**Advances and Applications in Mathematical Sciences**
Volume 21, Issue 1, November 2021, Pages 493-517


# ROUGH HESITANT BIPOLAR NEUTROSOPHIC SETS AND ITS APPLICATIONS IN GAME THEORY

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## Abstract

This paper proffers a rough hesitant bipolar valued neutrosophic set (RHBVNS) based on the combination of rough bipolar neutrosophic sets and hesitant fuzzy sets. The proposed set generalizes the notion of fuzzy set, intuitionistic fuzzy set, hesitant fuzzy set, single valued neutrosophic set, single valued neutrosophic hesitant fuzzy set, rough set, bipolar valued neutrosophic set, rough bipolar valued neutrosophic set. We define the basic operational laws union, intersection and complement for rough hesitant bipolar valued neutrosophic elements and study its properties. Some relevant examples are also given to provide a better understanding of the introduced concept. Two combination operators are developed based on RHBVNS which are the rough hesitant bipolar valued neutrosophic weighted averaging (RHBVNWA) and the rough hesitant bipolar valued neutrosophic weighted geometric (RHBVNWG). A decision making method is developed based on new sets and the proposed RHBVNW and RHBVNG operators. Finally an illustrative example is given to show the applicability of the proposed decision making method.

## 1. Introduction

The concept of neutrosophic set theory was introduced by Smarandache [1] in (1998), In (2005) Wang et al. [2] recommended the concept of single valued neutrosophic set (SVNS). Lee [3] introduced the concept of bipolar fuzzy sets, as an extension of fuzzy sets. In bipolar fuzzy set the degree of membership is extended from [0, 1] to [-1, 1]. In a bipolar fuzzy set if the degree of membership of an element is zero, then we say the element is

2020 Mathematics Subject Classification: 03E72, 91A35, 91B06.

Keywords: Rough hesitant bipolar valued neutrosophic set, rough hesitant bipolar valued neutrosophic weighted averaging operator and decision making.

Received August 14, 2021; Accepted October 22, 2021

unconnected to the corresponding property. The membership degree (0, 1] of an element satisfies that the element somewhat satisfies the property and the membership degree [-1, 0) of an element somewhat satisfies the implicit counter property.

In (2014), Broumi et al. [4] introduced the concept of rough neutrosophic set to deal indeterminacy in more adoptable way. Different applications of rough neutrosophic sets in decision making are proposed by Pramanik and Mondal [5].

Deli et al. [6] described bipolar neutrosophic set and revealed numerical example for multi-criteria decision making problem. As another generalization of fuzzy sets, the hesitant fuzzy sets (HF) was defined by Torra [7] which allows its membership function to have a set of feasible values. One of the implicit concept used to deal with imperfect information is hesitant fuzzy set. Ye [8] associates the advantages of single valued neutrosophic sets and hesistant fuzzy sets and introduced the concept of single valued neutrosophic hesitant fuzzy set which allows its membership function to have sets of possible values.

A. Awang et al. [13] proposed hesitant bipolar valued neutrosophic weighted averaging opertors and hesitant bipolar valued weighted geometric operator to aggregate criteria by considering their parameterization factor. Rough bipolarity and hesitancy are two different concepts that are constructed to model an easy going but accurate judgments when dealing with the imprecious and uncertainty in bipolar neutrosophic information. For this purpose, this paper proffered the rough hesitant valued neutrosophic sets. This set is the combination of rough bipolar valued neutrosophic sets and hesitant fuzzy sets. The rough hesitant bipolar valued neutrosophic set is implemented to obtain optimal decision for any decision making problems involving roughness, bipolarity and hesitancy. The basic operations for RHBVNS are presented including the union, intersection, complement, equality and inclusion. Also the properties related to the introduced definitions are also analysed.

The outline of this paper is arranged as follows, section 2 presents necessary basic definitions and concepts for developing the RHBVNS. Section 3 introduces the RHBVNS and its properties. Then a multiattribute decision making method is developed on the basis of RHVBNS in section 4. An illustrative numerical example is given in section 5. Finally section 6 concludes this paper.

## 2. Preliminaries

This section provides the related definitions that will be used to develop the proffered set.

**Neutrosopic Set:**

Let $X$ be a universe of discourse. Then a neutrosophic set is defined as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle / x \in X\}$ which is characterized by a truth membership function $T_A : X \to \left]0^-, 1^+\right[$, an indeterminacy membership function $I_A : X \to \left]0^-, 1^+\right[$ and a falsity membership function $F_A : X \to \left]0^-, 1^+\right[$.

There is no restriction on the sum of $T_A(x), I_A(x), F_A(x)$.

So $0^- \le \sup T_A(x) \le \sup I_A(x) \le \sup F_A(x) \le 3^+$.

**Single valued Neutrosopic Set:**

Let $X$ be a universe of discourse. A single valued neutrosophic set is defined as $A_{SNS} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle / x \in X\}$ which is characterized by a truth membership function $T_A : X \to [0, 1]$, an indeterminacy membership function $I_A : X \to [0, 1]$ and a falsity membership function $F_A : X \to [0, 1]$.

There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$.

So $0 \le \sup T_A(x) \le \sup I_A(x) \le \sup F_A(x) \le 3$.

**Bipolar Neutrosopic Set:**

A bipolar neutrosophic set $A$ in $X$ is defined as an object of the form $A = \{\langle x, T^+(x), I^+(x), F^+(x), I^-(x), F^-(x)\rangle / x \in X\}$ where $T^+, I^+, F^+ : X \to [0, 1], T^-, I^-, F^- : X \to [-1, 0]$.

The positive membership degree $T^{+}(x), I^{+}(x), F^{+}(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ corresponding to a bipolar neutrosophic set $A$ and the negative membership degree $T^{-}(x), I^{-}(x), F^{-}(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implicit counter property corresponding to a bipolar neutrosophic set $A$.

**Definition 2.1** (Inclusion). Let $A_1 = \{\langle x, T_1^{+}(x), I_1^{+}(x), F_1^{+}(x), T_1^{-}(x), I_1^{-}(x), F_1^{-}(x)\rangle / x \in X\}$ and $A_2 = \{\langle x, T_2^{+}(x), I_2^{+}(x), F_2^{+}(x), T_2^{-}(x), I_2^{-}(x), F_2^{-}(x)\rangle / x \in X\}$ be two bipolar neutrosophic sets. Then $A_1 \subseteq A_2$ if and only if $T_1^{+}(x) \leq T_2^{+}(x), I_1^{+}(x) \leq I_2^{+}(x), F_1^{+}(x) \geq F_2^{+}(x), T_1^{-}(x) \geq T_2^{-}(x), I_1^{-}(x) \geq I_2^{-}(x), F_1^{-}(x) \leq F_2^{-}(x)\ \forall x \in X$.

**Definition 2.2** (Equality). For the above two bipolar neutrosophic set (BNS), $A_1 = A_2$ if and only if $T_1^{+}(x) = T_2^{+}(x), I_1^{+}(x) = I_2^{+}(x) = I_2^{+}(x), F_1^{+}(x) = F_2^{+}(x)$ and $T_1^{-}(x) = T_2^{-}(x), I_1^{-}(x) = I_2^{-}(x), F_1^{-}(x) = F_2^{-}(x)\ \forall x \in X$.

**Definition 2.3** (Union).

$$A_1 \cup A_2(x) = \begin{cases} \max(T_1^{+}(x), T_2^{+}(x)), \dfrac{I_1^{+}(x) + I_2^{+}(x)}{2}, \min(F_1^{+}(x), F_2^{+}(x)) \\ \min(T_1^{-}(x), T_2^{-}(x)), \dfrac{I_1^{-}(x) + I_2^{-}(x)}{2}, \max(F_1^{-}(x), F_2^{-}(x)) \end{cases}$$

$\forall\ x \in X$.

**Definition 2.4** (Intersection).

$$A_1 \cap A_2(x) = \begin{cases} \min(T_1^{+}(x), T_2^{+}(x)), \dfrac{I_1^{+}(x) + I_2^{+}(x)}{2}, \max(F_1^{+}(x), F_2^{+}(x)) \\ \max(T_1^{-}(x), T_2^{-}(x)), \dfrac{I_1^{-}(x) + I_2^{-}(x)}{2}, \min(F_1^{-}(x), F_2^{-}(x)) \end{cases}$$

$\forall\ x \in X$.

**Definition 2.5** (Complement).

$$A^c = \begin{cases} \{1^+\} - T_A^+(x),\ \{1^+\} - I_A^+(x),\ \{1^+\} - F_A^+(x) \\ \{1^-\} - T_A^-(x),\ \{1^-\} - I_A^-(x),\ \{1^-\} - F_A^-(x) \end{cases}$$

$\forall\ x \in X$.

