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SOLUTIONS OF PELL'S EQUATION USING EISENSTEIN PRIMES

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Abstract

Pell's equation is any Diophantine equation of the form $x^2 - dy^2 = \pm N$, where d is a positive non-square integer and N is any fixed positive integer. In this paper, we search for solutions to the equation $x^2 - 53y^2 = -17^t$, $\forall\, t \in \mathbb{N}$. Here we choose d and N to be Eisenstein primes 53 and 17 respectively and search for the solutions to the equation for different choices of t given by (i) $t = 1$, (ii) $t = 3$, (iii) $t = 5$, (iv) $t = 2k$, (v) $t = 2k + 5$, $\forall\, k \in \mathbb{N}$. Finally the recurrence relations on the solutions are obtained.

1. Introduction

The Pell's equation is any Diophantine equation of the form $x^2 - dy^2 = 1$, where d is a given positive non-square integer and integer solutions are sought for x and y. Pell's equation is named after the Mathematician John Pell. Pell's equation has infinitely many distinct integer solutions.

The Pell's equation discussed here is a negative Pell's equation given as

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$x^2 - dy^2 = -N,$ to be solved in positive integers x and y. In this paper, we have chosen the Eisenstein primes as coefficients in negative Pell's equation to find the positive integer solutions. An Eisenstein prime is an Eisenstein integer of the form $z = a + bw,$ where $w = e^{\frac{2\pi i}{3}}$ that is irreducible. An Eisenstein integer is said to be an Eisenstein prime if it satisfies one of the following conditions:

(i) $b = 0$ and $a = p,$ where p is a prime with $p \equiv 2 (\mathrm{mod}\, 3).$

(ii) $a = 0, b = p,$ where p is a prime with $p \equiv 2 (\mathrm{mod}\, 3).$

(iii) $N(a + bw) = a^2 - ab + b^2 = p,$ where p is a prime such that $p = 3$ or $p \equiv 1 (\mathrm{mod}\, 3).$

In other words, a prime number p is an Eisenstein prime if it is equal to a natural prime of the form $3p - 1.$ In the Pell's equation $x^2 = 53y^2 - 17^t, t \in \mathbb{N};$ we use Eisenstein primes 53 and 17 and 2 search for its non-trivial integer solutions. In order to get the solutions, we have approached it with the choices of t given by (i) $t = 1$ (ii) $t = 3$ (iii) $t = 5$ (iv) $t = 2k$ and (v) $t = 2k + 5.$

Using Brahma Gupta lemma, we obtain the sequence of non-zero distinct integer solutions. Also we obtain recurrence relations on the solution.

2. Preliminaries

Theorem 2.1 [2]. *If* (x_1, y_1) *is the fundamental solution of* $x^2 - dy^2 = 1.$ *Then every positive solutions of the equation is given by* (x_n, y_n) *where* y_n *and* x_n *are the integers determined from*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n, \textit{ for } n = 1, 2, 3, \ldots$$

2.1 [12] Testing the solubility of the negative Pell equation

Suppose D is a positive integer, not a perfect square. Then the negative Pell equation $x^2 - Dy^2 = 1.$ is soluble if and only if D is expressible as

$D = a^2 + b^2$, $\gcd(a, b) = 1$, a and b positive, b is odd and the Diophantine equation $bV^2 + 2aVW + bW^2 = 1$ has a solution. (The case of solubility occurs for exactly one such (a, b)).

The Algorithm

(i) Find all expressions of D as a sum of two relatively-prime squares using Cornacchia's method. If none, exist - the negative Pell equation is not solvable.

(ii) For each representation $D = a^2 + b^2$, $\gcd(a, b) = 1$, a and b positive, b odd, test the solubility of $-bV^2 + 2aVW + bW^2 = 1$ using the Lagrange-Matthews algorithm. If soluble, exist - the negative Pell equation is solvable.

(iii) If each representation yields no solution, then the negative Pell equation is insolvable.

Theorem 2.2 [6]. *Let p be a prime. The negative Pell's equation $x^2 - py^2 = -1$ is solvable if and only if $p = 2$ or $p \equiv 1 \pmod 4$.*

This paper deals with a negative Pell equation

$$x^2 = 53y^2 - 17^t,\ t \in \mathbb{N}$$

For this particular equation, we consider the prime $p = 53$, which satisfies the conditions of Theorem 2.2. Therefore, we can substantiate the proof that the negative Pell's equation $x^2 = 53y^2 - 17^t$, $t \in \mathbb{N}$ is solvable in integers.

Using the Algorithm as in 2.1 and testing $(a, b) = (2, 7)$:

$-bV^2 + 2aVW + bW^2 = 1$ has a solution $(V, W) = (-3, 4)$, so $x^2 - 53y^2 = -1$ is soluble.

