Advances and Applications in Mathematical Sciences
Volume 21, Issue 1, November 2021, Pages 15-20

STUDY OF PARAMETRIC CONVERGENCE AND PROJECTIVE CONVERGENCE IN A FUNCTION SPACE

KARUNA KUMARI SHARMA

Department of Mathematics
Jai Prakash University
Chapra-841302 (Bihar), India
E-mail: karunaksharma24@gmail.com

Abstract

In this article we introduce the notions of parametric convergence, projective convergence, coordinate convergence and projective continuity in function spaces. We have introduced the notion of parametric limit. A necessary and sufficient condition has been given for $f_\lambda(x)$ a family of functions of x defined for all λ in $[0, \infty)$ (where λ is parameter) to be $\alpha(f)\,\beta(f)$-convergence as well as for $\alpha(f)\,\beta(f)$-continuous.

1. Definition of Co-ordinate Convergence

Let $f_\lambda(x)$ be a family of functions of x defined for all λ in $[0, \infty)$, where λ is a parameter.

If for almost all $x \geq 0$, $f_\lambda(x)$ tends to finite limit as $\lambda \to \infty$, i.e. if to every $\varepsilon > 0$ and to almost all $x \geq 0$, there corresponds a positive number $N(\varepsilon, x)$ such that $| f_\lambda(x) - f_{\lambda'}(x) | \leq \varepsilon$ for all $\lambda, \lambda' \geq N(\varepsilon, x)$ and for almost all $x \geq 0$, then the family $f_\lambda(x)$ is said to be **co-ordinate convergent** and we denote it in short as $c - cgt$. [cf. Infinite matrices and sequence spaces, p-283].

2. Definition of Parametric Convergence

If for almost all $x \geq 0$, $f_\lambda(x)$ converges uniformly to a finite limit as $\lambda \to \infty$ i.e., if to every $\varepsilon > 0$, there corresponds a positive number $N(\varepsilon)$,

---

2020 Mathematics Subject Classification: 54C35.

Keywords: Co-ordinate convergence, Parametric convergence, Parametric limit, Projective continuity, Projective convergence.

Received September 21, 2021; Accepted October 22, 2021

independent of x, such that, for almost all $x \geq 0$,

$$| f_\lambda(x) - f_{\lambda'}(x) | \leq \varepsilon \tag{2.1}$$

for all $\lambda, \lambda' \geq N(\varepsilon)$, then the family $\psi(x)$ is said to be **parametric convergent**. And in short, we denote it by $(\lambda - cgt)$. If for almost all $x \geq 0$, $f_\lambda(x)$ converges uniformly to $\psi(x)$ as $\lambda \to \infty$ i.e., if given any $\varepsilon > 0$, there exists a positive number $N(\varepsilon)$, independent of x, such that for almost all $x \geq 0$

$$| f_\lambda(x) - \psi(x) | \leq \varepsilon \tag{2.2}$$

for all $\lambda \geq N(\varepsilon)$, then $\psi(x)$ is called the **parametric limit** (λ-limit) of $f_\lambda(x)$, and we write $\lambda - \lim f_\lambda(x) = \psi(x)$.

Let $\psi_1(x) = \psi(x)$ for almost all $x \geq 0$

Then

$$| f_\lambda(x) - \psi_1(x) | = | f_\lambda(x) - \psi(x) + \psi(x) - \psi_1(x) |$$

$$\leq | f_\lambda(x) - \psi(x) | + | \psi(x) - \psi_1(x) | \tag{2.3}$$

Thus, to every $\varepsilon > 0$ there corresponds a positive number $N(\varepsilon)$, independent of x, such that by (2.2) and (2.3), for almost all $x \geq 0$

$$| f_\lambda(x) - \psi_1(x) | < \varepsilon$$

for all $\lambda \geq N(\varepsilon)$. Hence $\psi_1(x)$ is also a λ-limit of $f_\lambda(x)$. Thus, we observe that any function equal to $\psi(x)$ for almost all $x \geq 0$ is also a λ-limit of $f_\lambda(x)$.

Hence, we say that $\psi(x)$ is the parametric limit (λ-limit) of $f_\lambda(x)$, we mean that $\psi(x)$ is a λ-limit of $f_\lambda(x)$ and all functions equivalent to $\psi(x)$ in $[0, \infty)$ are λ-limits of $f_\lambda(x)$.

[A function $\varphi(x)$ is said to be equivalent to $\psi(x)$ in $[0, \infty)$ when $\varphi(x) = \psi(x)$ almost everywhere in $[0, \infty)$].

