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ITERATED FUNCTION SYSTEM CONSISTING OF CHU AND DIAZ MAPPING IN COMPLETE *b*-METRIC SPACE

PRIYANKA, S. C. SHRIVASTAVA and R. SHRIVASTAVA

Department of Applied Mathematics
Rungta College of Engineering and Technology
Bhilai-491024, Chhattisgarh, India
E-mail: ps183241@gmail.com
Subhash_2911@rediffmail.com

Department of Physics
Govt. Naveen College, Risali, Bhilai
Dist-Durg, Chhattisgarh, India
E-mail: ritu_10101010@rediffmail.com

Abstract

Intention of this research work is to introduce the notion of Chu and Diaz contraction mapping and to establish a new Iterated Function System known as "Chu and Diaz-IFS" in the complete b-Metric Spaces. Further, the results introduced in this article are an extension of Hutchinson-Barnsley theory in the new setting. As an application, Collage theorem established in the new framework.

1. Introduction

Fixed point theory assumes a most essential part in the improvement of non-linear analysis. Additionally, it has been broadly used in diverse fields of sciences and engineering. The metric be the pillar of the field of functional analysis, real analysis and complex analysis. Metric fixed point theory is an important part of mathematical investigation, as a result of its applications in diverse fields like approximation theory, improvement, variational inequalities and linear inequalities. It has been generalized and extended in numerous diverse directions by various researchers. For instance, 2-metric,

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D-metric, G-metric, S-metric, fuzzy metric, symmetric, b-metric etc.

b-metric space is another important development of metric space. Bourbaki [1] and Backhtin [2] gave an idea of b-metric space adopting that thought Czerwik [3] introduced a Banach fixed point hypothesis in the b-metric space in 1993. They provided a new concepts and theory for mathematicians.

One of the various fields that use the fixed point properties is fractals theory. It is very modern area in mathematics and has many applications in various technical and scientific areas. IFS (Iterated Function Systems), presented by Hutchinson [4] in 1981, which was the simple technique for producing fractals. It is consisting a set of constraints $(W_n)_{n=1}^{N}$ acting on a complete metric space X. Consider a non-empty compact set $A \subset X$ such that $A = \bigcup_{n=1}^{N} W_n(A)$, Where, A is normally called a fractal set and it is known as the attractor of the corresponding IFS.

In 1964-65, Chu and Diaz 5, [6] introduced a new mapping which is called Chu and Diaz-mapping. Later on, S. C. Shrivastava et al. [7] presented a new IFS namely "Chu and Diaz-IFS" using Chu and Diaz mapping in a complete metric spaces. The purpose of this research to generalize Chu and Diaz IFS introduced by S. C. Shrivastava et al. [7] in a complete b-metric space.

2. Preliminaries

Here, we define the following basic notations and concepts which is useful for establishment of our main consequences.

Definition 2.1 [3]**.** Consider a non-empty set X together with $s \in R$, where $s \geq 1$. Let $d : X \times X \to R^{+}$ is called a b-metric iff for every $x, y, z \in X$. If it fulfills the following axioms:

$[b_1] d(x, y) = 0 \; iff \; x = y.$

$[b_2] d(x, y) = d(y, x).$

$[b_3] d(x, z) \leq s[d(s, y) d(y, z)].$

The pair (X, d) is said to be a b-metric space.

Definition 2.2 ([3], [8]). Suppose a sequence $\{x_n\}_{n \in N}$ in a b-metric space (X, d) is known as:

i. Convergent iff for every $\varepsilon > 0$ and $n \in N \; \exists x \in X$ such that $d(x_n, x) < \varepsilon$ i.e. $d(x_n, x) \to 0$ as $n \to \infty$. Consider x be the limit of the sequence can be expressed as $\lim_{n \to \infty} x_n = x$.

ii. Cauchy if and only if for each $\varepsilon > 0$ and $n, m \in N \; \exists x \in X$ such that $d(x_n, x_m) < \varepsilon \; \forall n, m \geq N$ i.e. $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 2.3 [8]. Consider b-metric space (X, d) is complete if every Cauchy sequence is convergence in X.

