1 Mirror graph of pappus graph.

5. Construction of Shadow graph $D_2(G)$ of Pappus Graph

Step 1. Consider the pappus graph G with 18 vertices and 27 edges. Let G' and G'' be two copies of G. Let $V(G') = \{v_i/1 \leq i \leq 18\}$ be the vertex set of G' and $E(G') = \{v_iv_{i+1}/1 \leq i \leq 17\} \cup \{v_iv_{i+7}/i = 1, 3, 7, 9\} \cup \{v_iv_{i+5}/i = 6, 12\}$ $\cup\{v_iv_{i+11}/i = 2, 4\} \cup \{v_1v_{18}\} \cup \{v_5v_{18}\}$ be the edge set of G'.

Step 2. Let $V(G'') = \{u_i / 1 \le i \le 18\}$ be the vertex set of G'' and let $E(G'') = \{u_i u_{i+1} / 1 \le i \le 17\} \cup \{u_i u_{i+7} / i = 1, 3, 7, 9\} \cup \{u_i u_{i+5} / i = 6, 12\}$ $\cup \{u_i u_{i+11} / i = 2, 4\} \cup \{u_1 u_{18}\} \cup \{u_5 u_{18}\}$ be the edge set of G''.

Step 3. Let $D_2(G)$ be the shadow graph of G obtained by joining each vertex v_i in G' to the neighbours of corresponding vertex u_i in G''. The vertex set $V[D_2(G)] = \{v_i, u_i / 1 \le i \le 18\}$ is the vertex set of $D_2(G)$. $|V[D_2(G)]| = 36$.

The edge set $E[D_2(G)] = \{v_i v_{i+1} / 1 \le i \le 17\} \cup \{v_i v_{i+7} / i = 1, 3, 7, 9\}$

$\cup \{v_i v_{i+5} / i = 6, 12\} \cup \{v_i v_{i+11} / i = 2, 4\} \cup \{v_1 v_{18}\} \cup \{v_5 v_{18}\}$

$\cup \{u_i u_{i+1} / 1 \le i \le 17\} \cup \{u_i u_{i+7} / i = 1, 3, 7, 9\} \cup \{u_i u_{i+5} / i = 6, 12\}$

$\cup \{u_i u_{i+11} / i = 2, 4\} \cup \{u_1 u_{18}\} \cup \{u_5 u_{18}\} \cup \{v_i u_{i+1} / 1 \le i \le 17\}$

$\cup \{v_i u_{i+7} / i = 1, 3, 7, 9\} \cup \{v_i u_{i+5} / i = 6, 12\} \cup \{v_i u_{i+11} / i = 2, 4\}$

$\cup \{v_1 u_{18}\} \cup \{v_5 u_{18}\}$

The Shadow graph $D_2(G)$ is as in the following figure 5.1.1

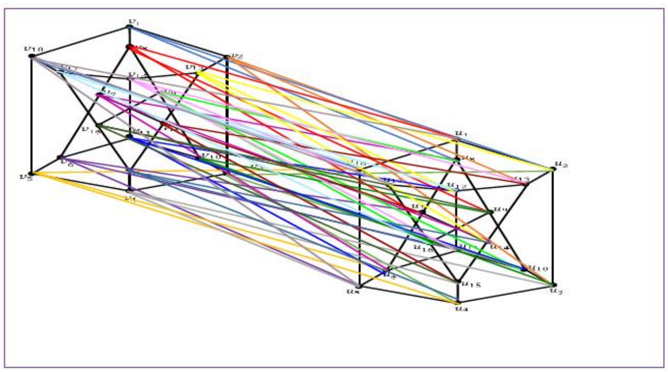


Figure 5.1.1 Shadow graph of pappus graph.

6. Duplication of a Vertex of Pappus Graph

Theorem 6.1. *The graph attained by duplication of any arbitrary vertex of pappus graph is a prime graph.*

Proof. Let G be the pappus graph with 18 vertices and 27 edges. The vertex set $V(G) = \{v_i / 1 \le i \le 18\}$ and $|V(G)| = 18$.

The Edge set $E(G) = \{v_i v_{i+1} / 1 \le i \le 17\} \cup \{v_i v_{i+7} / i = 1, 3, 7, 9\}$ $\cup \{v_i v_{i+5} / i = 6, 12\} \cup \{v_i v_{i+11} / i = 2, 4\} \cup \{v_1 v_{18}\} \cup \{v_5 v_{18}\}$ and $|E(G)| = 27$.

Let G_d represent the graph attained by duplication of any arbitrary vertex of pappus graph.