**Definition 2.6** (Addition).

$$A_1 \oplus A_2 = \left\langle \begin{matrix} T_1^+ + T_2^+ - T_1^+ \cdot T_2^+,\ I_1^+ \cdot I_2^+,\ F_1^+ \cdot F_2^+ \\ - T_1^- - T_2^-,\ -(- I_1^- - I_2^- - I_1^- \cdot I_2^-) - (- F_1^- - F_2^- - F_1^- \cdot F_2^-) \end{matrix} \right\rangle$$

**Definition 2.7** (Product).

$$A_1 \otimes A_2 = \left\langle \begin{matrix} T_1^+ \cdot T_2^+,\ I_1^+ + I_2^+ - I_1^+ \cdot I_2^+,\ F_1^+ + F_2^+ - F_1^+ \cdot F_2^+ \\ -(T_1^- - T_2^- - T_1^- \cdot T_2^-),\ -(I_1^- \cdot I_2^-),\ -(F_1^- \cdot F_2^-) \end{matrix} \right\rangle$$

**Definition 2.8** (Power).

$$A^P = \left\langle \begin{matrix} {T_1^+}^P,\ 1 - (1 - I_1^+)^P,\ 1 - (1 - F_1^+)^P \\ -(1 - (1 - (-T_1^-)^P),\ -(-I_1^-)^P,\ -(-F_1^-)^P \end{matrix} \right\rangle$$

where $p > 0$.

**Definition 2.9** (Scalar multiplication).

$$PA = \left\langle \begin{matrix} 1 - (1 - T^+)^P,\ (I^+)^P,\ (F^+)^P \\ -(-T_1^-)^P,\ -(1 - (1 - (-I_1^-))^P),\ -(1 - (1 - (-F_1^-))^P) \end{matrix} \right\rangle$$

**Hesitant Fuzzy Set (HFS):**

Let $X$ be a reference set and a hesitant fuzzy set $H$ on $X$ is defined as

$$H = \langle\ x,\ h_H(x)/x \in X\}$$

where $h = h_H(x)$ denoted as the hesitant fuzzy element and the values are in the range of [0, 1] and reject the possible membership values of an element $x \in X$ of the set $H$.

**Definition 2.10** Let $h, h_1$ and $h_2$ be three hesitant fuzzy elements. Then

(i) Complement $h^c = \bigcup_{\gamma \in h} (1 - \gamma)$

(ii) Union $h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$

(iii) Intersection $h_1 \cap h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$

(iv) Addition $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} [\gamma_1 + \gamma_2 - \gamma_1 \cdot \gamma_2]$

(v) Product $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} (\gamma_1, \gamma_2)$

(vi) Power $h^P = \bigcup_{\gamma \in h} \{\gamma^P\}$

(vii) Scalar multiplication $Ph = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^P\}$

**Rough Neutrosophic Sets:**

Let $X$ be a non-null set and $R$ be an equivalence relation on $X$ and let $A$ be a neutrosophic set in $X$ with the truth membership function $T_A$, indeterminacy function $I_A$ and non-membership or Falsity membership function $F_A$.

The lower and upper approximation of $A$ in the approximation $(X, R)$ denoted by $\underline{N}(A)$ and $\overline{N}(A)$ can be respectively defined as follows

$$\underline{N}(A) = \{\langle x, T_{\underline{N}(A)}(x), I_{\underline{N}(A)}(x), F_{\underline{N}(A)}(x) \rangle / x \in X\}$$

$$\overline{N}(A) = \{\langle x, T_{\overline{N}(A)}(x), I_{\overline{N}(A)}(x), F_{\overline{N}(A)}(x) \rangle / x \in X\}$$

where $T_{\underline{N}(A)}(x) = \wedge T_A(x)$. Similarly $T_{\overline{N}(A)}(x) = \vee T_A(x)$

$$I_{\underline{N}(A)}(x) = \wedge I_A(x) \qquad I_{\overline{N}(A)}(x) = \vee I_A(x)$$

$$F_{\underline{N}(A)}(x) = \wedge F_A(x) \qquad F_{\overline{N}(A)}(x) = \vee F_A(x)$$

so $0 \le T_{\underline{N}(A)}(x) + I_{\underline{N}(A)}(x) + F_{\underline{N}(A)}(x) \le 3$ and

$0 \le T_{\overline{N}(A)}(x) + I_{\overline{N}(A)}(x) + F_{\overline{N}(A)}(x) \le 3$ where

$\vee$ and $\wedge$ denote maximum and minimum operators respectively.

$T_A(x), I_A(x), F_A(x)$ are the true membership, indeterminacy and non-membership of $x$ with respect to $A$. It is easily checked $\underline{N}(A)$ and $\overline{N}(A)$ are two neutrosophic sets in $X$. The neutrosophic mapping $\underline{N}, \overline{N} : N(X) \to N(X)$ are respectively referred to as the lower and upper rough neutrosophic set operators and the pair $(\underline{N}(A), \overline{N}(A))$ is called the rough neutrosophic set in $(X, R)$.

If $\underline{N}(A) = \overline{N}(A)$ then $T_{\underline{N}(A)}(x) = T_{\overline{N}(A)}(x)\, I_{\underline{N}(A)}(x) = I_{\overline{N}(A)}$ and $F_{\underline{N}(A)}(x)$ $= F_{\overline{N}(A)}(x) \forall x \in X$.

**Definition 2.11** (Hesitant Bipolar Neutrosophic set (HBNS))**.** Let $X$ be a reference set and with a generic element in $X$ denoted by $x$. A hesitant bi-polar neutrosophic set $\hat{H}$ in $X$ is defined as

$$\hat{H} = \{x, \langle\, h_T^+(x), h_I^+(x), h_F^+(x), h_T^-(x), h_I^-(x), h_F^-(x)\, \rangle / x \in X\}$$

where $h_T^+(x), h_I^+(x), h_F^+(x) : X \to [0, 1]$

and $h_T^-(x), h_I^-(x), h_F^-(x) : X \to [0, 1]$.

The positive hesitant bi-polar neutrosophic elements $h_T^+, h_I^+, h_F^+$ denotes the possible satisfactory degree of truth, indeterminacy and falsity of an element $x \in X$ corresponding to a HBNS $\hat{H}$ respectively while the negative hesitant bipolar neutrosophic elements $h_T^-, h_I^-, h_F^-$ denote the possible satisfactory degree of truth, indeterminacy and falsity of an element $x \in X$ to the implicit counter property to the set $\hat{H}$ respectively.

In addition, a HBNS $\hat{H}$ must satisfy the conditions $0 \le \gamma_T^+, \gamma_I^+, \gamma_F^+ \le 1$ and $-1 \le \gamma_T^-, \gamma_I^-, \gamma_F^- \le 0$.

$0 \le \max\{\gamma_T^+\} + \max\{\gamma_I^+\} + \max\{\gamma_F^+\} \le 3$ and

$-3 \le \max\{\gamma_T^-\} + \max\{\gamma_I^-\} + \max\{\gamma_F^-\} \le 0$

in which $\gamma_T^+ \in h_T^+(x)$. Similarly $\gamma_T^- \in h_T^-(x)$

$$\gamma_I^+ \in h_I^+(x) \qquad \gamma_I^- \in h_I^-(x)$$

$$\gamma_F^+ \in h_F^+(x) \qquad \gamma_F^- \in h_F^-(x) \text{ for } x \in X.$$

For convenience, we use the symbol $\hat{H}$ to represent all the hesitant bipolar neutrosophic sets and $h = \{h_T^+, h_I^+, h_F^+, h_T^-, h_I^-, h_F^-\}$ for a hesitant bipolar neutrosophic element.

## 3. Proposed Rought Hesitant Bipolar Valued Neutrosophic Set (RHBVNS)

In this section, the rough hesitant bipolar neutrosophic set (RHBNS) is introduced. Then some basic properties and operational laws are investigated for the proposal.

**Definition 3.1.** Let $X$ be a reference set $R$ be an equivalence relation on $X$. Let $\hat{H}$ be a hesitant bipolar neutrosophic set with positive hesitant truth membership function $h_T^+$, indeterminacy function $h_I^+$ and falsity membership function $h_F^+$ and negative hesitant truth membership function $h_T^-$, indeterminacy function $h_I^-$ and falsity membership function $h_F^-$.