Definition 2.4 [8]. Suppose (X, d) is a b-metric space. Then $Y \subset X$ is known as:

i. Closed iff for every sequence $\{x_n\}_{n \in N}$ in Y which converges to an element $x \in Y$ (i.e. $Y = \overline{Y}$).

ii. Compact iff for every sequence in Y has a convergent subsequence in Y.

iii. Bounded iff $\delta(Y) = \sup\{d(a, b) : a, b \in Y\} < \infty$.

Definition 2.5 [9]. Consider X be a complete b-metric space and $H(X)$ represents the family of all non-empty compact subsets of X. Let $a, b \in X$ and $A, B \in H(X)$. Then the Hausdorff metric is expressed by;

$$h(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

Where; $d(a, B) = \inf\{d(a, b) : b \in B\}$. The pair $(H(X), h)$ is known as Hausdorff metric space (or Fractal space).

Lemma 2.1 [8]. *Consider (X, d) be a b-metric space. Then for every $A, B, C \in H(X)$, we know that*

$$h(A, C) \leq s[h(A, B) + h(B, C)].$$

Theorem 2.1 [8]. *If (X, d) be a complete b-metric space. Then $(H(X), h)$ is also a complete b-metric space.*

Theorem 2.2 [19]**.** *Consider* (X, d) *be a complete b-metric space together with constant* $s \geq 1$. *Suppose that* $T : X \to X$ *be a contraction mapping with contractivity ratio* $\alpha \in [0, 1)$ *in a complete b-metric space X. Then T possesses exactly one fixed point* $x^* \in X$. *Furthermore, for each point* $x \in X$, *the sequence* $\{T^n(x) : n = 0, 1, 2, 3, \ldots\}$ *converges to* x^*. *That is* $\lim_{n\to\infty} T^n(x) = x^*$, *for each* $x \in X$.

3. Main Results

In this portion, we develop the Chu and Diaz-IFS theory in the b-metric space. Chu and Diaz mapping defined as follows:

In 1964-65, Chu and Diaz ([5], [6]) presented a mapping, which was a development over contraction mapping, celebrated as Chu and Diaz mapping defined in the following way:

If ∃ a constant α, $0 < \alpha < 1$ such that $\forall\ x, y \in X$.

$$d(T^n x, T^n y) \leq \alpha d(x, y) \tag{3.1}$$

Then, T^n is called a Chu and Diaz mapping.

Proposition 3.1. *Suppose that* (X, d) *be a complete b-metric space. Let* $T^n : X \to X$ *is a Chu and Diaz mapping satisfying* (3.1). *Then* T^n *has a unique fixed point* x^* *in X. Furthermore, for any* $x_0 \in X$, *the successive sequence* $\{x_m\}$ *where* $x_m = T^n x_{m-1} (m = 1, 2, 3, \ldots)$ *converges to the fixed point* x^*.

Proof. Assume an arbitrary point x_0 in X. Suppose a sequence $\{x_m\}$ in X, so that

$$x_m = T^n x_{m-1},\ m = 1, 2, 3, \ldots.$$

Now, we shall show $\{x_m\}$ is a Cauchy sequence. From (3.1), we know

$$d(x_{m+1}, x_m) = d(T^n x_m, T^n x_{m-1})$$

$$\leq \alpha d(x_m, x_{m-1})$$

Repeating this procedure we can easily say that;

$$d(x_{m+1}, x_m) \leq \alpha d(x_m, x_{m-1}) \leq \alpha^2 d(x_{m-1}, x_{m-2}) \leq \ldots \leq \alpha^m d(Tx_0, x_0)$$

i.e., T^n be a contraction mapping.

Now, it is prove that $\{x_m\}$ be a Cauchy sequence in X.

Suppose $m, k > 0$ with $m > k$, we have

$$d(x_m, x_k) \leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_k)]$$

$$\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) + \ldots + s^{k-m} d(x_{m+k-1}, x_k)$$

$$\leq s\alpha^m d(Tx_0, x_0) + s^2 \alpha^{m+1} d(Tx_0, x_0) + s^3 \alpha^{m+2} d(Tx_0, x_0) + \ldots + s^{k-m} \alpha^{k-1} d(Tx_0, x_0)$$

$$\leq s\alpha^m [1 + s\alpha + (s\alpha)^2 + (s\alpha)^3 + \ldots + (s\alpha)^{k-m-1}] d(Tx_0, x_0)$$

$$\leq \frac{s\alpha^m}{1 - s\alpha} d(Tx_0, x_0)$$

Now, taking $m \to \infty$, we get $\lim_{m \to \infty} d(x_m, x_k) = 0$.

