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ALGORITHMIC APPROACH OF SPUR ON LAPLACIAN MATRIX FOR CERTAIN FUZZY GRAPHS

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Abstract

Laplacian matrix of certain fuzzy graphs is examined and explored for its structural and behavioral variations in terms of its spur or trace. Spur is another name for trace which emerged from the German word Spur which means track or trace. Spur plays an important role in linear operating models and is also a commanding analytical tool in cracking several optimization problems, but detection of spur for Laplacian matrix of fuzzy graph embraces several steps such as finding its degree matrix, adjacency matrix and then its difference. Hence finding spur becomes very monotonous, susceptible to miscalculation and time consuming. To enumerate it directly by using either μ or σ value of given fuzzy graph, without finding its corresponding Laplacian matrix, effectual formulae is deduced and constructed as an algorithm with execution time $\mathcal{O}(n^2)$, so as to increase the usage of spur of Laplacian matrix in many fields.

1. Introduction

Leonard Euler presented graph theory in 1736 [1]. Kaufmann in 1973 pervaded fuzzy sets into graph theory and presented fuzzy theory built on Zadeh's vision on fuzzy sets [3] [4]. Further, in 1975 fuzzy sets and its relations was established as well as extended as fuzzy graph theory, by Azriel

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Rosenfeld [2]. Emphasizing Laplacian matrix in the field of chemistry was done during 1994 [5]. Natalia, Juan and Mauricio discussed the Laplacian matrix of a digraph during 2018 for the trace norm of its upper bound [6]. Still, in fuzzy graph theory spur of Laplacian matrix has not been discussed due to its complexity and this indoctrinated us to investigate it. To start up, we have given the required definitions. Then, accentuates on finding spur or trace of Laplacian matrix with an algorithm.

2. Preliminaries

Let G be a simple connected graph with vertex set V with n number of vertices and edge set E with m number of edges where n, m greater than one.

A fuzzy graph, $G' : (\sigma, \mu)$ with n vertices and m edges where σ from V to [0, 1] is a fuzzy subset and μ from $V \times V$ to [0, 1] is a fuzzy relation on σ such that $\mu(x, y)$ is less than or equal to the minimum between $\sigma(x)$ and $\sigma(y)\ \forall x, y \in V$ [2]. The adjacency matrix $A(G') = [a_{ij}]$ where $a_{ij} = \mu(u_i, u_j)$ entry [8]. For any vertex of a fuzzy graph say u the degree of u is, $d_{G'}(u) = \sum_{uv \in E(G')} \mu(uv)$ and its degree matrix $D(G') = [d_{ij}]$ where $d_{ij} = \begin{bmatrix} d_{G'}(v_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{bmatrix}$ [8]. The Laplacian matrix is a matrix between the adjacency matrix deducted from the degree matrix. A fuzzy graph is complete if $\mu(x, y)$ is equal to the minimum between $\sigma(x)$ and $\sigma(y)\ \forall x, y \in V$ [7] [11]. If a cycle contains more than one weakest arc then it is a fuzzy cycle [12]. If each vertex has same degree in a fuzzy graph $G' : (\sigma, \mu)$ then it is said to be regular fuzzy graph [7] [9] [10]. $G' : (\sigma, \mu)$ is said to be an edge regular fuzzy graph if each edge has same degree [13]. The sum of elements on the main diagonal of A is defined as trace or spur of a square matrix A and is denoted by $tr(A)$.

3. Main Results

We started working on Laplacian spectrum of fuzzy graph and constructed an algorithm for computing Laplacian spectrum of complete fuzzy graph [14]. While working on other graphs we depicted an algorithm for the trace of certain Laplacian matrix of fuzzy graph. One of the important properties of trace is that, it is the same as totality of eigen values for the corresponding matrix. While working on complete fuzzy graphs, fuzzy cycle and regular graphs of both complete fuzzy graphs and fuzzy cycle we arrived at finding the subsequent results.

3.1 Trace of Laplacian matrix of complete fuzzy graph

On examining complete fuzzy graphs, we see that μ_i for the given complete fuzzy graph completely depends on σ_i, where $i = 1, 2, \ldots, n.$ That is, $\mu_{ij} = \mu(\sigma_i, \sigma_j) = \sigma_i \wedge \sigma_j \ \forall i, j \in n.$ Hence, we see that the degree matrix and the adjacency matrix also depend on σ_i, where $i = 1, 2, \ldots, n.$ Therefore, we examined behavior of Laplacian matrix and deduced a formula which uses only the σ_i values.