The lower and upper approximations of $A$ in the approximation $(X, R)$ denoted by $(\underline{\hat{H}})$ and $(\overline{\hat{H}})$ can be respectively defined as follows

$$\underline{\hat{H}} = \{\langle\ x, h_{\underline{T}}^+(x), h_{\underline{I}}^+(x), h_{\underline{F}}^+(x), h_{\underline{T}}^-(x), h_{\underline{I}}^-(x), h_{\underline{F}}^-(x)\ \rangle / x \in X\}$$

$$\overline{\hat{H}} = \{\langle\ x, h_{\overline{T}}^+(x), h_{\overline{I}}^+(x), h_{\overline{F}}^+(x), h_{\overline{T}}^-(x), h_{\overline{I}}^-(x), h_{\overline{F}}^-(x)\ \rangle / x \in X\}$$

where $h^{+}_{\underline{T}}(x) = \wedge h^{+}_{T}(x)$. Similarly $h^{+}_{\underline{T}}(x) = \vee h^{+}_{T}(x)$

$$h^{+}_{\underline{I}}(x) = \wedge h^{+}_{I}(x) \qquad h^{+}_{\overline{I}}(x) = \vee h^{+}_{I}(x)$$

$$h^{+}_{\underline{F}}(x) = \wedge h^{+}_{F}(x) \qquad h^{+}_{\overline{F}}(x) = \vee h^{+}_{F}(x)$$

$$h^{-}_{\underline{T}}(x) = \wedge h^{-}_{T}(x) \qquad h^{-}_{\overline{T}}(x) = \vee h^{-}_{T}(x)$$

$$h^{-}_{\underline{I}}(x) = \wedge h^{-}_{I}(x) \qquad h^{-}_{\overline{I}}(x) = \vee h^{-}_{I}(x)$$

$$h^{-}_{\underline{F}}(x) = \wedge h^{-}_{F}(x) \qquad h^{-}_{\overline{F}}(x) = \vee h^{-}_{F}(x)$$

In addition a RHBVNS must satisfy the condition

$$0 \le \gamma^{+}_{\underline{T}}, \gamma^{+}_{\underline{I}}, \gamma^{+}_{\underline{F}}, \gamma^{+}_{\overline{T}}, \gamma^{+}_{\overline{I}}, \gamma^{+}_{\overline{F}} \le 1 \text{ and } -1 \le \gamma^{-}_{\underline{T}}, \gamma^{-}_{\underline{I}}, \gamma^{-}_{\underline{F}}, \gamma^{-}_{\overline{T}}, \gamma^{-}_{\overline{I}}, \gamma^{-}_{\overline{F}} \le 0$$

$$0 \le \max\{\gamma^{+}_{\underline{T}}\} + \max\{\gamma^{+}_{\underline{I}}\} + \max\{\gamma^{+}_{\underline{F}}\} \le 3.$$

Similarly $0 \le \max\{\gamma^{+}_{\overline{T}}\} + \max\{\gamma^{+}_{\overline{I}}\} + \max\{\gamma^{+}_{\overline{F}}\} \le 3$

and $-3 \le \max\{\gamma^{-}_{\underline{T}}\} + \max\{\gamma^{-}_{\underline{I}}\} + \max\{\gamma^{+}_{\underline{F}}\} \le 0.$

Similarly $-3 \le \max\{\gamma^{-}_{\overline{T}}\} + \max\{\gamma^{-}_{\overline{I}}\} + \max\{\gamma^{-}_{\overline{F}}\} \le 0$

in which $\gamma^{+}_{\underline{T}}(x) \in h^{+}_{\underline{T}}(x)$. Similarly $\gamma^{-}_{\underline{T}}(x) \in h^{-}_{\underline{T}}(x)$

$$\gamma^{+}_{\underline{I}}(x) \in h^{+}_{\underline{I}}(x) \qquad \gamma^{-}_{\underline{I}}(x) \in h^{-}_{\underline{I}}(x)$$

$$\gamma^{+}_{\underline{F}}(x) \in h^{+}_{\underline{F}}(x) \qquad \gamma^{-}_{\underline{F}}(x) \in h^{-}_{\underline{F}}(x)$$

$$\gamma^{+}_{\overline{T}}(x) \in h^{+}_{\overline{T}}(x) \qquad \gamma^{-}_{\overline{T}}(x) \in h^{-}_{\overline{T}}(x)$$

$$\gamma^{+}_{\overline{I}}(x) \in h^{+}_{\overline{I}}(x) \qquad \gamma^{-}_{\overline{I}}(x) \in h^{-}_{\overline{I}}(x)$$

$$\gamma^{+}_{\overline{F}}(x) \in h^{+}_{\overline{F}}(x) \qquad \gamma^{-}_{\overline{F}}(x) \in h^{-}_{\overline{F}}(x).$$

For convenience, we use the $\underline{\hat{H}}$ to represent the lower approximation of $\hat{H}$, $\overline{\hat{H}}$ to represent the upper approximation of $\hat{H}$ and

$$\underline{h} = \langle h^{+}_{\underline{T}}, h^{+}_{\underline{I}}, h^{+}_{\underline{F}}, h^{-}_{\underline{T}}, h^{-}_{\underline{I}}, h^{-}_{\underline{F}} \rangle$$

$$\overline{h} = \langle h^{+}_{\overline{T}}, h^{+}_{\overline{I}}, h^{+}_{\overline{F}}, h^{-}_{\overline{T}}, h^{-}_{\overline{I}}, h^{-}_{\overline{F}} \rangle.$$

**Example 3.1.** Let $X = \{x_1, x_2, x_3\}$. A rough hesitant bipolar neutrosophic set $A$ is defined by

$$A = [\underline{\hat{H}}, \overline{\hat{H}}]$$

$$= \left\langle \begin{array}{l} \langle x_1, \{(0.5, 0.4, \{0.1, 0.3\}, -0.6, -0.3, -0.5) \\ \quad (0.6, \{0.3, 0.4\}, 0.5, -0.2, -0.5, -0.6)\}\rangle \\ \langle x_2, \{(\{0.5, 0.4\}, 0.6, 0.7, -0.3, -0.4, -0.6) \\ \quad (0.7, 0.8, \{0.5, 0.6\}, -0.2, -0.5, -0.7)\} \\ \langle x_3, \{(0.7, 0.8, 0.6, \{-0.1, -0.2\}, \{-0.5, -0.4\}, -0.6), \\ \quad (0.5, 0.3, 0.4, \{-0.3, -0.4\}, -0.7, -0.8)\}\rangle \end{array} \right\rangle$$

**Containment:**

Let $A = [\underline{\hat{H}_1}, \overline{\hat{H}_1}]$ and $B = [\underline{\hat{H}_2}, \overline{\hat{H}_2}]$ be two rough hesitant bipolar neutrosophic sets. Then $A \subseteq B$ if and only if the following conditions hold.

$h^{+}_{1\underline{T}}(x) \le h^{+}_{1\overline{T}}(x)$. Similarly $h^{-}_{1\underline{T}}(x) \ge h^{-}_{1\overline{T}}(x)$

$h^{+}_{1\underline{I}}(x) \le h^{+}_{1\overline{I}}(x)$ $\quad h^{-}_{1\underline{I}}(x) \ge h^{-}_{1\overline{I}}(x)$ and

$h^{+}_{1\underline{F}}(x) \le h^{+}_{1\overline{F}}(x)$ $\quad h^{-}_{1\underline{F}}(x) \ge h^{-}_{1\overline{F}}(x)$

$h^{+}_{2\underline{T}}(x) \le h^{+}_{2\overline{T}}(x)$ $\quad h^{-}_{2\underline{T}}(x) \ge h^{-}_{2\overline{T}}(x)$

$h^{+}_{2\underline{I}}(x) \le h^{+}_{2\overline{I}}(x)$ $\quad h^{-}_{2\underline{I}}(x) \ge h^{-}_{2\overline{I}}(x)$ and

$h^{+}_{2\underline{F}}(x) \le h^{+}_{2\overline{F}}(x)$ $\quad h^{-}_{2\underline{F}}(x) \ge h^{-}_{2\overline{F}}(x)$ $\forall x \in X$

**Equality:**

Two rough hesitant bipolar neutrosophic sets $A = [\underline{\hat{H}_1}, \overline{\hat{H}_1}]$ and $B = [\underline{\hat{H}_2}, \overline{\hat{H}_2}]$ are equal if the following conditions hold.

