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LIAR'S DOMINATION AND CONNECTED LIAR'S DOMINATION IN TRIANGLE-LIKE GRAPHS

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Abstract

A dominating set $L \subset V(G)$ is a liar's dominating set (LDS) if for any vertex $v \in V(G)$ if all or all but one of the vertices in $N[v] \cap L$ report v as the intruder location, and at most one vertex w in $N[v] \cap L$ either reports a vertex $x \in N[w]$ or fails to report any vertex, then vertex v can be correctly identified as intruder vertex. If the induced subgraph of subset L is connected then it is called a connected liar's dominating set. Liar's domination and connected liar's domination have gained attention of many researchers due to its various important applications. In this paper, we determine liar's domination and connected liar's domination for triangular snake graphs, double triangular snake graphs, Sierpiński triangle graphs and Sierpiński gasket graphs.

1. Introduction

Theory of domination has been the nucleus of research activity in graph theory. Among different variants of domination, connected domination and liar's domination is of great importance both theoretically and practically. Liar's domination was introduced by Slater in 2009. Liar's dominating set (LDS) can identify the location of the intruder correctly even when one device becomes faulty.

Given a graph $G(V, E)$, we denote the open neighborhoods of v in G by

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$N(v) = \{x \in V(G)/(x, v) \in E(G)\}$ and the closed neighbourhoods as $N(v) = N(v) \cup v$. A set S of vertices of a graph $G = (V, E)$ is a dominating set of G if every vertex in $V(G) - S$ is adjacent to some vertex of S. The minimum cardinality of a dominating set of G is $\gamma(G)$. Let $G[S]$ denote the induced subgraph of G on S then $S \in V$ is a connected dominating set of a graph G if (i) $| N[v] \cap S | \geq 1$ for all $v \in V$ and (ii) $G(S)$ is connected [8]. A vertex set $L \subseteq V(G)$ is a Liar's Dominating Set (LDS) if and only if (1) L double dominates every $v \in V(G)$ and (2) for every pair u, v of distinct vertices we have $| (N[u] \cup N[v]) \cap L | \geq 3$. In other words, $L \subseteq V(G)$ is a LDS if each component of $\langle L \rangle$, the subgraph generated by L contains at least three vertices and is called a connected LDS of a graph G, if $G[L]$ is connected. Thus every connected graph G having at least three vertices contains a connected LDS since V is itself a LDS of G. The liar's domination number and connected liar's domination number of a graph G are denoted by $\gamma_L(G)$ and $\gamma_{CL}(G)$ respectively.

LDS problem is NP-hard for general graphs and bipartite graphs. LDS for circulant networks, bounded degree graphs and p-claw free graphs have been studied and the characterization of graphs and trees for which $\gamma_L(G)$ is $| V |$ and $| V | - 1$ respectively have been provided. Approximation algorithms for minimum connected liar's domination problem have been proposed and hardness of approximation in general graphs has been investigated. The details of this brief literature survey are found in [9], [7], [4], [1], [6], [5], [10]. In this paper, LDS and connected LDS for triangular snake graphs, double triangular snake graphs, Sierpiński triangle graphs and Sierpiński gasket graphs are determined.

2. Liar's Dominating Set for Triangular Snake Graphs (TS_n)

A triangular snake graph TS_n is a connected graph obtained from a path $P = v_1, v_2, v_3, \ldots, v_{n+1}$ by joining v_i and v_{i+1} to a new vertex u_i, $i = 1, 2, \ldots, n$. It has $2n + 1$ vertices and $3n$ edges where n is the number of blocks.

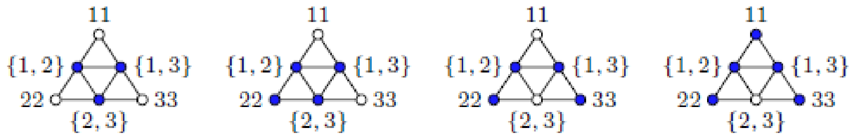

Figure 1. Liar's Dominating Set of TS_n and DTS_n.

Theorem 2.1. *Let G be a triangular snake graph* TS_n *with* $n \geq 3$. *Then* $\gamma_L(TS_n) = n + 3$.