Theorem 1. *Let* $G' : (\sigma, \mu)$ *be the given complete fuzzy graph with n vertices. Arrange* σ_i *such that* $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n$. *Then the spur or trace(tr) of the Laplacian matrix of given graph is*

$$tr(L(G')) = 2[\sigma_1(n-1) + \sigma_2(n-2) + \ldots + \sigma_{n-1}].$$

Proof. Given $G' : (\sigma, \mu)$. $L(G')$ will be a $n \times n$ matrix, since the row and column depend on σ_i, $i = 1, 2, \ldots, n.$ Let us compute the degree of each σ_i.

$$\deg(\sigma_1) = (\sigma_1 \wedge \sigma_2) + (\sigma_1 \wedge \sigma_3) + (\sigma_1 \wedge \sigma_4) + \ldots + (\sigma_1 \wedge \sigma_n)$$

$$= \sigma_1 \times (n-1) (\because \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n)$$

$$\deg(\sigma_2) = (\sigma_2 \wedge \sigma_1) + (\sigma_2 \wedge \sigma_3) + (\sigma_2 \wedge \sigma_4) + \ldots + (\sigma_2 \wedge \sigma_n)$$

$$= \sigma_1 + (\sigma_2 \times (n-2)) (\because \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n)$$

$$\deg(\sigma_3) = (\sigma_3 \wedge \sigma_1) + (\sigma_3 \wedge \sigma_2) + (\sigma_3 \wedge \sigma_4) + \ldots + (\sigma_3 \wedge \sigma_n)$$

$$= \sigma_1 + \sigma_2 + (\sigma_3 \times (n-3)) (\because \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n)$$

$$\ldots \ \ldots \ \ldots$$

$$\deg(\sigma_n) = (\sigma_n \wedge \sigma_1) + (\sigma_n \wedge \sigma_2) + (\sigma_n \wedge \sigma_3) + \ldots + (\sigma_n \wedge \sigma_{n-1})$$

$$= \sigma_1 + \sigma_2 + \ldots + \sigma_{n-1} (\because \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n)$$

$$D(G') = \begin{pmatrix} \deg(\sigma_1) & 0 & 0 & 0 & \ldots & 0 \\ 0 & \deg(\sigma_2) & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 & \ldots & \deg(\sigma_n) \end{pmatrix}$$

$$A(G') = \begin{pmatrix} 0 & \mu_{12} & \mu_{13} & \ldots & \mu_{1n} \\ \mu_{21} & 0 & \mu_{23} & \ldots & \mu_{2n} \\ & \vdots & \vdots & \vdots & \\ \mu_{n1} & \mu_{n2} & \mu_{n3} & \ldots & 0 \end{pmatrix}$$

$$L(G') = \begin{pmatrix} \deg(\sigma_1) & -\mu_{12} & -\mu_{13} & \ldots & -\mu_{1n} \\ -\mu_{21} & \deg(\sigma_2) & -\mu_{23} & \ldots & -\mu_{2n} \\ \vdots & \vdots & \vdots & & \\ -\mu_{n1} & -\mu_{n2} & -\mu_{n3} & \ldots & \deg(\sigma_n) \end{pmatrix}$$

We observe that trace depends only on the degrees of each vertex. Hence,

$$tr(L(G')) = \deg(\sigma_1) + \deg(\sigma_2) + \deg(\sigma_3) + \ldots + \deg(\sigma_n).$$

$$= \sigma_1 \times (n-1) + \sigma_1 + (\sigma_2 \times (n-2)) + \sigma_1 + \sigma_2 + \sigma_3 \times (n-3) + \ldots + \sigma_1$$

$$\sigma_2 + \ldots + \sigma_{n-1}$$

$$= 2[\sigma_1 \times (n-1)] + 2[\sigma_2 \times (n-2)] + \ldots + 2\sigma_{n-1}$$

$$= 2[(\sigma_1 \times (n-1))] + (\sigma_2 \times (n-2)) + \ldots + \sigma_{n-1}].$$

Hence the theorem.