$h^{+}_{1\underline{T}}(x) = h^{+}_{2\overline{T}}(x)$. Similarly $h^{-}_{1\underline{T}}(x) = h^{-}_{2\overline{T}}(x)$

$h^{+}_{1\underline{I}}(x) = h^{+}_{2\bar{I}}(x)$ $\qquad h^{-}_{1\underline{I}}(x) = h^{-}_{2\bar{I}}(x)$

$h^{+}_{1\underline{F}}(x) = h^{+}_{2\overline{F}}(x)$ $\qquad h^{-}_{1\underline{F}}(x) = h^{-}_{2\overline{F}}(x)\ \forall x \in X.$

**Union:**

Let $A = [\underline{\hat{H}_1}, \overline{\hat{H}_1}]$ and $B = [\underline{\hat{H}_2}, \overline{\hat{H}_2}]$ be two rough hesitant bipolar neutrosophic sets. Then $A \cup B$ is defined as

$$(A \cup B)(x) = \begin{bmatrix} \left( \max\,(h^{+}_{1\underline{T}}(x),\, h^{+}_{2\underline{T}}(x)),\, \dfrac{h^{+}_{1\underline{I}}(x),\, h^{+}_{2\underline{I}}(x)}{2},\, \min\,(h^{+}_{1\underline{F}}(x),\, h^{+}_{2\underline{F}}(x)) \right) \\ \left( \min\,(h^{-}_{1\underline{T}}(x),\, h^{-}_{2\underline{T}}(x)),\, \dfrac{h^{-}_{1\underline{I}}(x) + h^{-}_{2\underline{I}}(x)}{2},\, \max\,(h^{-}_{1\underline{F}}(x),\, h^{-}_{2\underline{F}}(x)) \right) \\ \left( \max\,(h^{+}_{1\overline{T}}(x),\, h^{+}_{2\overline{T}}(x)),\, \dfrac{h^{+}_{1\bar{I}}(x) + h^{+}_{2\bar{I}}(x)}{2},\, \min\,(h^{+}_{1\overline{F}}(x),\, h^{+}_{2F}(x)) \right) \\ \left( \min\,(h^{-}_{1\overline{T}}(x),\, h^{-}_{2\overline{T}}(x)),\, \dfrac{h^{-}_{1\bar{I}}(x),\, h^{-}_{2\bar{I}}(x)}{2},\, \max\,(h^{-}_{1\overline{F}}(x),\, h^{-}_{2\overline{F}}(x)) \right) \end{bmatrix}.$$

**Example 3.2.** Let $X = \{x_1, x_2, x_3\}$. Two rough hesitant bipolar neutrosophic sets $A$ and $B$ on $X$ are

$$A = \left\langle \begin{array}{l} \langle\, x_1,\ \{(0.5,\ 0.4,\ \{0.1,\ 0.3\},\ -0.6,\ -0.3,\ -0.5), \\ \qquad (0.6,\ \{0.3,\ 0.4\},\ 0.5,\ -0.2,\ -0.5,\ -0.6)\}\ \rangle \\ \langle\, x_2,\ \{(\{0.5,\ 0.4\},\ 0.6,\ 0.7,\ -0.3,\ -0.4,\ -0.6) \\ \qquad (0.7,\ 0.8,\ \{0.5,\ 0.6\},\ -0.2,\ -0.5,\ -0.7)\}\ \rangle \\ \langle\, x_3,\ \{(0.7,\ 0.8,\ 0.6,\ \{-\ 0.1,\ -0.2\},\ \{-\ 0.5,\ -0.4\},\ -0.6), \\ \qquad (0.5,\ 0.3,\ 0.4,\ \{-\ 0.3,\ -0.4\},\ -0.7,\ -0.8)\}\ \rangle \end{array} \right\rangle$$

and

$$B = \left\langle \begin{array}{c} \langle\, x_1,\ \{(0.8,\ 0.9,\ 0.6,\ \{-0.6,\ -0.7\},\ -0.5,\ -0.4), \\ (0.6,\ \{0.7,\ 0.8\},\ 0.9,\ -0.4,\ -0.6,\ -0.3)\}\ \rangle \\ \langle\, x_2,\ \{(\{0.5,\ 0.7\},\ 0.3,\ 0.4,\ -0.6,\ -0.5,\ -0.8) \\ (0.3,\ 0.4,\ \{0.7,\ 0.8\},\ -\ 0.4,\ -0.6,\ -0.3)\}\ \rangle \\ \langle\, x_3,\ \{(0.3,\ 0.4,\ 0.5,\ \{-0.6,\ -0.7\},\ \{-0.3,\ -0.4\},\ -0.8), \\ (\{0.5,\ 0.6\},\ 0.7,\ 0.8,\ \{-0.3,\ -0.4\},\ -0.1,\ -0.2)\}\ \rangle \end{array} \right\rangle$$

$$A \cup B = \left\langle \begin{array}{c} \langle\, x_1,\ \{(0.8,\ 0.65,\ 0.1,\ -0.6,\ -0.4,\ -0.4), \\ (0.6,\ 0.65,\ 0.5,\ -0.4,\ -0.5,\ -0.3)\}\ \rangle \\ \langle\, x_2,\ \{(0.5,\ 0.45,\ 0.4,\ -0.6,\ -0.45,\ -0.6), \\ (0.7,\ 0.6,\ 0.6\ -0.7,\ -0.7,\ -0.5)\}\ \rangle \\ \langle\, x_3,\ \{(0.7,\ 0.6,\ 0.5,\ -0.7,\ 0.45,\ -0.6), \\ (0.5,\ 0.45,\ 0.4,\ -0.4,\ -0.4,\ -0.2)\}\ \rangle \end{array} \right\rangle$$

**Intersection:**

$$(A \cap B)(x) = \left[ \begin{array}{c} \left( \min\ (h^{+}_{1\underline{T}}(x),\ h^{+}_{2\underline{T}}(x)),\ \dfrac{h^{+}_{1\underline{I}}(x),\ h^{+}_{2\underline{I}}(x)}{2},\ \max\,(h^{+}_{1\underline{F}}(x),\ h^{+}_{2\underline{F}}(x)) \right) \\ \left( \max\ (h^{-}_{1\underline{T}}(x),\ h^{-}_{2\underline{T}}(x)),\ \dfrac{h^{-}_{1\underline{I}}(x),\ h^{-}_{2\underline{I}}(x)}{2},\ \min\,(h^{-}_{1\underline{F}}(x),\ h^{-}_{2\underline{F}}(x)) \right) \\ \left( \min\ (h^{+}_{1\overline{T}}(x),\ h^{+}_{2\overline{T}}(x)),\ \dfrac{h^{+}_{1\bar{I}}(x),\ h^{+}_{2\bar{I}}(x)}{2},\ \max\,(h^{+}_{1\overline{F}}(x),\ h^{+}_{2F}(x)) \right) \\ \left( \max\ (h^{-}_{1\overline{T}}(x),\ h^{-}_{2\overline{T}}(x)),\ \dfrac{h^{-}_{1\bar{I}}(x),\ h^{-}_{2\bar{I}}(x)}{2},\ \min\,(h^{-}_{1\overline{F}}(x),\ h^{-}_{2\overline{F}}(x)) \right) \end{array} \right].$$

**Example 3.3.** Let $X = \{x_1,\ x_2,\ x_3\}$. For the above two rough hesitant bipolar valued neutrosophic sets $A$ and $B$

$$A \cap B = \left\langle \begin{array}{c} \langle\, x_1,\ \{(0.5,\ 0.65,\ 0.6,\ -0.6,\ -0.4,\ -0.5), \\ (0.6,\ 0.6,\ 0.9,\ -0.2,\ -0.55,\ -0.6)\}\ \rangle \\ \langle\, x_2,\ \{(0.4,\ 0.45,\ 0.7,\ -0.3,\ -0.45,\ -0.8), \\ (0.3,\ 0.6,\ 0.8\ -0.2,\ -0.7,\ -0.7)\}\ \rangle \\ \langle\, x_3,\ \{(0.3,\ 0.6,\ 0.6,\ -0.1,\ 0.45,\ -0.8), \\ (0.5,\ 0.5,\ 0.8,\ -0.3,\ -0.4,\ -0.8)\}\ \rangle \end{array} \right\rangle$$

**Complement:**

If $A = [\underline{\hat{H}_1}, \overline{\hat{H}_1}]$ is a rough hesitant bipolar neutrosophic set, the complement of $A$, $A^c$ is also rough hesitant bipolar neutrosophic set which is defined by $A^c = [\underline{\hat{H}}^c, \overline{\hat{H}}^c]$

$$\underline{\hat{H}}^c = \left\langle \begin{array}{l} \langle\, x, 1 - h^+_{\underline{T}}(x), 1 - h^+_{\underline{I}}(x), 1 - h^+_{\underline{F}}(x), \\ -1 - h^-_{\underline{T}}(x), -1 - h^-_{\underline{I}}(x), -1 - h^-_{\underline{F}}(x) \,\rangle / x \in x \end{array} \right\rangle$$

$$\overline{\hat{H}}^c = \left\langle \begin{array}{l} \langle\, x, 1 - h^+_{\overline{T}}(x), 1 - h^+_{\overline{I}}(x), 1 - h^+_{\overline{F}}(x), \\ -1 - h^-_{\overline{T}}(x), -1 - h^-_{\overline{I}}(x), -1 - h^-_{\overline{F}}(x) \,\rangle / x \in x \end{array} \right\rangle$$

**Example 3.4.** Let $X = \{x_1, x_2, x_3\}$. For the above rough hesitant bipolar neutrosophic sets $A$, the complement of $A$, $A^c$ is given as follows.