Proof. Let $L = \{v_i : 1 \leq i \leq n+1\}$. Since $N(u_i) = \{v_i, v_{i+1}\}; L$ double dominates every $v \in V(G)$. In order to prove that for every pair of distinct vertices in TS_n to be triple dominated, we consider the following cases:

Case 1: Consider $u_i, u_j \in TS_n$. For $i \neq j$, v_i and v_{i+1} are adjacent to u_i and v_j and v_{j+1} are adjacent to u_j. Since $\{v_i, v_{i+1}, v_j, v_{j+1}\}$ belong to the LDS L, we observe that $|(N[u] \cup N[v]) \cap L| \geq 3$.

Case 2: Consider $v_i, v_j \in TS_n$. For $i \neq j$, since all v's belong to L, it dominates itself and is dominated by its adjacent vertices. Thus we have $\{v_{i-1}, v_i, v_{i+1}, v_{j-1}, v_j, v_{j+1}\}$ in L. Even if $i = j-1$ or $j = i+1$ it is clear that $|(N[u] \cup N[v]) \cap L| \geq 3$.

Case 3: Consider $u_i, v_j \in TS_n$. (i) For all i, j except $i = j = 1$ or n, $\{v_i, v_{i+1}, v_{j-1}, v_j, v_{j+1}\}$ in L. We observe that even if $i = j-1$ or $j = i+1$ we have $|(N[u] \cup N[v]) \cap L| \geq 3$. (ii) For $i = j = 1$ or n, $(N[u] \cup N[v])$ $= \{v_i, v_{i+1}\}$. This implies $|(N[u] \cup N[v]) \cap L| < 3$. Therefore both u_1 and $u_n \in L$. Also $L = \{v_i : 1 \leq i \leq n+1\} \cup \{u_1, u_n\}$ is a minimum LDS. Thus $\gamma_L(TS_n) = n + 3$.

In the view of the above result, LDS becomes connected LDS and hence we have the following result.

Theorem 2.2. *Let G be a triangular snake graph* TS_n *with* $n \geq 3$. *Then connected liar's domination* $\gamma_{CL}(TS_n) = n + 3$.

3. Liar's Dominating Set for Double Triangular Snake Graphs (DTS_n)

A double triangular snake (DTS_n) has $3n+1$ vertices and $5n$ edges where n represents the number of blocks in it. Adding vertices $w_1, w_2, w_3, \ldots w_n$ below the triangular snake TS_n and by joining v_j and v_{j+1} to a new vertex w_k, we get a double triangular snake DTS_n.

Theorem 3.1. *Let G be a double triangular snake graph* DTS_n *with* $n \geq 3$. *Then* $\gamma_L(DTS_n) = 2n+1$.

Proof. Let $L = \{v_i : 1 \leq i \leq n+1\}$. Consider the pair of distinct vertices u_i and w_k in DTS_n. If $i \neq k$ then we have $|(N[u] \cup N[w]) \cap L| \geq 3$. But for $i = k$, $N[u_i] \cup N[w_i] = \{v_i, v_{i+1}\}$. Thus either all u_i's or w_i's should be in L so that, $|(N[u] \cup N[v]) \cap L| \geq 3$. Therefore, $L = \{v_i : 1 \leq i \leq n+1\} \cup \{u_i : 1 \leq i \leq n\}$. Hence, $\gamma_L(DTS_n) = 2n+1$.

Theorem 3.2. *Let G be a triangular snake graph* DTS_n *with* $n \geq 3$. *Then connected liar's domination* $\gamma_{CL}(DTS_n) = 2n+1$.

4. Liar's Dominating Set for Sierpiński Triangle Graphs and Sierpiński Gasket Graphs

The generalized *Sierpiński* triangle graph $S(n, k)$ where $n \geq 1, k \geq 1$ which has vertex set $\{1, 2, \ldots, k\}^n$, and there is an edge between two vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ iff there is an $h \in \{1, 2, \ldots, n\}$ such that $u_j = v_j$ for $j = 1, \ldots, h-1; u_h \neq v_h;$ and $u_j = v_h; v_j = u_h$ for $j = h+1, \ldots, n$. Here vertex $(u_1, u_2, \ldots u_n)$ is represented as $(u_1 u_2, \ldots, u_n)$. The vertices $(1, \ldots, 1), (2, \ldots, 2), \ldots, (k, \ldots, k)$ are the extreme vertices of $S(n, k)$ [2]. Sierpinski gasket graphs $S_n, n \geq 1$ is obtained by contracting all edges of $S(n, 3)$ that do not form K_3. If $(u_1, \ldots, u_r ij, \ldots j)$ and $(u_1, \ldots u_r ji, \ldots i)$ are end vertices of that edge, then we denote the corresponding vertex of S_n by $(u_1, \ldots u_r)\{i, j\}, \leq n-2$. Thus, S_n is the graph with three special vertices $(1, \ldots, 1), (2, \ldots, 2)$ and $(3, \ldots, 3)$ which are extreme vertices of S_n.

