**Advances and Applications in Mathematical Sciences**
Volume 21, Issue 11, September 2022, Pages 6469-6478


# CERTAIN DEFINITE INTEGRAL INVOLVING ERROR FUNCTIONS

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## Abstract

In this paper we aim to evaluate some definite integrals in terms of error function. We have also derived some special cases of the formulae.

## 1. Introduction

Error function is defined as

$$er\,f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{1.1}$$

and has properties

$$er\,f(-\infty) = -1,\; er\,f(+\infty) = 1$$

$$er\,f(-\,x) = -er\,f(x),\; er(x^*) = [er\,f(x)]^*$$

where the asterisk denotes complex conjugation.

The basic operations of Boolean algebra are as follows:

2020 Mathematics Subject Classification: 33B10, 33B20, 03C80.

Keywords: Error Function, Boolean Algebra.

Received December 12, 2021; Accepted March 22, 2022

AND (conjunction), denoted $p \wedge q$, satisfies $p \wedge q = 1$ if $p = q = 1$, and $p \wedge q = 0$ otherwise.

OR (disjunction), denoted $p \vee q$, satisfies $p \vee q = 0$ if $p = q = 0$, and $p \vee q = 1$ otherwise.

NOT (negation), denoted $\neg p$, satisfies $\neg p = 0$ if $p = 1$ and $\neg p = 1$ if $p = 0$.

Yurry A. Brychkov [Brychkov p. 116(4.1.2.3 - 4.1.2.5)] has developed the following formulae

$$\int_0^a e^{bx(a-x)} dx = \sqrt{\frac{\pi}{b}} e^{\frac{a^2 b}{4}} erf\left(\frac{a\sqrt{b}}{2}\right). \tag{1.2}$$

$$\int_0^a x\ e^{bx(a-x)} dx = \frac{a}{2} \sqrt{\frac{\pi}{b}} e^{\frac{a^2 b}{4}} erf\left(\frac{a\sqrt{b}}{2}\right). \tag{1.3}$$

$$\int_0^a x^2 e^{bx(a-x)} dx = \frac{a}{2b}\left[\frac{a^2 b + 2}{2a} \sqrt{\frac{\pi}{b}} e^{\frac{a^2 b}{4}} erf\left(\frac{a\sqrt{b}}{2}\right) - 1\right]. \tag{1.4}$$

## 2. Main Formulae of the Integration

$$\int_0^a x^3\ e^{bx(a-x)} dx = \frac{a[\sqrt{\pi}(a^2 b + 2) e^{\frac{a^2 b}{4}} erf\left(\frac{a\sqrt{b}}{2}\right) - 6a\sqrt{b}]}{8b^{\frac{3}{2}}}. \tag{2.1}$$

$$\int_0^a x^4\ e^{bx(a-x)} dx = \frac{\sqrt{\pi}(a^4 b^2 + 12a^2 b + 12) e^{\frac{a^2 b}{4}} erf\left(\frac{a\sqrt{b}}{2}\right) - 2a\sqrt{b}(7a^2 b + 6)}{16b^{\frac{5}{2}}}. \tag{2.2}$$

$$\int_0^a x^5\, e^{bx(a-x)}dx = \frac{a[\sqrt{\pi}(a^4b^2 + 20\,a^2b + 60)\,e^{\frac{a^2b}{4}}\,er\,f\left(\frac{a\sqrt{b}}{2}\right) - 30\,a\sqrt{b}(a^2b + 2)]}{32b^{\frac{5}{2}}}. \tag{2.3}$$

$$\int_0^a x^6\, e^{bx(a-x)}dx = \frac{1}{64b^{\frac{7}{2}}}[\sqrt{\pi}\{a^2b(a^2b + 30) + 180) + 120\}\,e^{\frac{a^2b}{4}}\,er\,f\left(\frac{a\sqrt{b}}{2}\right)$$