$$A^c = \left\langle \begin{array}{l} \langle\, x_1, \{(0.5, 0.6, \{0.9, 0.7\}, -0.4, -0.7, -0.5), \\ \quad (0.4, \{0.7, 0.6\}, 0.5, -0.8, -0.5, -0.4)\} \,\rangle \\ \langle\, x_2, \{(\{0.5, 0.6\}, 0.4, 0.3, -0.7, -0.6, -0.4) \\ \quad (0.3, 0.2, \{0.5, 0.4\}, -0.8, -0.5, -0.3)\} \,\rangle \\ \langle\, x_3, \{(0.3, 0.2, 0.4, \{-0.9, -0.8\}, \{-0.5, -0.6\}, -0.4), \\ \quad (0.5, 0.7, 0.6, \{-0.7, -0.6\}, -0.3, -0.2)\} \,\rangle \end{array} \right\rangle$$

**Definition 3.2.** If $A = [\underline{\hat{H}_1}, \overline{\hat{H}_1}]$ and $B = [\underline{\hat{H}_2}, \overline{\hat{H}_2}]$ be two rough hesitant bipolar neutrosophic sets in $X$, then we define the following

(1) $A = B$ if and only if $\underline{\hat{H}_1} = \underline{\hat{H}_2}$ and $\overline{\hat{H}_1} = \overline{\hat{H}_2}$

(2) $A \subseteq B$ if and only if $\underline{\hat{H}_1} = \underline{\hat{H}_2}$ and $\overline{\hat{H}_1} \subseteq \overline{\hat{H}_2}$

(3) $A \cup B$ if and only if $\langle \underline{\hat{H}_1} \cup \underline{\hat{H}_2}, \overline{\hat{H}_1} \cup \overline{\hat{H}_2} \rangle$

(4) $A \cap B$ if and only if $\langle \underline{\hat{H}_1} \cap \underline{\hat{H}_2}, \overline{\hat{H}_1} \cap \overline{\hat{H}_2} \rangle$

(5) $A + B$ if and only if $\langle \underline{\hat{H}_1} + \underline{\hat{H}_2}, \overline{\hat{H}_1} + \overline{\hat{H}_2} \rangle$

(6) $A \cdot B$ if and only if $\langle \underline{\hat{H}_1} \cdot \underline{\hat{H}_2}, \overline{\hat{H}_1} \cdot \overline{\hat{H}_2} \rangle$

**Proposition 3.1.** *If* $A = [\underline{\hat{H}_1}, \overline{\hat{H}_1}]$, $B = [\underline{\hat{H}_2}, \overline{\hat{H}_2}]$ *and* $C = [\underline{\hat{H}_3}, \overline{\hat{H}_3}]$ *are rough hesitant bipolar neutrosophic sets in* $(X, R)$ *then the following propositions hold.*

(1) $(A^c)^c = A$.

(2) $\underline{\hat{H}_1} \subseteq \overline{\hat{H}_1}$

(3) $[\underline{\hat{H}_1} \cup \underline{\hat{H}_2}]^c = (\underline{\hat{H}_1})^c \cap (\underline{\hat{H}_2})^c$

(4) $[\underline{\hat{H}_1} \cap \underline{\hat{H}_2}]^c = (\underline{\hat{H}_1})^c \cup (\underline{\hat{H}_2})^c$

(5) $[\overline{\hat{H}_1} \cup \overline{\hat{H}_2}]^c = (\overline{\hat{H}_1})^c \cap (\underline{\overline{\hat{H}_2}})^c$

(6) $[\overline{\hat{H}_1} \cup \overline{\hat{H}_2}]^c = (\overline{\hat{H}_1})^c \cup (\underline{\overline{\hat{H}_2}})^c$

(7) $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$

(8) $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$

**Proof of (1)**

If $A = [\underline{\hat{H}_1}, \overline{\hat{H}_1}]$ is a rough hesitant bipolar neutrosophic set in $(X, R)$, the complement of $A$ is the rough hesitant bipolar neutrosophic set defined as follows

$$A^c = [\underline{\hat{H}_1}^c, \overline{\hat{H}_1}^c]$$

$$(A^c)^c = [(\underline{\hat{H}_1}^c)^c, (\overline{\hat{H}_1}^c)^c]$$

$$(\underline{\hat{H}}_1{}^c)^c = \left\langle \begin{matrix} \langle\ x, 1-(1-h^+_{\underline{T}}(x)), 1-(1-h^+_{\underline{I}}(x), 1-(1-h^+_{\underline{F}}(x)), \\ -1-(\ -1-h^-_{\underline{T}}(x)), -1-(-1-h^-_{\underline{I}}(x), -1-(\ -1-h^-_{\underline{F}}(x))\ \rangle / x \in x \end{matrix} \right\rangle$$

$$= \left\langle x, h^+_{\underline{T}}(x), h^+_{\underline{I}}(x), h^+_{\underline{F}}(x), h^-_{\underline{T}}(x), h^-_{\underline{I}}(x), h^-_{\underline{F}}(x)/x \in x \right\rangle$$

$= \underline{\hat{H}_1}$

Similarly $(\overline{H_1}^c)^c = \overline{\hat{H}_1}$ which implies $(A^c)^c = A$.

**Proof of (2)**

From the definition of rough hesitant bipolar valued neutrosophic set, we can have $\underline{\hat{H}_1} \subseteq \overline{\hat{H}_1}$.

**Proof of (3)**

Let $x \in [\underline{\hat{H}_1} \cup \underline{\hat{H}_2}]^c \Rightarrow x \in \underline{\hat{H}_1}$ and $x \in \underline{\hat{H}_2}^c$

$$\Rightarrow x \in [(\underline{\hat{H}_1})^c \cap (\underline{\hat{H}_2})^c]$$

$$\Rightarrow [\underline{\hat{H}_1} \cup \underline{\hat{H}_2}]^c \subseteq (\underline{\hat{H}_1})^c \cap (\underline{\hat{H}_2})^c$$

Similarly $(\underline{\hat{H}_1})^c \cap (\underline{\hat{H}_2})^c \subseteq (\underline{\hat{H}_1} \cup \underline{\hat{H}_2})^c$

**Proof of (4)**

Similar proof of (3).

We can easily prove (5), (6), (7) and (8) results.

**Definition 3.3.** Let $A = [\underline{\hat{H}}_1, \overline{\hat{H}_1}]$ and $B = [\underline{\hat{H}}_1, \overline{\hat{H}_1}]$

$$A = \left\langle \begin{array}{l} \langle\, x, (h^+_{1\underline{T}}(x), h^+_{1\underline{I}}(x), h^+_{1\underline{F}}(x), h^-_{1\underline{T}}(x), h^-_{1\underline{I}}(x), h^-_{1\underline{F}}(x)) \\ (h^+_{1\overline{T}}(x), h^+_{1\overline{I}}(x), h^+_{1\overline{F}}(x), h^-_{1\overline{T}}(x), h^-_{1\overline{I}}(x), h^-_{1\overline{F}}(x))\,\rangle \end{array} \right\rangle$$

$$B = \left\langle \begin{array}{l} \langle\, x, (h^+_{2\underline{T}}(x), h^+_{2\underline{I}}(x), h^+_{2\underline{F}}(x), h^-_{2\underline{T}}(x), h^-_{2\underline{I}}(x), h^-_{2\underline{F}}(x)) \\ (h^+_{2\overline{T}}(x), h^+_{2\overline{I}}(x), h^+_{2\overline{F}}(x), h^-_{2\overline{T}}(x), h^-_{2\overline{I}}(x), h^-_{2\overline{F}}(x))\,\rangle \end{array} \right\rangle$$

(1) Addition

$$A \oplus B = \left\langle \begin{pmatrix} h^{+}_{1\underline{T}}(x) + h^{+}_{2\underline{T}}(x) - h^{+}_{1\underline{T}}(x) \cdot h^{+}_{2\underline{T}}(x), h^{+}_{1\underline{I}}(x) \cdot h^{+}_{2\underline{I}}(x), h^{+}_{1\underline{F}}(x) \cdot h^{+}_{2\underline{F}}(x), \\ -(h^{-}_{1\underline{T}}(x) \cdot h^{-}_{2\underline{T}}(x)), -(-h^{-}_{1\underline{I}}(x) - h^{-}_{2\underline{I}}(x) - h^{-}_{1\underline{I}}(x) \cdot h^{-}_{2\underline{I}}(x)), \\ -(-h^{-}_{1\underline{F}}(x) - h^{-}_{2\underline{F}}(x) - h^{-}_{1\underline{F}}(x) \cdot h^{-}_{2\underline{F}}(x)) \end{pmatrix} \begin{pmatrix} h^{+}_{1\overline{T}}(x) + h^{+}_{2\overline{T}}(x) - h^{+}_{1\overline{T}}(x) \cdot h^{+}_{2\overline{T}}(x), h^{+}_{1\overline{I}}(x) \cdot h^{+}_{2\overline{I}}(x), h^{+}_{1\overline{F}}(x) \cdot h^{+}_{2\overline{F}}(x), \\ -(h^{-}_{1\overline{T}}(x) \cdot h^{-}_{2\overline{T}}(x)), -(-h^{-}_{1\overline{I}}(x) - h^{-}_{2\overline{I}}(x) - h^{-}_{1\overline{I}}(x) \cdot h^{-}_{2\overline{I}}(x)), \\ -(-h^{-}_{1\overline{F}}(x) - h^{-}_{2\overline{F}}(x) - h^{-}_{1\overline{F}}(x) \cdot h^{-}_{2\overline{F}}(x)) \end{pmatrix} \right\rangle$$