$\Rightarrow \{x_m\}$ be a Cauchy sequence in X. Since X is a complete.

Suppose that $\{x_m\}$ converges to x^*.

Next, we have to prove that x^* is fixed point of T^n.

We know that,

$$d(x^*, T^n(x^*)) \leq s[d(x^*, x_m) + d(x_m, T^n(x^*))]$$

$$\leq s[d(x^*, x_m) + d(T^n(x_{m-1}), T^n(x^*))]$$

$$\leq s[d(x^*, x_m) + \alpha d(x_{m-1}, x^*)]$$

Taking $\lim_{m \to \infty}$, we get

$\lim_{m\to\infty} d(x^*, T^n(x^*)) = 0.$

$\Rightarrow x^* = T^n(x^*)$

$\Rightarrow x^*$ is the fixed point of T^n.

Uniqueness:

Assume *x* and *y* be two fixed points of T^n.

$\Rightarrow x^* = T^n(x^*), y^* = T^n(y^*).$

$d(x^*, y^*) = d(T^n(x^*), T^n(y^*))$

$d(x^*, y^*) \leq \alpha d(x^*, y^*)$

Which gives $(1-\alpha)d(x^*, y^*) \leq 0.$

Thus, $d(x^*, y^*) = 0.$

So, $x^* = y^*.$

That is to say, x^* be a unique fixed point of T^n in *X*.

In order to present the Chu and Diaz-IFS in the b-metric space, we need the verification of subsequent lemmas to establishment of Chu and Diaz-IFS in the new framework.

Lemma 3.1. *Suppose* (X, d) *be a complete b-metric space. Let the mapping* $T^n : X \to X$ *is continuous and satisfies*

$$d(T^n x, T^n y) \leq \alpha d(x, y)$$

For every $x, y \in X,$ *where* $0 < \alpha < 1.$

Then $T^n : H(X) \to H(X)$ *defined by* $T^n(B) = \{T^n(x) : x \in B\}$ *for each* $B, C \in H(X)$ *also satisfies*

$$h(T^n(B), T^n(C)) \leq \alpha h(B, C) \tag{3.2}$$

Proof. Since T^n is a continuous mapping and $T^n : H(X) \to H(X)$.

Let $B, C \in H(X)$. Then

$$h(T^n(B), T^n(C)) = d(T^n(B), T^n(C)) \vee d(T^n(C), T^n(B))$$
$$\leq \alpha\{d(B, C) \vee d(C, B)\}$$
$$= \alpha d(B, C)$$

Therefore, $h(T^n(B), T^n(C)) \leq \alpha h(B, C)$

Lemma 3.2. *Suppose* (X, d) *be a complete b-metric space. Let* $T_r^n : r = 1, 2, 3, \ldots, N$ *be continuous Chu and Diaz mappings which maps* $(H(X), h)$ *into* $(H(X), h)$*. Suppose, the mappings* T_r^n *satisfy*

$$h(T_r^n(B), T_r^n(C)) \leq \alpha(h(B, C))$$

for all $B, C \in H(X)$, *where* $0 < \alpha_r < 1$. *Define* $T_r^n : H(X) \to H(X)$ *by* $T_r^n(B) = T_1^n(B) \cup T_2^n(B) \cup T_3^n(B) \cup \ldots \cup T_N^n(B) = \cup_{r=1}^N T_r^n(B)$ *for each* $B \in H(X)$. *Then* T^n *also satisfies*

$$h(T^n(B), T^n(C)) \leq \alpha(h(B, C)),$$

Where, $B, C \in H(X), \alpha = \max\{\alpha_r : r = 1, 2, \ldots, N\}$.

Proof. By mathematical induction method,

The statement is obviously valid for $N = 1$.