$$- 2a\sqrt{b}\{a^2b(31a^2b + 100) + 60\}]. \tag{2.4}$$

$$\int_0^a x^7\, e^{bx(a-x)}dx = \frac{1}{128b^{\frac{7}{2}}}\,a[\sqrt{\pi}\{a^2b(a^2b + 42) + 420) + 840\}\,e^{\frac{a^2b}{4}}\,er\,f\left(\frac{a\sqrt{b}}{2}\right)$$

$$- 14a\sqrt{b}\{a^2b(9a^2b + 40) + 60\}]. \tag{2.5}$$

$$\int_0^a x^8\, e^{bx(a-x)}dx =$$

$$\frac{1}{256b^{\frac{9}{2}}}\,a[\sqrt{\pi}(a^2b(a^2b(a^2b(a^2b + 56) + 840) + 3360) + 1680)\,e^{\frac{a^2b}{4}}\,er\,f\left(\frac{a\sqrt{b}}{2}\right)$$

$$- 2a\sqrt{b}(a^2b(a^2b(127a^2b + 714) + 1820) + 840)]. \tag{2.6}$$

$$\int_0^a x^9\, e^{bx(a-x)}dx =$$

$$\frac{1}{512b^{\frac{9}{2}}}\,a[\sqrt{\pi}\{a^2b(a^2b(a^2b(a^2b + 72) + 1512) + 10080) + 15120\}\,e^{\frac{a^2b}{4}}\,er\,f\left(\frac{a\sqrt{b}}{2}\right)$$

$$- 6a\sqrt{b}\{a^2b(a^2b(85a^2b + 574) + 2100) + 2520\}]. \tag{2.7}$$

$$\int_0^a x^{10}\, e^{bx(a-x)}dx =$$

$$\frac{1}{1024b^{\frac{11}{2}}}[\sqrt{\pi}(a^2b(a^2b(a^2b(a^2b(a^2b+90)+2520)+25200)+75600)+30240)$$

$$e^{\frac{a^2b}{4}}erf\left(\frac{a\sqrt{b}}{2}\right)-2a\sqrt{b}(a^2b(a^2b(a^2b(511a^2b+4008)+19152)+40320)+15120].$$

(2.8)

$$\int_0^a x^{11}e^{bx(a-x)}dx=$$

$$\frac{1}{2048b^{\frac{11}{2}}}a[\sqrt{\pi}\{a^2b(a^2b(a^2b(a^2b(a^2b+110)+3960)+55440)+277200)+332640\}$$

$$e^{\frac{a^2b}{4}}erf\left(\frac{a\sqrt{b}}{2}\right)-66a\sqrt{b}\{a^2b(a^2b(a^2b(31a^2b+276)+1624)+5040)+5040\}].$$

(2.9)

$$\int_0^a x^{12}e^{bx(a-x)}dx=\frac{1}{4096b^{\frac{13}{2}}}[\sqrt{\pi}(a^2b(a^2b(a^2b(a^2b(a^2b(a^2b+132)+5940)$$

$$+110880)+831600)+1995840)+665280)e^{\frac{a^2b}{4}}erf\left(\frac{a\sqrt{b}}{2}\right)$$

$$-2a\sqrt{b}(a^2b(a^2b(a^2b(a^2b(2047a^2b+20350)+141768)+587664)$$

$$+1053360)+332640]. \quad (2.10)$$

$$\int_0^a x^{13}e^{bx(a-x)}dx=\frac{1}{8192b^{\frac{13}{2}}}a[\sqrt{\pi}\{a^2b(a^2b(a^2b(a^2b(a^2b(a^2b+156)+8580)$$

$$+205920)+2162160)+8648640)+8648640\}e^{\frac{a^2b}{4}}erf\left(\frac{a\sqrt{b}}{2}\right)$$

$$-26a\sqrt{b}\{a^2b(a^2b(a^2b(a^2b(315a^2b+3454)+27720)+144144)+388080)+332640\}].$$