(2) Multiplication

$$A \otimes B = \left\langle \begin{pmatrix} h^{+}_{1\underline{T}}(x) \cdot h^{+}_{2\underline{T}}(x), h^{+}_{1\underline{I}}(x) + h^{+}_{2\underline{I}}(x) - h^{+}_{1\underline{I}}(x) \cdot h^{+}_{2\underline{I}}(x), \\ h^{+}_{1\underline{F}}(x) + h^{+}_{2\underline{F}}(x) - h^{+}_{1\underline{F}}(x) \cdot h^{+}_{2\underline{F}}(x), \\ -(-h^{-}_{1\underline{T}}(x) - h^{-}_{2\underline{T}}(x) - h^{-}_{1\underline{T}}(x) \cdot h^{-}_{2\underline{T}}(x)), \\ -(h^{-}_{1\underline{I}}(x) \cdot h^{-}_{2\underline{I}}(x)), -h^{-}_{1\underline{F}}(x) \cdot h^{-}_{2\underline{F}}(x)) \end{pmatrix} \begin{pmatrix} h^{+}_{1\overline{T}}(x) \cdot h^{+}_{2\overline{T}}(x), h^{+}_{1\overline{I}}(x) + h^{+}_{2\overline{I}}(x) - h^{+}_{1\overline{I}}(x) \cdot h^{+}_{2\overline{I}}(x), \\ h^{+}_{1\overline{F}}(x) + h^{+}_{2\overline{F}}(x) - h^{+}_{1\overline{F}}(x) \cdot h^{+}_{2\overline{F}}(x), \\ -(-h^{-}_{1\overline{T}}(x) - h^{-}_{2\overline{T}}(x) - h^{-}_{1\overline{T}}(x) \cdot h^{-}_{2\overline{T}}(x)), \\ -(h^{-}_{1\overline{I}}(x) \cdot h^{-}_{2\overline{I}}(x)), -h^{-}_{1\overline{F}}(x) \cdot h^{-}_{2\overline{F}}(x)) \end{pmatrix} \right\rangle$$

(3) Power

$$A^{P} = \left\langle \begin{pmatrix} (h^{+}_{1\underline{T}})^{p}, (1-(1-h^{+}_{1\underline{I}})^{p}), (1-(1-h^{+}_{1\underline{F}})^{p}, \\ -(1-(1-(-h^{+}_{1\underline{T}})^{p})), -(-h^{+}_{1\underline{I}})^{p}, -(h^{+}_{1\underline{F}})^{p} \end{pmatrix} \begin{pmatrix} (h^{+}_{1\overline{T}})^{p}, (1-(1-h^{+}_{1\overline{I}})^{p}), (1-(1-h^{+}_{1\overline{F}})^{p}, \\ -(1-(1-(-h^{+}_{1\overline{T}})^{p})), -(-h^{+}_{1\overline{I}})^{p}, -(h^{+}_{1\overline{F}})^{p} \end{pmatrix} \right\rangle$$

(4) Scalar multiplication:

$$PA = \left\langle \begin{pmatrix} 1-(1-h^{+}_{1\underline{T}})^{p}, (h^{+}_{1\underline{I}})^{p}, (h^{+}_{1\underline{F}})^{p}, \\ -(-h^{-}_{1\underline{T}})^{p})), -(1-(1-(-h^{-}_{1\underline{I}})^{p}), -(1-(1-(-h^{-}_{1\underline{F}})^{p})) \end{pmatrix} \begin{pmatrix} 1-(1-h^{+}_{1\overline{T}})^{p}, (h^{+}_{1\overline{I}})^{p}, (h^{+}_{1\overline{F}})^{p}, \\ -(-h^{-}_{1\overline{T}})^{p})), -(1-(1-(-h^{-}_{1\overline{I}})^{p}), -(1-(1-(-h^{-}_{1\overline{F}})^{p})) \end{pmatrix} \right\rangle$$

**Definition 3.4** (Score function)**.** Let $A$ be a rough hesitant bipolar neutrosophic set, then the score function $S(A)$ is defined as follows.

$$S(A)=\frac{\left[\begin{array}{l}6+\left(\frac{1}{\ell_{h_{\underline{T}^+}}}\sum\limits_{\gamma_{\underline{T}^+}\in h_{\underline{T}^+}}\gamma_{\underline{T}^+}+\frac{1}{\ell_{h_{\overline{T}^+}}}\sum\limits_{\gamma_{\overline{T}^+}\in h_{\overline{T}^+}}\gamma_{\overline{T}^+}\right)\\ -\left(\frac{1}{\ell_{h_{\underline{I}^+}}}\sum\limits_{\gamma_{\underline{I}^+}\in h_{\underline{I}^+}}\gamma_{\underline{I}^+}+\frac{1}{\ell_{h_{\overline{I}^+}}}\sum\limits_{\gamma_{\overline{I}^+}\in h_{\overline{I}^+}}\gamma_{\overline{I}^+}\right)\\ -\left(\frac{1}{\ell_{h_{\underline{F}^+}}}\sum\limits_{\gamma_{\underline{F}^+}\in h_{\underline{F}^+}}\gamma_{\underline{F}^+}+\frac{1}{\ell_{h_{\overline{F}^+}}}\sum\limits_{\gamma_{\overline{F}^+}\in h_{\overline{F}^+}}\gamma_{\overline{F}^+}\right)\\ +\left(\frac{1}{\ell_{h_{\underline{T}^-}}}\sum\limits_{\gamma_{\underline{T}^-}\in h_{\underline{T}^-}}\gamma_{\underline{T}^-}+\frac{1}{\ell_{h_{\overline{T}^-}}}\sum\limits_{\gamma_{\overline{T}^-}\in h_{\overline{T}^-}}\gamma_{\overline{T}^-}\right)\\ -\left(\frac{1}{\ell_{h_{\underline{I}^-}}}\sum\limits_{\gamma_{\underline{I}^-}\in h_{\underline{I}^-}}\gamma_{\underline{I}^-}+\frac{1}{\ell_{h_{\overline{I}^-}}}\sum\limits_{\gamma_{\overline{I}^-}\in h_{\overline{I}^-}}\gamma_{\overline{I}^-}\right)\\ -\left(\frac{1}{\ell_{h_{\underline{F}^-}}}\sum\limits_{\gamma_{\underline{F}^-}\in h_{\underline{F}^-}}\gamma_{\underline{F}^-}+\frac{1}{\ell_{h_{\overline{F}^-}}}\sum\limits_{\gamma_{\overline{F}^-}\in h_{\overline{F}^-}}\gamma_{\overline{F}^-}\right)\end{array}\right]}{12}$$

For any two RHBNS $A_1$ and $A_2$, if $S(A_1) > S(A_2)$ then $A_1$ is bigger than $A_2$ and if $S(A_1) = S(A_2)$ then $A_1$ is equivalent to $A_2$.

**Definition 3.5.** Let $A_i = \langle\, x, (h^{+}_{i\underline{T}}(x), h^{+}_{i\underline{I}}(x), h^{+}_{i\underline{F}}(x), h^{-}_{i\underline{T}}(x), h^{-}_{i\underline{I}}(x), h^{-}_{i\underline{F}}(x)), (h^{+}_{i\overline{T}}(x), h^{+}_{i\overline{I}}(x), h^{+}_{i\overline{F}}(x), h^{-}_{i\overline{T}}(x), h^{-}_{i\overline{I}}(x), h^{-}_{i\overline{F}}(x))\,\rangle$ be a family of RHBN sets. A mapping RHBNW $A_w : \hat{H}_n \to \hat{H}$ is called rough hesitant bipolar neutrosophic weighted average operator if it satisfies.