For example, let a complete fuzzy graph with $n = 11$ vertices be given, say, $\sigma_1(x) = 0.1$, $\sigma_2(x) = 0.1$, $\sigma_3(x) = 0.2$, $\sigma_4(x) = 0.3$, $\sigma_5(x) = 0.3$, $\sigma_6(x) = 0.4$, $\sigma_7(x) = 0.5$, $\sigma_8(x) = 0.5$, $\sigma_9(x) = 0.6$, $\sigma_{10}(x) = 0.7$, $\sigma_{11}(x) = 0.8$ arranged in ascending order. Then trace is

$$= 2[(0.1 \times 10) + (0.1 \times 9) + (0.2 \times 8) + (0.3 \times 7) + (0.3 \times 6) + (0.4 \times 5) + (0.5 \times 4)$$

$$+ (0.5 \times 3) + (0.6 \times 2) + (0.7 \times 1)] = 29.6.$$

Remarks. (i) The highest σ value is not taken into consideration, since μ

value is the smallest of its corresponding σ values

(ii) The above theorem holds good even though two or more σ values are same

(iii) This theorem is applicable for complete bipartite fuzzy graphs

(iv) Also, we observe that trace of Laplacian matrix of fuzzy graph depends only on the degrees of each vertex of the given fuzzy graph.

3.2 Trace of Laplacian matrix of fuzzy cycle

On inspecting fuzzy cycle, we observe that the values of Laplacian matrix depend only on the edge membership function μ_i, where $i = 1 \ldots n$. Here, we study only the peripheral fuzzy cycle.

Theorem 2. *Let* $G' : (\sigma, \mu)$ *be given fuzzy cycle with n vertices. Then, the spur or trace of the corresponding Laplacian matrix of the fuzzy cycle is* $tr(L(G')) = 2\sum_{i=1}^{n} \mu_i \ \forall\, i = 1 \ldots n.$

Proof. Given $G' : (\sigma, \mu)$ a fuzzy cycle with *n* vertices.

We observed that the trace of Laplacian matrix depends only on degree of each vertex. Hence, $\deg(\sigma_1) = \mu_1 + \mu_2$, $\deg(\sigma_2) = \mu_2 + \mu_3$, $\deg(\sigma_3) = \mu_3 + \mu_4$, $\ldots$, $\deg(\sigma_n) = \mu_1 + \mu_n$.

Therefore, $tr(L(G')) = \deg(\sigma_1) + \deg(\sigma_2) + \deg(\sigma_3) + \ldots + \deg(\sigma_n)$

$$= (\mu_1 + \mu_2) + (\mu_2 + \mu_3) + (\mu_3 + \mu_4) + \ldots + (\mu_{n-1} + \mu_n) + (\mu_1 + \mu_n)$$

$$= 2\mu_1 + 2\mu_2 + 2\mu_3 + 2\mu_4 + \ldots + 2\mu_{n-1} + 2\mu_n$$

$$= 2(\mu_1 + \mu_2 + \mu_3 + \mu_4 + \ldots + \mu_{n-1} + \mu_n)$$

$$= 2\sum_{i=1}^{n} \mu_i.$$

Hence the proof.

For example, consider a fuzzy cycle with five vertices which is labelled as $\mu_1 = 0.6$, $\mu_2 = 0.9$, $\mu_3 = 0.3$, $\mu_4 = 0.1$, $\mu_5 = 0.3$. Then by using the above theorem, trace $= 2\sum_{i=1}^{4} \mu_i = 2[0.6 + 0.9 + 0.3 + 0.1 + 0.3] = 4.4.$

3.3 Trace of Laplacian matrix of certain regular fuzzy graphs

A fuzzy cycle can be made vertex regular or edge regular by labelling its edge membership function in such a way that it satisfies the definition of regular fuzzy graph [10] [13].

Theorem 3. *Let $G' : (\sigma, \mu)$ be given vertex regular even fuzzy cycle with n vertices. Then spur or trace of the corresponding Laplacian matrix of the vertex regular fuzzy cycle is*

$$tr(L(G')) = \begin{cases} n(2\mu) & when\, \mu = \mu_1 = \mu_2 = \ldots = \mu_n \\ n(\mu' + \mu'') & when\, \mu' = \mu_1 = \mu_3 = \ldots = \mu_{n-1} \\ & and\, \mu'' = \mu_2 = \mu_4 = \ldots = \mu_n. \end{cases}$$

Proof. Let $G' : (\sigma, \mu)$ be given vertex regular fuzzy cycle with n vertices.