(2.11)

$$\int_0^a x^{14}\, e^{bx(a-x)}dx = \frac{1}{16384\, b^{\frac{13}{2}}}\,[\sqrt{\pi}\{a^2b(a^2b(a^2b(a^2b(a^2b(a^2b(a^2b + 182) + 12012)$$

$$+ 360360) + 5045040) + 30270240) + 60540480) + 17297280\}\, e^{\frac{a^2b}{4}}\, erf\left(\frac{a\sqrt{b}}{2}\right)$$

$$- 2a\sqrt{b}\{a^2b(a^2b(a^2b(a^2b(a^2b(8191a^2b + 98124) + 889460) + 5559840)$$

$$+ 20324304) + 31711680) + 8648640\}]. \quad (2.12)$$

$$\int_0^a x^{15}\, e^{bx(a-x)}dx = \frac{1}{32768\, b^{\frac{15}{2}}}\,[\sqrt{\pi}\{a^2b(a^2b(a^2b(a^2b(a^2b(a^2b(a^2b + 210) + 16380)$$

$$+ 600600) + 10810800) + 90810720) + 302702400) + 259459200\}\, e^{\frac{a^2b}{4}}\, erf\left(\frac{a\sqrt{b}}{2}\right)$$

$$- 6a\sqrt{b}\{a^2b(a^2b(a^2b(a^2b(a^2b(5461a^2b + 70928) + 715572) + 5216640)$$

$$+ 24264240) + 57657600) + 43243200\}]. \quad (2.13)$$

## 3. Special Cases of the Formulae

$$\int_0^1 e^{bx(a-x)}dx = \frac{\sqrt{\pi}\, e^{\frac{a^2b}{4}}\left[erf\left(\frac{1}{2}(2-a)\sqrt{b}\right) + erf\left(\frac{a\sqrt{b}}{2}\right)\right]}{2\sqrt{b}}. \quad (3.1)$$

$$\int_0^1 x\, e^{bx(a-x)}dx = \frac{\sqrt{\pi}a\sqrt{b}\, e^{\frac{a^2b}{4}}\left[erf\left(\frac{1}{2}(2-a)\sqrt{b}\right) + erf\left(\frac{a\sqrt{b}}{2}\right)\right] - 2\, e^{(a-1)b} + 2}{4b}.$$

(3.2)

$$\int_0^1 x^2\, e^{bx(a-x)}dx =$$

$$\frac{\sqrt{\pi}(a^2b+2)e^{\frac{a^2b}{4}}\left[erf\left(\frac{1}{2}(2-a)\sqrt{b}\right)+erf\left(\frac{a\sqrt{b}}{2}\right)\right]+2\sqrt{b}(a-(a+2)e^{(a-1)b})}{8b^{3/2}}$$

(3.3)

$$\int_0^1 x^3\, e^{bx(a-x)}dx = \frac{1}{16b^2}(\sqrt{\pi}a\sqrt{b}(a^2b+6)e^{\frac{a^2b}{4}}(erf(\frac{1}{2}(2-a)\sqrt{b})+erf(\frac{a\sqrt{b}}{2}))$$

$$+2a^2b-2((a(a+2)+4)b+4)e^{(a-1)b}+8). \tag{3.4}$$

$$\int_0^1 x^4\, e^{bx(a-x)}dx = \frac{1}{32b^{5/2}}(\sqrt{\pi}(a^2b(a^2b+12)+12)e^{\frac{a^2b}{4}}(erf(\frac{1}{2}(2-a)\sqrt{b})$$

$$+\sqrt{\pi}(a^2b(a^2b+12)+12)e^{\frac{a^2b}{4}}erf(\frac{a\sqrt{b}}{2})+2a\sqrt{b}(a^2b+10)$$

$$-2\sqrt{b}(a^3b+2a^2b+2a(2b+5)+8b+12)e^{(a-1)b}). \tag{3.5}$$

$$\int_0^1 x^5\, e^{bx(a-x)}dx =$$

$$\frac{1}{64b^3}(2a^4b^2+36a^2b+\sqrt{\pi}a\sqrt{b}(a^4b^2+20a^2b+60)e^{\frac{a^2b}{4}}(erf(\frac{1}{2}(2-a)\sqrt{b})$$