RHBNW $A_w(A_1, A_2, \ldots, A_n)$

$$= \sum_{i=1}^{n} w_i A_i$$

$$= \left\langle \begin{pmatrix} 1-\prod_{i=1}^{n}(1-h_{i\underline{T}}^{+})^{w_i}, 1-\prod_{i=1}^{n}(h_{i\underline{I}}^{+})^{w_i}, 1-\prod_{i=1}^{n}(h_{i\underline{F}}^{+})^{w_i}, -\prod_{i=1}^{n}(-h_{i\underline{T}}^{-})^{w_i}, \\ -(1-\prod_{i=1}^{n}(1-(-h_{i\underline{I}}^{-})^{w_i})), -(1-\prod_{i=1}^{n}(1-(-h_{i\underline{F}}^{-})^{w_i})) \end{pmatrix} \atop \begin{pmatrix} 1-\prod_{i=1}^{n}(1-h_{i\overline{T}}^{+})^{w_i}, \prod_{i=1}^{n}(h_{i\overline{I}}^{+})^{w_i}, -\prod_{i=1}^{n}(-h_{i\overline{F}}^{+})^{w_i}, -\prod_{i=1}^{n}(-h_{i\underline{T}}^{-})^{w_i}, \\ -(1-\prod_{i=1}^{n}(1-(-h_{i\overline{I}}^{-})^{w_i})), -\prod_{i=1}^{n}(-h_{i\overline{I}}^{-})^{w_i}, -\prod_{i=1}^{n}(-h_{i\overline{F}}^{-})^{w_i} \end{pmatrix} \right\rangle$$

where $w_i$ is the weight of $A_i(1 = 1, 2, \ldots, n)$, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

**Definition 3.6** (Rough hesitant bipolar neutrosophic weighted geometric (RHBNWG) operator)**.**

Let $\{A_i\}$ be a family of RHBNs. A mapping RHBNW $G_w : \hat{H}_n \to \hat{H}$ is called rough hesitant bipolar neutrosophic weighted geometric operator if it satisfies

$$RHBNWG(A_1, A_2, \ldots, A_n)$$

$$= \sum_{i=1}^{n}(A_i)^{w_i}$$

$$= \left\langle \begin{pmatrix} \prod_{i=1}^{n}(h_{i\underline{T}}^{+})^{w_i}, 1-\prod_{i=1}^{n}(h_{i\underline{I}}^{+})^{w_i}, 1-\prod_{i=1}^{n}(h_{i\underline{F}}^{+})^{w_i}, \\ -(-1-\prod_{i=1}^{n}(1-(-h_{i\underline{T}}^{-})^{w_i}, -\prod_{i=1}^{n}(-h_{i\underline{I}}^{-})^{w_i}, -\prod_{i=1}^{n}(-h_{i\underline{F}}^{-})^{w_i} \end{pmatrix} \atop \begin{pmatrix} \prod_{i=1}^{n}(h_{i\overline{T}}^{+})^{w_i}, 1-\prod_{i=1}^{n}(h_{i\overline{I}}^{+})^{w_i}, 1-\prod_{i=1}^{n}(h_{i\overline{F}}^{+})^{w_i}, \\ -(-1-\prod_{i=1}^{n}(1-(-h_{i\overline{T}}^{-})^{w_i}, -\prod_{i=1}^{n}(-h_{i\overline{I}}^{-})^{w_i}, -\prod_{i=1}^{n}(-h_{i\overline{F}}^{-})^{w_i} \end{pmatrix} \right\rangle$$

where $w_i$ is the weight of $A_i(i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Then $RHBNWG_w(A_{i1}, A_{i2}, \ldots, A_{in})$ is called RHBNWG.

## 4. Multi-criteria Decision Making Problem

Let $B_i = \{B_1, B_2, \ldots, B_m\}$ and $C_j = \{C_1, C_2, \ldots, C_n\}$ be a set of alternatives and attributes respectively. An evaluation is collected with rough hesitant bipolar neutrosophic elements $A_{ij}$ which represents the decision makers judgement on $i^{\text{th}}$ alternative with respect to $j^{\text{th}}$ attribute.

The general decision making form is as below.

$$D = \langle A_{ij} \rangle_{m\times n} = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{m2} & \ldots & A_{mn} \end{bmatrix}$$

The evaluation can be analysed using the RHBVNWA operators. The algorithm of this method is given below.

**Step 1.** Compute the weighted average values RHBNW $G_w(A_{i1}, A_{i2}, \ldots, A_{in})$ by using the RHBNWA operator or the weighted geometric values RHBNW $G_w(A_{i1}, A_{i2}, \ldots, A_{in})$ by using the RHBNWG operator.

**Step 2.** Calculate the score function $S(A_i)\, i = 1, 2, 3, 4$ for each $A_i = (i = 1, 2, 3, 4)$.

**Step 3.** Rank the alternatives in descending order based on the obtained score function.

## 5. Numerical Example

Let us consider the following decision making problem.

A helping organization wants to give 4 set of books in a Government arts and science college by considering four major students (Four alternatives) which are (1) Mathematics (2) Computer science (3) Physics (4) Chemistry. There is a panel who is ready to select one student from each group to give that 4 set of books according to a set of attributes 1) Academic report $(C_1)$ 2)

Interest in book reading $(C_2)$ (3) Economical background of the student $(C_3)$ (4) Suggestion of teachers $(C_4)$.

The weights of attribute are $(0.4, 0.3, 0.2, 0.1)^T$. The decision making evaluation of the alternative based on the set of attributes $C_j (j = 1, 2, 3, 4)$ is obtained. The evaluation information is represented in the form of RHBN elements. Then according to the proposed MCDM approach, we have the decision matrix as shown in Table 1.

$$
\begin{array}{cc}
 & \begin{array}{ll} \qquad A_1 & \qquad\qquad A_2 \\ \qquad A_3 & \qquad\qquad A_4 \end{array} \\
\begin{array}{c} C_1 \\ \\ \\ \\ C_2 \\ \\ \\ \\ \\ C_3 \\ \\ \\ \\ \\ C_4 \end{array} &
\left(\begin{array}{ll}
\begin{array}{l} \langle(0.5, 0.6, \{0.2, 0.4\}, -0.5, -0.4, -0.7) \\ (0.7, \{0.3, 0.4\}, 0.6, -0.6, -0.7, -0.8)\rangle \end{array} & \begin{array}{l} \langle(0.8, 0.9, 0.6, \{-0.6, -0.7\}, -0.5, -0.4) \\ (0.6, \{0.7, 0.8\}, 0.9, -0.6, -0.7, -0.5)\rangle \end{array} \\
\begin{array}{l} \langle(0.7, 0.6, 0.5, \{-0.3, -0.2\}, -0.4, -0.5) \\ (0.5, 0.6, 0.7, -0.7, -0.5, -0.6)\rangle \end{array} & \begin{array}{l} \langle(0.7, 0.8, 0.9, -0.5, -0.6, -0.7) \\ (0.7, 0.4, 0.6, -0.4, -0.5, -0.6)\rangle \end{array} \\
\begin{array}{l} \langle(\{0.5, 0.4\}, \{0.4, 0.5\}, 0.7 \\ \{-0.3, -0.4\}, -0.5, -0.6) \\ (0.7, 0.8, \{0.5, 0.6\}, -0.2, -0.5, -0.7)\rangle \end{array} & \begin{array}{l} \langle(0.6, 0.5, 0.4, -0.6, -0.7, -0.8) \\ (0.4, 0.3, 0.5, -0.5, -0.6, -0.7)\rangle \end{array} \\
\begin{array}{l} \langle(0.6, 0.5, 0.4, -0.6, -0.7, -0.8) \\ (0.5, 0.6, 0.7, -0.3, -0.4, -0.5)\rangle \end{array} & \begin{array}{l} \langle(0.5, 0.6, 0.7, -0.6, -0.7, -0.8) \\ (0.6, 0.7, 0.8, -0.7, -0.8, -0.9)\rangle \end{array} \\
\begin{array}{l} \langle(, 0.7, 0.8, 0.6, \{-0.1, -0.2\}, \\ \{-0.5, -0.6\}, -0.7) \\ (0.6, 0.5, 0.4, -0.5, -0.4, -0.3)\rangle \end{array} & \begin{array}{l} \langle(0.5, 0.6, 0.7, -0.8, -0.9, -0.7) \\ (0.4, 0.3, 0.5, -0.7, -0.8, -0.6)\rangle \end{array} \\
\begin{array}{l} \langle(0.5, 0.7, 0.8, -0.6, -0.7, -0.8) \\ (0.3, 0.2, 0.1, -0.3, -0.2, -0.1)\rangle \end{array} & \begin{array}{l} \langle(0.4, 0.5, 0.6, -0.4, -0.5, -0.6) \\ (0.7, 0.8, 0.9, -0.5, -0.6, -0.7)\rangle \end{array} \\
\begin{array}{l} \langle(0.6, 0.5, 0.4, -0.4, -0.3, -0.5) \\ (0.7, 0.5, 0.9, -0.4, -0.5, -0.6)\rangle \end{array} & \begin{array}{l} \langle(0.8, 0.7, 0.6, -0.6, -0.7, -0.8) \\ (0.5, 0.6, 0.7, -0.4, -0.3, -0.2)\rangle \end{array} \\
\begin{array}{l} \langle(0.5, 0.4, 0.3, -0.6, -0.5, -0.4) \\ (0.8, 0.7, 0.6, -0.7, -0.5, -0.6)\rangle \end{array} & \begin{array}{l} \langle(0.5, 0.6, 0.7, -0.5, -0.6, -0.7) \\ (0.5, 0.6, 0.7, -0.4, -0.6, -0.7)\rangle \end{array}
\end{array}\right)
\end{array}
$$