Case (i) Let $\mu = \mu_1 = \mu_2 = \ldots = \mu_n$. Then, by theorem 2,

$\deg(\sigma_1) = \mu_1 + \mu_2 = \mu + \mu + 2\mu$, $\deg(\sigma_2) = \mu_2 + \mu_3 = \mu + \mu = 2\mu$,

$\deg(\sigma_3) = \mu_3 + \mu_4 = \mu + \mu + 2\mu, \ldots, \deg(\sigma_n) = \mu_1 + \mu_n = \mu + \mu = 2\mu$

Therefore, $tr(L(G')) = \deg(\sigma_1) + \deg(\sigma_2) + \deg(\sigma_3) + \ldots + \deg(\sigma_n)$

$= 2\mu + 2\mu + \ldots + 2\mu(n \text{ times}) = n(2\mu).$

Case (ii). Let $\mu' = \mu_1 = \mu_3 = \ldots = \mu_{n-1}$ and $\mu'' = \mu_2 = \mu_4 = \ldots = \mu_n$.

Using theorem 2, $\deg(\sigma_1) = \mu_1 + \mu_2 = \mu' + \mu''$, $\deg(\sigma_2) = \mu_2 + \mu_3 = \mu' + \mu''$, $\deg(\sigma_3) = \mu_3 + \mu_4 = \mu' + \mu'', \ldots, \deg(\sigma_n) = \mu_1 + \mu_n = \mu' + \mu''$.

Therefore, $tr(L(G')) = \deg(\sigma_1) + \deg(\sigma_2) + \deg(\sigma_3) + \ldots + \deg(\sigma_n)$

$$= (\mu' + \mu'') + (\mu' + \mu'') + \ldots + (\mu' + \mu'')(n \text{ times})$$

$$= n(\mu' + \mu'').$$

Corollary 1. *Let $G' : (\sigma, \mu)$ be given vertex regular odd fuzzy cycle with n vertices. Then spur or trace for the corresponding Laplacian matrix of the edge regular fuzzy cycle is*

$$tr(L(G')) = 2(2\mu) \text{ where } \mu = \mu_1 = \mu_2 = \ldots = \mu_n.$$

Proof follows from theorem 3, case (i).

Theorem 4. *Let* $G' : (\sigma, \mu)$ *be given edge regular fuzzy cycle with n vertices. Then the spur or trace of the corresponding Laplacian matrix of the edge regular fuzzy cycle is*

$$tr(L(G')) = n(2\mu) \text{ where } \mu = \mu_1 = \mu_2 = \ldots = \mu_n.$$

Proof follows from theorem 3, case (i).

Remark. Even edge regular fuzzy cycle with n vertices cannot be labelled as $\mu' = \mu_1 = \mu_3 = \ldots = \mu_{n-1}$ and $\mu'' = \mu_2 = \mu_4 = \ldots = \mu_n$.

3.4 Trace of Laplacian matrix of regular complete fuzzy graph

After converting, a complete fuzzy graph into a vertex regular or edge regular fuzzy graph we analyzed the spur and the following theorem emerged which also holds good for regular complete bipartite fuzzy graph.

Theorem 5. *A complete fuzzy graph* $G' : (\sigma, \mu)$ *is vertex regular as well as edge regular only when* $\sigma_1 = \sigma_2 = \ldots = \sigma_n = \mu = \mu_1 = \mu_2 = \ldots = \mu_n$. *Then the spur or trace of its corresponding Laplacian matrix regular complete fuzzy graph is,* $tr(L(G')) = n\mu(n-1)$.