$$+\sqrt{\pi}a\sqrt{b}(a^4b^2+20a^2b+60)e^{\frac{a^2b}{4}}erf(\frac{a\sqrt{b}}{2})-2(2(9a^2+14a+16)b$$

$$+(a^4+2a^3+4a^2+8a+16)b^2+32)e^{(a-1)b}+64). \tag{3.6}$$

$$\int_0^\infty e^{bx(a-x)}dx = \frac{1}{2\sqrt{b}}[\pi e^{\frac{a^2b}{4}}\left(erf\left(\frac{a\sqrt{b}}{2}\right)+1\right)], \text{ for } (\mathrm{Re}(b)\geq 0 \wedge \mathrm{Re}(ab)<0)$$

$$\vee \mathrm{Re}(b) > 0. \tag{3.7}$$

$$\int_0^\infty x\, e^{bx(a-x)}dx = \frac{1}{4b}[\pi a\sqrt{b}\, e^{\frac{a^2b}{4}}\left(erf\left(\frac{a\sqrt{b}}{2}\right)+1\right)+2], \text{ for}$$

$$(\mathrm{Re}(b) \geq 0 \wedge \mathrm{Re}(ab) < 0) \vee \mathrm{Re}(b) > 0. \tag{3.8}$$

$$\int_0^\infty x^2 e^{bx(a-x)} dx = \frac{1}{8b^{3/2}} [\sqrt{\pi}(a^2b+2)e^{\frac{a^2b}{4}}\left(erf\left(\frac{a\sqrt{b}}{2}\right)+1\right)+2a\sqrt{b}], \text{ for}$$

$$(\mathrm{Re}(b) \geq 0 \wedge \mathrm{Re}(ab) < 0) \vee \mathrm{Re}(b) > 0. \tag{3.9}$$

$$\int_0^\infty x^3 e^{bx(a-x)} dx = \frac{1}{16b^2} [\sqrt{\pi}a\sqrt{b}(a^2b+2)e^{\frac{a^2b}{4}}\left(erf\left(\frac{a\sqrt{b}}{2}\right)+1\right)+2a^2b+8],$$

$$(\mathrm{Re}(b) \geq 0 \wedge \mathrm{Re}(ab) < 0) \vee \mathrm{Re}(b) > 0. \tag{3.10}$$

$$\int_0^\infty x^4 e^{bx(a-x)} dx = \frac{1}{32b^{5/2}} [\sqrt{\pi}(a^2b(a^2b+12)+12)e^{\frac{a^2b}{4}}\left(erf\left(\frac{a\sqrt{b}}{2}\right)+1\right)$$

$$+ 2a\sqrt{b}(a^2b+10)], \text{ for } (\mathrm{Re}(b) \geq 0 \wedge \mathrm{Re}(ab) < 0) \vee \mathrm{Re}(b) > 0. \tag{3.11}$$

$$\int_0^\pi e^{bx(a-x)} dx = \frac{1}{2\sqrt{b}} [\sqrt{\pi}\, e^{\frac{a^2b}{4}} (erf\left(\frac{a\sqrt{b}}{2}\right) + erf\,(\frac{1}{2}(2\pi - a)\sqrt{b}))]. \tag{3.12}$$

$$\int_0^\pi x\, e^{bx(a-x)} dx = \frac{1}{4b} [\sqrt{\pi}a\sqrt{b}e^{\frac{a^2b}{4}} (erf\left(\frac{a\sqrt{b}}{2}\right) + erf\,(\frac{1}{2}(2\pi - a)\sqrt{b})) - 2\, e^{\pi(a-\pi)b} + 2]. \tag{3.13}$$