The vertex set $V(G_d) = \{v_i / 1 \le i \le 18\} \cup \{v_i' / i = 1 \text{ or } 2 \text{ or } \cdots 18\}$ and $|V(G_d)| = 19$

The edge set $E(G_d) = \{v_i v_{i+1} / 1 \le i \le 17\} \cup \{v_i v_{i+7} / i = 1, 3, 7, 9\}$

$\cup \{v_i v_{i+5} / i = 6, 12\} \cup \{v_i v_{i+11} / i = 2, 4\} \cup \{v_1 v_{18}\} \cup \{v_5 v_{18}\} \cup$ {the 3 edges of v_1' adjacent to all those vertices which are adjacent to v_1} and $|E(G_d)]| = 30$.

The duplication of vertex v_1 and v_{12} of pappus graph is as in the following figure 6.1.1 and 6.1.2 respectively.

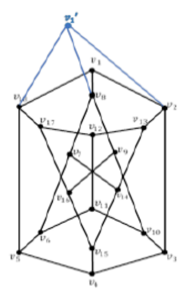


Figure 6.1.1. Prime labeling of duplication vertex v_1 of pappus graph.

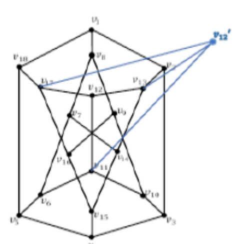


Figure 6.1.2. Prime labeling of Duplication of vertex v_{12} of pappus graph.

Let us define a labeling $f : V(G) \to \{1, 2, 3, \cdots, 19\}$ by $f(v_i) = i$, $1 \le i \le 13$ and $i = 15, 17, 18$ $f(v_{14}) = 16$ and $f(v_{16}) = 14$ $f(v_i') = 19$.

In view of the above pattern of labeling, G_d admits prime labeling. Therefore G_d is a prime graph.

Illustration 6.1.

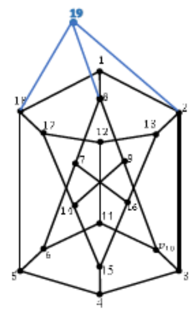


Figure 6.1.3. Prime labeling of Duplication vertex v_1 of pappus graph.

Illustration 6.2.

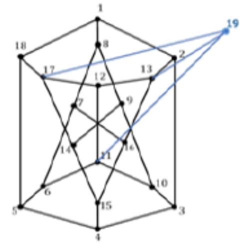


Figure 6.1.4. Prime labeling of Duplication of vertex v_{12} of pappus graph.

7. Switching of a Vertex of Pappus Graph

Theorem 7.1. *The graph attained by Switching of vertex* v_1 *of a pappus graph is a prime graph.*

Proof. Let G_s represent the graph attained by Switching of vertex v_1 of pappus graph.

The vertex set $V(G_s) = \{v_i / 1 \le i \le 18\}$ and $|V(G_d)| = 18$.

The edge set $E(G_s) = \{ v_iv_{i+1} / 1 \le i \le 17 \} \cup \{ v_iv_{i+7} / i = 1, 3, 7, 9 \}$

$\cup \{ v_iv_{i+5} / i = 6, 12\} \cup \{ v_iv_{i+11} / i = 2, 4\} \cup \{v_1v_{18}\} \cup \{v_5v_{18}\}$

$\cup \{ v_iv_i / i = 3 \le i \le 17, i \ne 8\}$ and $|E(G)| = 38$.

Let us define a labeling $f : V(G) \to \{1, 2, 3, \cdots, 18\}$ by

$f(v_i) = i, 1 \le i \le 13$ and $i = 15, 17, 18$

$f(v_{14}) = 16$ and $f(v_{16}) = 14$.

In view of the above pattern of labeling, G_s admits prime labeling. Therefore G_s is a prime graph.

The switching of vertex v_1 of pappus graph and its prime labeling are as in the following figures 6.1.1 and 6.1.2 respectively.

Illustration 7.1.

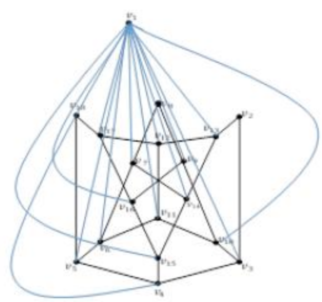


Figure 7.1.1. Switching of vertex v_1 of pappus graph.

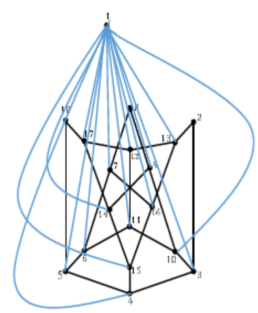


Figure 7.1.2. Prime labeling of Switching of vertex v_1 of pappus graph.