Sierpinski gasket graph S_n has $\frac{3}{2}[3^{n-1}+1]$ vertices, 3^n edges with diameter 2^{n-1}[3].

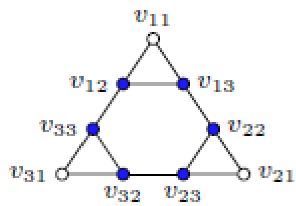

Figure 2. Liar's dominating set of $S(2, 3)$.

Lemma 4.1. *Let G be a Sierpiński triangle graph* $S(2, 3)$. *Then* $\gamma_L(S(2, 3)) = 6$.

Proof. The vertices of $S(2, 3)$. are labeled as shown in Figure 2. Since $d(v_{i1})=2$, to satisfy $|N[v_{i1}] \cap L| \geq 2$, any two among (v_{i1}, v_{i2}, v_{i3}), $i = 1, 2, 3$ should be in L. In other words, $G[v_{i1}, v_{i2}, v_{i3}]$ should have two vertices in L. Hence $|L| \geq 6$. We now prove that for $\gamma_L(S(2, 3)) = 6$, we first prove that no corner vertices belong to minimum LDS. Suppose if $v_{11} \in L$ then either v_{12} or v_{13} should be in L and so $|(N[v_{21}] \cup N[v_{22}]) \cap L| = 2$ or $|(N[v_{31}] \cup N[v_{32}]) \cap L| = 2$ respectively since L includes $v_{22}, v_{23}, v_{32}, v_{33}$. Therefore no corner vertices should be in L. Thus, $L = \{v_{12}, v_{13}, v_{22}, v_{23}, v_{31}, v_{32}\}$.

Lemma 4.2. *Let G be a Sierpiński triangle graph* $S(3, 3)$. *Then* $\gamma_L(S(3, 3)) = 3(\gamma_L(S(2, 3))) = 18$.

Proof. The vertices of $S(2, 3)$ are labeled in such a way that 111, 222, 333 are the corner vertices. We first prove that corner vertices does not belong to L, minimum LDS. Suppose if $111 \in L$ then either 112 or 113 should be in L in order to satisfy $|N[111] \cap L| = 2$. Without loss of generality and by symmetry let $112 \in L$. Then either (131 and 133) $\in L$ or (131 and 132) $\in L$.

In both cases $311 \in L$. Also either (312 or 313) $\in L$. If $313 \in L$ then either (331 and 332) or (331 and 333) $\in L$. By doing so, $323 \in L$ or $332 \in L$ respectively. In the later case, $|L|$ gets increased whereas in the first case either (321 or 322) $\in L$. But $322 \notin L$ since $|(N[312] \cup N[311]) \cap L| = 2$ thus $321 \in L$.

Then by identifying 311 as 233 and 333 as 222 we found that $122 \in L$. Here, either 121 or $123 \in L$ in which case we have $|(N[131] \cup N[132]) \cap L| < 3$ or $|(N[111] \cup N[112]) \cap L| < 3$ which proves that no corner vertices should be in L.

Therefore let L have $\{112, 113, 221, 223, 331, 332\}$ in order to satisfy $|N[iii] \cap L| = 2, i = 1, 2, 3$. This makes both 121 and 131 to be in L. Suppose either 121 or 131 but not both belong to L, let $121 \in L$. Then $|(N[111] \cup N[113]) \cap L| = 2$. Thus L will further include $\{121, 131, 212, 232, 313, 323\}$. Now let the sets of vertices that are dominated only once are $\{123, 132, 312, 321, 231, 213\}$, $\{122\ 213, 233, 321, 311, 132\}$ and $\{123, 133, 211, 231, 322, 312\}$. Thus we include any of these sets in L in order to obtain a minimum LDS for $S(3, 3)$.