$$\int_0^\pi x^2\, e^{bx(a-x)} dx = \frac{1}{8b^{3/2}} [\sqrt{\pi}(a^2b+2)e^{\frac{a^2b}{4}} (erf\left(\frac{a\sqrt{b}}{2}\right) + erf\,(\frac{1}{2}(2\pi - a)\sqrt{b}))$$

$$+ 2\sqrt{b}(a - (a-2\pi)\, e^{\pi(a-\pi)b})]. \tag{3.14}$$

$$\int_0^\pi x^3\, e^{bx(a-x)} dx = \frac{1}{16b^2} [\sqrt{\pi}a\sqrt{b}(a^2b+6)e^{\frac{a^2b}{4}} (erf\left(\frac{a\sqrt{b}}{2}\right) + erf\,(\frac{1}{2}(2\pi - a)\sqrt{b}))$$

$$+ 2a^2b - 2\, e^{\pi(a-\pi)b}((a^2 + 2\pi a + 4\pi^2)b + 4) + 8]. \tag{3.15}$$

$$\int_0^\pi x^4\, e^{bx(a-x)} dx = \frac{1}{32b^{5/2}} [\sqrt{\pi}(a^2b(a^2b+12)+12)e^{\frac{a^2b}{4}} (erf\left(\frac{a\sqrt{b}}{2}\right)$$

$$+ erf\left(\frac{a\sqrt{b}}{2}\right) + \sqrt{\pi}(a^2b(a^2+12)+12)e^{\frac{a^2b}{4}} erf\left(\frac{1}{2}(2\pi - a)\sqrt{b}\right)$$

$$+ 2a\sqrt{b}(a^2b+10) - 2\sqrt{b}e^{\pi(a-\pi)b}(a^3b + 2\pi a^2b + 4\pi^2 ab + 10a + 8\pi^3 b + 12\pi)]. \quad (3.16)$$

$$\int_0^{\pi} x^5 e^{bx(a-x)}dx = \frac{1}{64b^3}[2a^2b^2 + \sqrt{\pi}a\sqrt{b}(a^2b(a^2b+20)+60)e^{\frac{a^2b}{4}}\left(erf\left(\frac{a\sqrt{b}}{2}\right)\right.$$

$$+ \sqrt{\pi}a\sqrt{b}(a^2b(a^2+20)+60)e^{\frac{a^2b}{4}} erf\left(\frac{1}{2}(2\pi - a)\sqrt{b}\right) + 36a^2b$$

$$- 2e^{\pi(a-\pi)b}(b(a^4b + 2\pi a^3b + 2a^2(2\pi^2b+9) + 4\pi a(2\pi^2b+7)$$

$$+ 16\pi^2(\pi^2b+2)) + 32) + 64]. \quad (3.17)$$

$$\int_0^{\frac{\pi}{2}} e^{bx(a-x)}dx = \frac{1}{2\sqrt{b}}\left[\sqrt{\pi}e^{\frac{a^2b}{4}}\left(erf\left(\frac{a\sqrt{b}}{2}\right) + erf\left(\frac{1}{2}(\pi - a)\sqrt{b}\right)\right)\right]. \quad (3.18)$$

$$\int_0^{\frac{\pi}{2}} x e^{bx(a-x)}dx = \frac{1}{4b}e^{-1/4\pi(\pi-2a)b}[\sqrt{\pi}a\sqrt{b}e^{1/4(a-\pi)^2b}\left(erf\left(\frac{a\sqrt{b}}{2}\right)\right.$$

$$+ erf\left(\frac{1}{2}(\pi - a)\sqrt{b}\right)) + 2(e^{1/4\pi(\pi-2a)b} - 1]. \quad (3.19)$$

$$\int_0^{\frac{\pi}{2}} x^2 e^{bx(a-x)}dx = \frac{1}{8b^{3/2}}[\sqrt{\pi}(a^2b+2)e^{\frac{a^2b}{4}}\left(erf\left(\frac{a\sqrt{b}}{2}\right) + erf\left(\frac{1}{2}(\pi - a)\sqrt{b}\right)\right)$$