8. Fusing of Two Vertices of Pappus Graph

Theorem 8.1. *The graph attained by fusing of vertices* v_1 *and* v_{18} *of a pappus graph is a prime graph.*

Proof. Let G_f represent the graph attained by fusing the vertices v_1 and v_{18} as one vertex u in a pappus graph.

The vertex set $V(G_f) = \{v_i / 2 \le i \le 17\} \cup \{u\}$ and $|V(G_f)| = 17$.

The edge set $E(G_s) = \{v_i v_{i+1} / 2 \le i \le 16\} \cup \{v_i v_{i+7} / i = 3, 7, 9\}$

$\cup \{v_i v_{i+5} / i = 6, 12\} \cup \{v_i v_{i+11} / i = 2, 4\} \cup \{v_1 v_{18}\} \cup \{v_5 v_{18}\}$

Let us define a labeling $f : V(G) \to \{1, 2, 3, \cdots, 17\}$ by $f(u) = 1$ $f(v_i) = i,\ 2 \le i \le 13$ and $i = 15, 17$

$f(v_{14}) = 16$ and $f(v_{16}) = 14$

In view of the above pattern of labeling, G_f admits prime labeling. Therefore G_f is a prime graph.

The graph attained by fusing the vertices v_1 and v_{18} as one vertex u in a pappus graph and its prime labeling are as in the following figures 8.1.1 and 8.1.2 respectively.

Illustration 8.1.

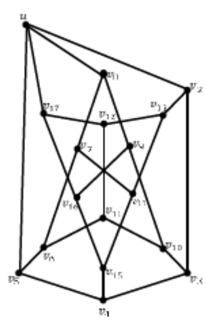


Figure 8.1.1. Fusing of vertices v_1 and v_{18} of pappus graph.

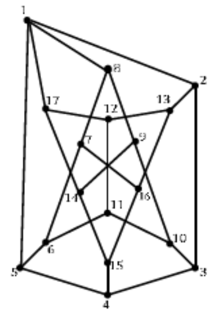


Figure 8.1.2. Fusing of vertices v_1 and v_{18} of pappus graph and its prime labeling.

9. Some Application Ideas

Application Ideas of Duplication of a vertex 9.1.

In biology of molecules, the biological process of DNA replication of producing two identical replicas of DNA from one original DNA molecule can be carried using the concept of duplication of graph theory. DNA replication occurs in all living organisms acting as the most essential part for biological inheritance. The concept of duplication in graph theory may be applied by considering DNA and chromosomes as vertices and duplicating these vertices by using duplication concept of graph theory and this may help in biological process of duplicating the DNA and chromosomes which may be carried for further research using graph labeling and this may help in the treatments of various ailments.

Application Ideas of Switching of a vertex 9.2.

The concept of switching in graph theory may be applied in computer networks, circuit theory, etc. In large networks, there can be multiple paths from sender to receiver. The best route for data transmission can be decided by using the concept of switching technique of graph theory. They may be used to connect the systems for making one-to-one communication. It also has various applications in circuit switching which helps to build a communication network.

Application Ideas of Fusing of vertices 9.3.

The concept of fusing of vertices in graph theory may be applied in chemical industry where the fusing of elements can be carried out by representing the elements as vertices. The concept of fusion may also help in the study of data fusion which is performed by constructing the connectivity networks that represent each clustering and forming.

10. Conclusion

In this research article we have proved that the pappus graph admits prime labeling. We have also constructed the mirror graph and shadow graph of the pappus graph which can be used for further research and exploration of its applications. We have also established prime labeling in the framework of some graph operations such as duplication, switching and fusion and have also discussed few of their application ideas.

References

[1] Burkard Polster and Hendrik van Maldeghem, Some constructions of small generalized polygons, Journal of Combinatorial Theory, Series A. 96 (2001), 162-179

[2] A. Edward Samuel and S. Kalaivani, Prime labeling to brush graphs, International Journal of Mathematics Trends and Technology 55(4) 2018.

[3] Joseph A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, (2019).

[4] S. K. Vaidya and N. B. Vyas, E-cordial labeling of some mirror graphs, International Journal of Contemporary Advanced Mathematics (IJCM) 2 (2011), 22-27

[5] S. K. Vaidya and N. H. Shah, Prime cordial labeling of some graphs, Open Journal of Discrete Mathematics (2012), 11-16.

[6] S. K. Vaidya and N. J. Kothari, Line gracefulness of some path related graphs, International Journal of Mathematics and Scientific Computing 4(1) (2014), 15-18.