Theorem 4.3. *Let G be a Sierpiński triangle graph $S(n, 3)$ for $n \geq 2$. Then $\gamma_L(S(n, 3)) = 3(\gamma_L(S(n-1, 3))) = 2(3^{n-1})$.*

Proof. Let us prove this result by the method of induction. In view of Lemma 4.1 and 4.2 the result holds good for $n = 2$ and $n = 3$. Assume that the result is true for $S(k, 3)$, $k \leq n-1$. Let $k = n$. Since $S(k, 3)$ comprises of 3 copies of $S(k-1, 3)$ namely $S_1(k-1, 3)$ whose corner vertices are $111, \ldots, 1, 122, \ldots, 133, \ldots, 3$ of $S(k, 3)$. Similarly $S_2(k-1, 3)$ are identified as $211, \ldots, 1, 222, \ldots, 2, 233, \ldots, 3$ and $S_3(k-1, 3)$ are identified as $311, \ldots, 1, 322, \ldots, 2, 333, \ldots, 3$. By induction hypothesis, corner vertices of $S_1(k-1, 3)$, $S_2(k-1, 3)$, $S_3(k-1, 3)$ does not belong to L and $S(k, 3)$ is obtained by joining vertices $(211, \ldots, 1, 122, \ldots, 2)$; $(311, \ldots, 1, 133, \ldots 3)$; $(233, \ldots, 3, 322, \ldots, 2)$ with an edge. Therefore LDS for $S(n, 3)$ is $3[\gamma_L(n-1, 3)]$. Hence $\gamma_L(S(n, 3)) = 3(\gamma_L(S(n-1, 3))) = 2(3^{n-1})$.

Remark 4.4. For the Sierpiński triangle graph $S(3, 3)$ LDS given by

Lemma 4.2. *is* $L = \{112, 113, 221, 223, 331, 332, 123, 132, 312, 321, 231, 213\}$. *To make it connected, instead of selecting vertices* $\{123, 132, 312, 321, 231, 213\}$ *we select* $\{122, 133, 211, 233, 311, 322\}$ *respectively. Without loss of generality instead of* $123 \in L$ *we take* $122 \in L$ *but still* $|(N[122] \cup N[123]) \cap L| \geq 3$. *Thus for* $S(3, 3)$, $\gamma_L(S(3, 3)) = \gamma_{CL}(S(3, 3)) = 18$.

Theorem 4.5. *Let G be a Sierpiński triangle graph* $S(n, 3)$ *for* $n \geq 4$. *Then connected liar's domination* $\gamma_{CL}(S(n, 3)) = \gamma_L(S(n, 3)) = 2(3^{n-1})$.

Proof. LDS for $S(n, 3)$ is obtained by Theorem 4.3. In view of Remark 4.4 to make it connected instead of selecting vertices {123,..., 3, 132, ..., 2, 312, ..., 2, 321, ..., 1, 231, ..., 1, 213, ..., 3} to be included in the minimum LDS we select {122, ..., 2, 133, ..., 3, 211, ..., 1,233, ..., 3, 311, ..., 1, 322, ..., 2}. Thus $\gamma_{CL}[(S(n, 3)) = \gamma_L(S(n, 3))] = 2(3^{n-1})$.

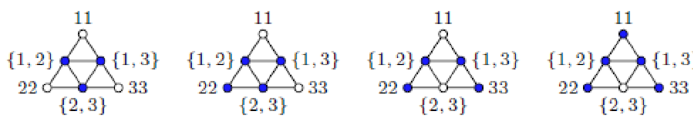

Figure 3. Liar's Dominating Set for S_2.

Remark 4.6. Sierpiński Gasket Graph (S_n): It is clear that for S_1, minimum LDS includes all of its elements. S_2 is obtained by joining 3 copies of K_3 to S_1 along its edges. Then we have the following:

(1) Since corner vertices ii are adjacent to $\{i, j\}$ and $\{i, k\}$ where $\{i, j, k\} = \{1, 2, 3\}$ we get $\gamma_L(S_2) = 3$ in which no corner vertices are selected. Here, let $L_1 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

(2) Suppose one corner vertex belongs to LDS L, let a corner vertex L $ii \in L$ then any one of its adjacent vertices should belong to L but for the other two corner vertices both its adjacent vertices are included so let $L_2 = \{11, \{1, 3\}, \{2, 3\}, \{1, 2\}\}$ and $|L_2| = 4$.

(3) If any two corner vertices belong to L then its adjacent vertices are

included in such a way that for the other corner vertex both its adjacent vertices belong to L. So, let $L_3 = 22, 33, \{1, 2\}, \{1, 3\}$ and $| L_3 | = 4$.