Now, for $N = 2$, we observe that

$$h(T^n(B), T^n(C))$$
$$= h(T_1^n(B) \cup T_2^n(B), T_1^n(C) \cup T_2^n(C))$$
$$= \alpha_1[h(B, C)] \vee \alpha_2[h(B, C)]$$
$$\leq (\alpha_1 \vee \alpha_2)h(B, C)$$
$$= \alpha h(B, C)$$

Therefore, $h(T^n(B), T^n(C)) \le \alpha h(B, C)$.

Hence, we explain Chu and Diaz-IFS with the help of above mention results and definitions.

Now, we present are markable result for Chu and Diaz-IFS in the new setting. IFS $\{T_r^n\}_{r=1}^N$ consisting of Chu and Diaz mappings defined as

$$d(T_r^n(x), T_r^n(y)) \le \alpha_r d(x, y). \tag{3.3}$$

for every $x, y \in X$, where a_r are constants with $0 < \alpha < 1$.

Definition 3.1. A Chu and Diaz-IFS consists of a b-metric space together with a finite set of contraction mappings $T_r^n : X \to X$ with contractivity factor a_r, for $r = 1, 2, \ldots, N$.

The representation for the IFS just introduced is $\{X : T_r^n; r = 1, 2, 3, \ldots, N\}$ and its contractivity factor is $\alpha = \max\{a_r : r = 1, 2, 3, \ldots, N\}$.

Theorem 3.1. *Suppose* $\{X : T_r^n; r = 1, 2, \ldots, N\}$ *be a Chu and Diaz-IFS with Chu and Diaz-contractivity factors* a_r. *Then the transformation* $T^n : H(X) \to H(x)$ *defined by* $T^n(B) = \cup_{r=1}^N T_r^n(B)$ *for all* $B \in H(X)$ *is a Chu and Diaz contraction mapping on the complete b-metric space* $(H(X), h)$ *with contractivity factor* α. *i.e.*

$$h(T^n(B), T^n(C)) \le \alpha h(B, C)$$

By Banach contraction principle, it has the unique fixed point $A \in H(X)$, *known as the attractor of the IFS.*

$$A = T^n(A) = \bigcup_{r=1}^{N} T_r^n(A)$$

Obey and is given by $A = \lim_{r\to\infty}(T_{or}^n)(B)$ for any $B \in H(X)$, where $(T_{or}^n)(A)$ denotes the r^{th} iteration of T^n.

4. Application

As an application, we formulate Collage theorem for Chu and Diaz-FS. Following Lemma is required in mathematical formulation of Collage theorem.

Lemma 4.1. *Let $T^n : X \to X$ be a continuous Chu and Diaz mapping on a complete b-metric space (X, d) with contractivity factor* α *and let $x^* \in X$ be the fixed point of T^n. Then*

$$d(x, x^*) \leq \frac{s}{(1 - s\alpha)} d(x, T^n(x)), \forall x \in X.$$

Finally, we formulate Collage theorem for Chu and Diaz-IFS in b-metric space.

Theorem 4.1. *Suppose that (X, d) be a complete b-metric space. Assume $L \in H(X)$ be given and $\varepsilon \geq 0$ be given. Choose a Chu and Diaz IFS $\{X : T_r^n; r = 1, 2, 3, \ldots, N\}$ with contractivity factor* α, *such that*

$$h\left(L, \bigcup_{r=0,\, r=1}^{N} T_r^n(A)\right) \leq \varepsilon$$

Then,

$$h(L, A) \leq \varepsilon \cdot \frac{s}{(1 - s\alpha)}$$

Where, A is the attractor of the Chu and Diaz Iterated Function System. Equivalently

$$h(L, A) \leq \frac{s}{(1 - s\alpha)} \cdot h\left(L, \bigcup_{r=0,\, r=1}^{N} T_r^n(A)\right)$$

Remark 4.1. Result (3.1) and Result (4.1) is an extension of [5, Theorem 3.7 and Theorem 3.8] on the complete b-metric space.

Remark 4.2. Result (3.1) and Result (4.1) is a valuable addition to the main results of the literature [4, 5, 6, 7].

5. Conclusion

In the present work we investigate the Chu and Diaz -IFS in complete b-metric space and establish the Hutchinson-Barnsley results for an IFS of Chu and Diaz mapping by the new setting of the space. As an application we established Collage theorem in the new framework.

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