3. Definition of Projective Convergence

In defining projective convergence in the case of sequence spaces, the condition $\beta \geq \varphi$ has been taken to ensure that projective convergence implies co-ordinate convergence; [see Allen [1], p.312]. But it will presently be seen that in the case of function spaces, even if we take $\beta(f) \geq \varphi(f)$, projective convergence does not necessarily imply co-ordinate convergence. Hence, when defining projective convergence in the case of function spaces, we do not prescribe the condition $\beta(f) \geq \varphi(f)$.

$$\text{Let } \alpha^*(f) \geq \beta(f) \text{ and } F_g(\lambda) = \int_0^\infty f_\lambda(x)\, g(x)\, dx \tag{3.1}$$

where $f_\lambda(x) \in \alpha(f)$ and $g(x) \in \beta(f)$,

then (i) if $F_g(\lambda)$ tends to a finite limit as $\lambda \to \infty$ for every $g(x)$ in $\beta(f)$, we say that $f_\lambda(x)$ is **Projective convergent**. In short, we denote it by (p-cgt) relative to $\beta(f)$ or $\alpha(f)\,\beta(f)$-cgt, and we say that $f_\lambda(x)$ is p-cgt in $\alpha(f)$, or $\alpha(f)$-cgt when $\beta(f) = a^*(f)$, [cf. Infinite matrices and sequence spaces, p-283].

(ii) If $F_g(\lambda)$ is uniformly continuous in λ in $[0, \infty)$ for every $g(x)$ in $\beta(f)$, then $f_\lambda(x)$ is said to be **projective continuous** (p-continuous) relative to $\beta(f)$, or $\alpha(f)\,\beta(f)$-continuous, and it is said to be p-continuous in $\alpha(f)$, or $\alpha(f)$-continuous when $\beta(f) = a^*(f)$.

Thus form the definitions themselves we have the following two results.

4. Some Theorems on Projective Convergence

Theorem 4.1. *A necessary and sufficient condition for the* $\alpha(f)\,\beta(f)$-*convergence of* $f_\lambda(x)$ *is that to every* $g(x)$ *in* $\beta(f)$, *and to every* $\varepsilon > 0$, *there corresponds a positive number* $N(\varepsilon, g)$ *such that, for all* $\lambda, \lambda' \geq N(\varepsilon, g)$,

$$\left| \int_0^\infty g(x)\{f_\lambda(x) - f_{\lambda'}(x)\}\, dx \right| \leq \varepsilon \tag{4.1}$$

[cf. Infinite matrices and sequence spaces, p-283].

Theorem 4.2. *A necessary and sufficient condition that* $f_\lambda(x)$ *should be* $\alpha(f)\,\beta(f)$*-continuous is that to every* $g(x)$ *in* $\beta(f)$, *and to every* $\varepsilon > 0$ *there corresponds a positive number* $\delta(\varepsilon, g)$ *such that* (4.1) *holds for all non-negative* λ *and* λ' *satisfying* $|\lambda - \lambda'| \leq \delta(\varepsilon, g)$.

Proof. We know that in the case of sequence spaces (see I.M., 283), $\alpha\beta$-convergence always implies co-ordinate convergence (c-convergence), but in the case of function spaces, as we shall see below $\alpha(f)\,\beta(f)$-convergence does not necessarily imply c-convergence and hence also does not imply parametric convergence.

For example, Let $f_\lambda(x) = \cos \lambda x,$ so that $f_\lambda(x) \in \sigma_\infty(f).$

Let $g(x)$ be any function in $\sigma_1(f),$ then by the Riemann-Lebesgue Theorem, we have

$$\lim_{\lambda \to \infty} \int_0^\infty g(x) \cos \lambda x \; dx = 0$$

i.e.,

$$\lim_{\lambda \to \infty} \int_0^\infty f_\lambda(x) \; g(x) \; dx = 0$$

And therefore, by the definition of projective convergence $f_\lambda(x)$ is $\sigma_\infty(f)$-cgt. But $\cos \lambda x$ does not tend to a limit as $\lambda \to \infty$. Hence $f_\lambda(x) \equiv \cos \lambda x$ is not c-cgt, and so is not λ-cgt. Thus $\sigma_\infty(f)$-convergence does not imply c-convergence and so in general $\alpha(f)\,\beta(f)$-convergence does not necessarily imply co-ordinate convergence. It is remarked that $\sigma_1(f) > \varphi(f);$ thus, even the fulfilment of the condition $\beta(f) \geq \varphi(x)$ does not ensure that $\alpha(f)\,\beta(f)$-convergence necessarily implies c-convergence.