$$+ 2\sqrt{b}(a - (a+\pi)e^{1/4\pi(\pi-2a)b})]. \quad (3.20)$$

$$\int_0^{\frac{\pi}{2}} x^3 e^{bx(a-x)}dx = \frac{1}{16b^2}[\sqrt{\pi}a\sqrt{b}(a^2b+6)e^{\frac{a^2b}{4}}\left(erf\left(\frac{a\sqrt{b}}{2}\right) + \sqrt{\pi}a\sqrt{b}(a^2b+6)\right.$$

$$e^{\frac{a^2b}{4}} erf\left(\frac{1}{2}(\pi - a)\sqrt{b}\right) + 2a^2b - 2e^{-1/4\pi(\pi-2a)b}((a^2 + \pi a + \pi^2)b + 4) + 8]. \quad (3.21)$$

$$\int_0^{\frac{\pi}{2}} x^4 e^{bx(a-x)}dx = \frac{1}{32b^{5/2}}[\sqrt{\pi}(a^2b(a^2b+12)+12)e^{\frac{a^2b}{4}} erf\left(\frac{a\sqrt{b}}{2}\right)$$

$$+\sqrt{\pi}\,(a^2b(a^2b+12)+12)\,e^{\frac{a^2b}{4}}\,erf\left(\frac{1}{2}(\pi-a)\sqrt{b}\right)+2a\sqrt{b}(a^2b+10)$$

$$-2\sqrt{b}e^{-1/4\pi(\pi-2a)b}(a^3b+\pi a^2b+\pi^2ab+10a+\pi^3b+6\pi)]. \quad (3.22)$$

$$\int_0^{\frac{\pi}{2}} x^5\, e^{bx(a-x)}dx = \frac{1}{64b^{5/2}}[2a^4b^2+\sqrt{\pi}a\sqrt{b}(a^2b(a^2b+20)+60)e^{\frac{a^2b}{4}}erf\left(\frac{a\sqrt{b}}{2}\right)$$

$$+\sqrt{\pi}a\sqrt{b}\,(a^2b(a^2b+20)+60)\,e^{\frac{a^2b}{4}}\,erf\left(\frac{1}{2}(\pi-a)\sqrt{b}\right)+36a^2b-2e^{-1/4\pi(\pi-2a)b}$$

$$(b(a^4b+\pi a^3b+a^2(\pi^2b+18)+\pi a(\pi^2b+14)+\pi^2(\pi^2b+8))+32)+64]. \quad (3.23)$$

## 4. Derivations of the Formulae

Derivation of (2.1)

$$\int_0^a x^3\, e^{bx(a-x)}dx = [\frac{1}{16b^2}(\sqrt{\pi}a\sqrt{b}(a^2b+6)e^{\frac{a^2b}{4}}\,erf(\frac{1}{2}\sqrt{b}(2x-a))-2\,e^{bx(a-x)}$$

$$(a^2b+2abx+4bx^2+4))]_0^a = \frac{a[\sqrt{\pi}(a^2b+6)\,e^{\frac{a^2b}{4}}\,erf(\frac{a\sqrt{b}}{2})-6a\sqrt{b}]}{8b^{\frac{3}{2}}}$$

Derivation of (2.2)

$$\int_0^a x^4\, e^{bx(a-x)}dx = [\frac{1}{32b^{5/2}}(\sqrt{\pi}(a^4b^2+12a^2b+12)e^{\frac{a^2b}{4}}\,erf(\frac{1}{2}\sqrt{b}(2x-a))$$

$$-2\sqrt{b}\,e^{bx(a-x)}(a^3b+2a^2bx+4abx^2+10a+8bx^3+12x))]_0^a$$

$$=\frac{\sqrt{\pi}(a^4b^2+12a^2b+12)\,e^{\frac{a^2b}{4}}\,erf(\frac{a\sqrt{b}}{2})-2a\sqrt{b}(7a^2b+6)}{16b^{\frac{5}{2}}}$$

On this same way other formulae can be derived.

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