Proof. Let $G' : (\sigma, \mu)$ be given vertex regular fuzzy cycle with n vertices.

$$\deg(\sigma_1) = (\sigma_1 \wedge \sigma_2) + (\sigma_1 \wedge \sigma_3) + (\sigma_1 \wedge \sigma_4) + \ldots + (\sigma_1 \wedge \sigma_n) = \mu \times (n-1)$$

$$\deg(\sigma_2) = (\sigma_2 \wedge \sigma_1) + (\sigma_2 \wedge \sigma_3) + (\sigma_2 \wedge \sigma_4) + \ldots + (\sigma_2 \wedge \sigma_n) = \mu \times (n-1)$$

$$\deg(\sigma_3) = (\sigma_3 \wedge \sigma_1) + (\sigma_3 \wedge \sigma_2) + (\sigma_3 \wedge \sigma_4) + \ldots + (\sigma_3 \wedge \sigma_n) = \mu \times (n-1)$$

$$\ldots \ldots \ldots$$

$$\deg(\sigma_n) = (\sigma_n \wedge \sigma_1) + (\sigma_n \wedge \sigma_2) + (\sigma_n \wedge \sigma_3) + \ldots + (\sigma_n \wedge \sigma_{n-1}) = \mu \times (n-1)$$

Therefore, $tr(L(G')) = \deg(\sigma_1) + \deg(\sigma_2) + \deg(\sigma_3) + \ldots + \deg(\sigma_n)$

$$= \mu(n \text{ times}) \times (n-1) = n(n-1)\mu.$$

Hence the theorem.

3.5 Interpretation of algorithm

Using above theorems, we constructed the following algorithm which can be converted into a programming language.

Let $G' : (\sigma, \mu)$ be given fuzzy graph with n vertices.

Step 1. Mention whether it is a complete fuzzy graph, fuzzy cycle, vertex regular/edge regular complete fuzzy graph or vertex regular/edge regular fuzzy cycle with the number of vertices, say n.

Step 2. If complete fuzzy graph input the values of $\sigma_i \ \forall\, i = 1 \ldots n$ then go to step 3

If fuzzy cycle input the values of $\mu_i \ \forall\, i = 1 \ldots n$ then got to step 4

If vertex regular or edge regular complete fuzzy graph input σ_1 value then go to step 5

If vertex regular even fuzzy cycle input μ_1 and μ_2 values then go to step 6

If vertex regular odd fuzzy cycle or edge regular fuzzy cycle (even and odd) input the μ_1 value, then go to step 7.

Step 3. Arrange σ_i, such that $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n$ then calculate

spur $= 2[\sigma_1(n-1) + \sigma_2(n-2) + \ldots + \sigma_{n-1}]$.

Step 4. Spur $= 2\sum_{i=1}^{n} \mu_i \ \forall\, i = 1 \ldots n$.

Step 5. Spur $= n(n-1)\sigma_1$.

Step 6. If $\mu_1 = \mu_2$ then spur $= n(2\mu_1)$ else spur $= n(\mu_1 + \mu_2)$.

Step 7. Spur $= n(2\mu_1)$.

Step 8. Stop

Proof of correction. Let $G' : (\sigma, \mu)$ be given fuzzy graph with n vertices. If the given graph is a complete fuzzy graph, then using theorem 1, arrange σ_i, such that $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n$, calculate spur $= 2[\sigma_1(n-1)$

$+\sigma_2(n-2)+\ldots+\sigma_{n-1}]$. If it is a fuzzy cycle then get the values of $\mu_i\ \forall\, i=1\ldots n$, using theorem 2, spur $=2\sum_{i=1}^{n}\mu_i\ \forall\, i=1\ldots n$. If it is vertex regular or edge regular complete fuzzy graph then get σ_1 value, using theorem 5, spur $=n(n-1)\sigma_1$. If vertex regular even fuzzy cycle input μ_1 and μ_2 values, using theorem 3, if $\mu_1=\mu_2$ then spur $=n(2\mu_1)$ else spur $=n(\mu_1+\mu_2)$. If vertex regular odd fuzzy cycle or edge regular fuzzy cycle (even and odd) then input the μ_1 value, using theorem 4, spur $=n(2\mu_1)$.

This algorithm is of polynomial complexity $\mathcal{O}(n^2)$ which can be developed as an application such as MATLAB which will be of greater use in future.

4. Conclusions

This paper will make a way for many researches in the field of Laplacian matrix of fuzzy graph, which will be the major upcoming concept in the field of machine learning, engineering etc. Further exertion is taken to extend this paper in symmetric interconnection networks particularly in discipline pattern fuzzy graphs so as to develop an application which will enhance the approach towards Laplacian matrix and its spectrum in future.

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