3. Method of Analysis

3.1. Choice 1. $t = 1$.

The Pell equation is

$$x^2 = 53y^2 - 17 \tag{1}$$

Let (x_0, y_0) be the initial solution of (1) given by

$$x_0 = 6; y_0 = 1$$

To find the other solutions of (1), consider the more general Pell equation

$$x^2 = 53y^2 + 1 \tag{2}$$

whose initial solution is (66249,9100) and the general solution (x_n, y_n) is given by theorem 2.1 as

$$x_n = \frac{1}{2} f_n$$

$$y_n = \frac{1}{2\sqrt{73}} g_n$$

where $f_n = (66249 + 9100\sqrt{53})^{n+1} + (66249 + 9100\sqrt{53})^{n+1}$

$$g_n = (66249 + 9100\sqrt{53})^{n+1} + (66249 + 9100\sqrt{53})^{n+1},\ n = 0, 1, 2 \ldots$$

Applying Brahma Gupta lemma between (x_0, y_0) and (x_n, y_n) the sequence of non-zero distinct integer solutions to (1) are obtained as

$$x_{n+1} = x_0 x_n + d y_0 y_n, y_{n+1} = x_0 y_n + y_0 x_n \ (3)$$

$$x_{n+1} = \frac{1}{2}[6f_n + \sqrt{53} g_n]$$

$$y_{n+1} = \frac{1}{2\sqrt{53}}[\sqrt{53} f_n + 6f_n] \tag{4}$$

The recurrence relations satisfied by the solutions of (1) are given by

$$x_{n+2} - 132498x_{n+1} + x_n = 0$$

$$y_{n+2} - 132498y_{n+1} + y_n = 0 \tag{5}$$

3.2 Choice 2. $t = 3$

The Pell equation is

$$x^2 = 53y^2 - 17^3 \tag{6}$$

Let (x_0, y_0) be the initial solution of (6) given by

$$x_0 = 102; y_0 = 17$$

Applying Brahma Gupta lemma between (x_0, y_0) and (x_n, y_n) the sequence of non-zero distinct integer solutions to (6) are obtained by equation (3) as

$$x_{n+1} = \frac{1}{2}[102f_n + 17\sqrt{53}g_n]$$

$$y_{n+1} = \frac{1}{2}[17\sqrt{53}f_n + 102g_n] \tag{7}$$

The recurrence relations satisfied by the solutions of (6) are given by

$$x_{n+2} = 132498x_{n+1} + x_n = 0$$

$$y_{n+2} = 132498y_{n+1} + y_n = 0 \tag{8}$$

3.3 Choice 3. $t = 5$

The Pell equation is

$$x^2 = 53y^2 - 17^5 \tag{9}$$

Let (x_0, y_0) be the initial solution of (9) given by

$$x_0 = 1734; y_0 = 289$$

Applying Brahma Gupta lemma between (x_0, y_0) and (x_n, y_n) the sequence of non-zero distinct integer solutions to (9) are obtained by equation (3) as

$$x_{n+1} = \frac{1}{2}[1734f_n + 289\sqrt{53}g_n]$$

$$y_{n+1} = \frac{1}{\sqrt{53}}[289\sqrt{53}f_n + 1734g_n] \tag{10}$$

The recurrence relations satisfied by the solutions of (9) are given by

$$x_{n+2} - 132498x_{n+1} + x_n = 0$$

$$y_{n+2} - 132498y_{n+1} + y_n = 0 \qquad (11)$$

3.4 Choices 4. $t = 2k,\ k \in \mathbb{N}$

The Pell equation is

$$x^2 = 53y^2 - 17^{2k},\ k \in \mathbb{N} \qquad (12)$$

Let $(x_0,\ y_0)$ be the initial solution of (12) given

$$x_0 = 182(17)^k;\ y_0 = 25(17)^k$$

Applying Brahma Gupta lemma between $(x_0,\ y_0)$ and (x_n, y_n) the sequence of non-zero distinct integer solutions to (12) are obtained by equation (3) as

$$x_{n+1} = \frac{17^k}{2}[182f_n + 25\sqrt{53}g_n]$$

$$y_{n+1} = \frac{17^k}{2\sqrt{53}}[25\sqrt{53}f_n + 182g_n] \qquad (13)$$

The recurrence relations satisfied by the solutions of (12) are given by

$$x_{n+2} - 132498x_{n+1} + x_n = 0$$

$$y_{n+2} - 132498y_{n+1} + y_n = 0 \qquad (14)$$

3.5 Choices 5. $t = 2k,\ 5,\ k \in \mathbb{N}$

The Pell equation is

$$x^2 = 53y^2 - 17^{2k+5},\ k \in \mathbb{N} \qquad (15)$$

Let $(x_0,\ y_0)$ be the initial solution of (15) given by

$$x_0 = 29478(17)^{k-1};\ y_0 = 4913(17)^{k-1}$$

Applying Brahma Gupta lemma between $(x_0,\ y_0)$ and (x_n, y_n) the sequence of non-zero distinct integer solutions to (15) are obtained by equation (3) as

$$x_{n+1} = \frac{17^{k-1}}{2}\left[2947 f_n + 4913\sqrt{53} g_n\right]$$

$$y_{n+1} = \frac{17^{k-1}}{2\sqrt{53}}\left[4913\sqrt{53} f_n + 29478 g_n\right] \quad (16)$$

The recurrence relations satisfied by the solutions of (15) are given by

$$x_{n+2} - 132498 x_{n+1} + x_n = 0$$

$$y_{n+2} - 132498 y_{n+1} + y_n = 0 \quad (17)$$

4. Conclusion

Solving a negative Pell's equation involving the Eisenstein primes has provided a powerful tool for finding solutions of equations of similar types. It is possible to determine the solvability of Pell-like equation using current methods.

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