Now

RHBNW $A_w(A_1, A_2, A_3, A_4)$

$$= \sum_{i=1}^{4} A_i w_i$$

$$= \left\langle \begin{pmatrix} 1-\prod_{i=1}^{4}(1-h_{i\underline{T}}^{+})^{w_i}, \prod_{i=1}^{4}(h_{i\underline{I}}^{+})^{w_i}, \prod_{i=1}^{4}(h_{i\underline{F}}^{+})^{w_i}, -\prod_{i=1}^{4}(-h_{i\underline{T}}^{-})^{w_i}, \\ -(1-\prod_{i=1}^{4}(-h_{i\underline{I}}^{-})^{w_i} -(1-\prod_{i=1}^{4}(1-(h_{i\underline{F}}^{-})^{w_i})) \end{pmatrix} \atop \begin{pmatrix} 1-\prod_{i=1}^{4}(1-h_{i\overline{T}}^{+})^{w_i}, \prod_{i=1}^{4}(h_{i\overline{I}}^{+})^{w_i}, \prod_{i=1}^{4}(h_{i\overline{F}}^{+})^{w_i} \\ -\prod_{i=1}^{4}(-h_{i\overline{T}}^{-})^{w_i}, -(1-\prod_{i=1}^{4}(-h_{i\overline{I}}^{-})^{w_i}, -(1-\prod_{i=1}^{4}(1-(h_{i\overline{F}}^{-})^{w_i})) \end{pmatrix} \right\rangle$$

Using this, we get

$h_{A_1} = \langle(0.6,\ 0.7,\ 0.4,\ -0.44,\ -0.46,\ -0.59)(0.6,\ 0.5,\ 0.7,\ -0.52,\ 0.65,$

$-0.68)\rangle$

$h_{A_2} = \langle(0.53,\ 0.5,\ 0.5,\ -0.6,\ -0.64,\ -0.74)(0.59,\ 0.54,\ 0.59,\ -0.32,$

$0.56,\ -0.72)\rangle$

$h_{A_3} = \langle(0.6,\ 0.68,\ 0.65,\ -0.4,\ -0.75,\ -0.725)(0.52,\ 0.42,\ 0.34,\ -0.48,\ 0.58,$

$-0.45)\rangle$

$h_{A_4} = \langle\ (0.66,\ 0.52,\ 0.44,\ -0.49,\ -0.54,\ -0.64)(0.67,\ 0.56,\ 0.73,\ -0.44,$

$0.48,\ -0.53)\ \rangle$

$RHBNWG_w(A_1,\ A_2,\ A_3,\ A_4)$

$$= \sum_{i=1}^{4}(A_i)^{w_i}$$

$$= \left\langle \begin{array}{l} \left( \begin{array}{l} \prod_{i=1}^{4} (h_{i\underline{T}}^{+})^{w_i}, 1 - \prod_{i=1}^{4} (1 - h_{i\underline{I}}^{+})^{w_i}, 1 - \prod_{i=1}^{4} (h_{i\underline{F}}^{+})^{w_i}, \\ -(1 - \prod_{i=1}^{4} (1 - (h_{i\underline{T}}^{-})^{w_i}, - \prod_{i=1}^{4} (-h_{i\underline{I}}^{-})^{w_i}, - \prod_{i=1}^{4} (-(h_{i\underline{F}}^{-})^{w_i})) \end{array} \right) \\ \left( \begin{array}{l} \prod_{i=1}^{4} (h_{i\overline{T}}^{+})^{w_i}, 1 - \prod_{i=1}^{4} (1 - h_{i\overline{I}}^{+})^{w_i}, 1 - \prod_{i=1}^{4} (h_{i\overline{F}}^{+})^{w_i}, \\ -(1 - \prod_{i=1}^{4} (1 - (h_{i\overline{T}}^{-})^{w_i}), - \prod_{i=1}^{4} (-h_{i\overline{I}}^{-})^{w_i}, - \prod_{i=1}^{4} (-(h_{i\overline{F}}^{-})^{w_i})) \end{array} \right) \end{array} \right\rangle$$

Using this, we get

$h_{G_1} = \langle (0.62, 0.24, 0.47, -0.56, -0.47, -0.54)(0.47, 0.62, 0.76, -0.6,$

$0.66, -0.63) \rangle$

$h_{G_2} = \langle (0.49, 0.49, 0.59, -0.54, -0.65, -0.69)(0.54, 0.66, 0.59, -0.40,$

$0.57, -0.65) \rangle$

$h_{G_3} = \langle (0.55, 0.72, 0.69, -0.56, -0.75, -0.69)(0.47, 0.45, 0.50, -0.54,$

$0.56, -0.32) \rangle$

$h_{G_4} = \langle (0.61, 0.57, 0.497, -0.52, -0.53, -0.57)(0.63, 0.59, 0.79, -0.49,$

$0.46, -0.43) \rangle$

Using the score function formula for RHBNS, $S(h_i)$, $i = 1, 2, 3, 4$ for $h_i$'s which are calculated using RHBNWA operator, we get

$S(h_{A_1}) = 0.5266, S(h_{A_2}) = 0.5608, S(h_{A_3}) = 0.5541, S(h_{A_4}) = 0.4783$

That is

$$S(h_{A_2}) > S(h_{A_3}) > S(h_{A_1}) > S(h_{A_4})$$

$$h_{A_2} > h_{A_3} > h_{A_1} > h_{A_4}.$$

score function $S(h_i)$ $i = 1, 2, 3, 4$ for $h_i$'s which are calculated using RHBNWG operator.

$$S(h_{G_1}) = 0.5133, S(h_{G_2}) = 0.5266, S(h_{G_3}) = 0.49, S(h_{G_4}) = 0.4808$$

$$S(h_{G_2}) > S(h_{G_3}) > S(h_{G_1}) > S(h_{G_4})$$

$$h_{G_2} > h_{G_3} > h_{G_1} > h_{G_4}.$$

In both cases $h_2$ is best, $h_4$ is worst.

## 6. Comparative Analysis

To verify the feasibility of the proposed decision-making method, a comparative analysis is presented.

According to Ye [8]'s numerical example, the proposed rough hesitant bipolar valued neutrosophic set decision making method is compared to the existing methods under different information. Interval valued intuitionistic fuzzy set (IVIFS), interval valued Pythagorean fuzzy set (IVPFS), simplified neutrosophic set (SNS), interval valued neutrosophic set (INS) and single valued neutrosophic hesitant fuzzy set (SVNHFS). The following table shows the ranking order of the existing group decision making methods compared to the introduced method.

| Method | Set | Ranking order |
|---|---|---|
| IVIFWA [9] | IVIFS | $A_2$ is best alternative |
| IVPFWA [10] | IVPFS | $A_2$ is best alternative |
| GSNNWA [11] | SNS | $A_4$ is best alternative |
| GSNNWG [11] | SNS | $A_4$ is best alternative |
| similarity measures [12] | INS | $A_4$ is best alternative |
| SVNHFWA [8] | SVNHFS | $A_4$ is best alternative |
| SVNHFWA [8] | SVNHFS | $A_2$ is best alternative |
| HBVNWA [13] | HBVNS | $A_4$ is best alternative |
| HBVNWG [13] | HBVNS | $A_2$ is best alternative |
| RHBVNWA | RHBVNS | $A_2$ is best alternative |

| | | |
|---|---|---|
| RHBVNWG | RHBVNS | $A_2$ is best alternative |

The optimal solution of the introduced method are quite close to the optimal solution of the existing methods.

## 7. Conclusion

This paper has successfully established a concept of rough hesitant bipolar valued neutrosophic set. There are several significant features of the introduced set. By adopting the characteristics of the rough sets, bipolar valued neutrosophic sets and hesitant fuzzy sets, the proposed set gives a lenient way of judgemental process instead of agitative concrete judgements. Some basic properties of RHBVNS such as union, intersection, complement and power are studied. The preposition related to the properties of proposed set is investigated and proved. This paper proposed two aggregation operators. RHBVNWA and RHBVNWG operators to aggregate the rough hesitant bipolar valued neutrosophic elements. A MADM method is developed under the RHBVN environment in which the evaluation values are in the form of RHBVN elements. The proposed RHBVNWA and RHBVNWG operators can be applied to obtain the best alternative.

An illustrative numerical example is given and a comparative analysis is presented to show the feasibility of the proposed approach. In future more MADM methods can be developed based on RHBVNS to solve real application in the areas such as information fusion system, group decision making, support system, expert system.

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