However, Theorem 4.3 established below gives sufficient conditions under which projective convergence implies parametric convergence.

We shall say that $f_\lambda(x) \in m(f)_\lambda$ when for almost all $x \geq 0$, $f_\lambda(x)$ is always either **monotonic increasing** or **monotonic decreasing** with respect to λ in $[0, \infty)$.

Theorem 4.3. *If* $\beta(f) \geq \varphi(x)$ *and* $f_\lambda(x) \in m(f)_\lambda$ *and* $f_\lambda(x)$ *is* $\alpha(f)\,\beta(f)$-*cgt, then* $f_\lambda(x)$ *is* λ-*cgt.*

Proof. Suppose that $f_\lambda(x) \in m(f)_\lambda$, $\beta(f) \geq \varphi(f)$ and that $f_\lambda(x)$ is $\alpha(f)\,\beta(f)$-cgt.

Since $f_\lambda(x)$ is $\alpha(f)\,\beta(f)$-cgt to every $g(x)$ in $\beta(f)$ and to every $\varepsilon > 0$, there corresponds by (2.1), a positive number $N(\varepsilon, g)$ such that for all $\lambda, \lambda' \geq N(\varepsilon, g)$,

$$\left| \int_0^\infty g(x)\{f_\lambda(x) - f_{\lambda'}(x)\}\, dx \right| \leq \varepsilon \tag{4.2}$$

If possible, suppose that there exist $\varepsilon > 0$ and an unbounded set $F \subset [0, \infty)$ such that, for all λ, λ' in F,

$$| f_\lambda(x) - f_{\lambda'}(x) | > \varepsilon \tag{4.3}$$

for all x in E, where E is a subset of $[0, \infty)$ and is of finite positive measure, say $m(E)$.

Take $g(x) = \begin{cases} \dfrac{1}{m(E)}, & \text{for } x \text{ in } E \\ 0, & \text{elsewhere,} \end{cases}$

Then $g(x) \in \varphi(f)$, and consequently $g(x) \in \beta(f)$, by the hypothesis Then for this g and that ε for which (4.3) is true, there exist N, such that by (4.2), for all $\lambda, \lambda' \geq N$,

$$\left| \int_E \frac{1}{m(E)} \{f_\lambda(x) - f_{\lambda'}(x)\}\, dx \right| \leq \varepsilon,$$

i.e.,

$$\frac{1}{m(E)} \int_E | f_\lambda(x) - f_{\lambda'}(x) | \, dx \leq \varepsilon \tag{4.4}$$

Since $f_\lambda(x) \in m(f)_\lambda$, by hypothesis (4.4) is obviously true for all λ, λ' in $F \subset [N, \infty)$.

But for all λ, λ' in $F \cap [N, \infty)$, by (4.3),

$$\frac{1}{m(E)} \int_E | f_\lambda(x) - f_{\lambda'}(x) | \, dx > \frac{1}{m(E)} \cdot \varepsilon \cdot m(E) = \varepsilon \tag{4.5}$$

Thus (4.5) contradicts (4.4), hence (4.3) is impossible and therefore

$$| f_\lambda(x) - f_{\lambda'}(x) | \leq \varepsilon$$

for almost all $x \geq 0$ and all $\lambda, \lambda' \geq N(\varepsilon)$ for every $\varepsilon > 0$ and so $f_\lambda(x)$ is λ-cgt.

Acknowledgement

Author wish to thank Dr. Lal Babu Singh, Professor, PG Department of Mathematics, Jai Prakash University, Chapra, Bihar, India, for his valuable suggestions to improve this paper a successful one.

References

[1] H. S. Allen, Projective convergence and limit in sequence spaces, P. L. M. S. 2-48(1) (1945), 310-338.

[2] H. S. Allen, Transformation of sequence spaces, Journal London Math Soc. 31(3) (1956), 374-376.

[3] R. G. Cooke, Infinite Matrices and Sequence Spaces, MacMillan, London, (1950).

[4] R. G. Cooke, Linear Operators, Macmillan, London, (1953).

[5] R. E. Edward, Functional Analysis, Holt, Rinehart and Winston, Inc., New York, (1965).

[6] C. Goffman and G. Pedrick, First Course in Functional Analysis, Prentice Hall, India, New Delhi, (1974).

[7] P. K. Kamthan and M. Gupta, Sequence Space and Series, Marcel Dekker, New York, (1981).