(4) Now, if all corner vertices belong to L then we include two vertices adjacent to any two pair of corner vertices such that it forms a LDS with $L_4 = \{11, 22, 33, \{1, 2\}, \{1, 3\}\}$. See Figure 3.

Theorem 4.7. *Let G be a Sierpiński gasket graph* S_3. *Then* $\gamma_L(S_3) = 9$.

Proof. S_3 consists of three copies of S_2 say S_2^i, $i = 1, 2\ 3$. Now, we discuss the following cases:

Case (i): Dominate L by L_1 in all S_2^i. Identify the vertices $\{111, \{1, 2\}, \{1, 3\}\}$; $\{\{1, 2\}, 222, \{2, 3\}\}$; $\{\{1, 3\}, \{2, 3\}, 333\}\}$ by $\{11, 22, 33\}$ then $| L | = 9$.

Case (ii): Dominate L by L_2 in all S_2^i. Identify the vertices $\{111, \{1, 2\}, \{1, 3\}\}$; $\{\{2, 3\}, \{1, 2\}, 222\}$; $\{\{2, 3\}, 333, \{1, 3\}\}$ by $\{11, 22, 33\}$. In this case, $| L | = 11$.

Case (iii): Dominate L by L_3 in all S_2^i. Identify the vertices $\{111, \{1, 2\}, \{1, 3\}\}$; $\{222, \{2, 3\}, \{1, 2\}\}$; $\{333, \{1, 3\}, \{2, 3\}\}$ by $\{11, 22, 33\}$ then $| L | = 9$.

Case (iv): Dominate L by L_4 in all S_2^i. Identify the vertices $\{111, \{1, 2\}, \{1, 3\}\}$; $\{222, \{2, 3\}, \{1, 2\}\}$; $\{333, \{1, 3\}, \{2, 3\}\}$ by $\{11, 22, 33\}$ then $| L | = 12$.

If we obtain L by L_i, $j = 1, 2, 3, 4$ in S_2^i, $i = 1, 2, 3$ then also $| L | \geq 9$. Thus minimum cardinality for the LDS L for S_3 is 9 that is $\gamma_L(S_3) = 9$.

Theorem 4.8. *Let G be a Sierpiński gasket graph* S_4. *Then* $\gamma_L(S_4) = 24$.

Proof. Let minimum LDS of S_4, L be obtained by using case (i) namely $L = \{1\{1, 2\}, 1\{1, 3\}, 1\{2, 3\}, 2\{1, 2\}, 2\{1, 3\}, 2\{2, 3\}, 3\{1, 2\}, 3\{2, 3\}, 3\{1, 3\}\}$. Since S_4 consists of three copies of S_3 namely S_3^1, S_3^2, S_3^3 identify each of

$\{(1111, \{1, 2\}, \{1, 3\}); (\{1, 2\}, 2222, \{2, 3\}); (\{1, 3\}, \{2, 3\}, 3333)\}$ of S_4 with $(111, 222, 333)$ of S_3. The vertex $\{1, 2\}$ is common to both S_3^1, S_3^2. Hence instead of $(21\{1, 3\}$ and $12\{2, 3\})$ to be in L we take $\{1, 2\}$. Similarly instead of $(31\{1, 3\}$ and $13\{2, 3\})$ and $(23\{1, 3\}$ and $32\{1, 2\})$ we take $\{1, 3\}$ and $\{2, 3\}$ respectively. Thus $\gamma_L(S_4) = 3(\gamma_L(S_3)) - 3 = 24$.

Theorem 4.9. *Let G be a Sierpiński gasket graph* S_n. *Then* $\gamma_L(S_n) = 3(\gamma_L(S_{n-1})) - 3$ *for* $n \geq 5$.

Proof. By Theorem 4.8, the result is true for $n = 4$. Let us assume that the result is true for S_k, $k \leq n$. Let $k = n$. Since S_n consists of 3 copies of S_{n-1} namely $S_{n-1}^i, i = 1, 2, 3$ identify the vertices $111, \ldots, 1, \{1, 2\}, \{1, 3\}$; $222, \ldots 2, \{1, 2\}, \{1, 3\}$; $333, \ldots, 3, \{1, 2\}, \{2, 3\}$ respectively by $\{111, \ldots, 1, 222, \ldots, 333, \ldots, 3\}$ of S_{n-1}. Since $\{1, 2\}$ is in both S_{n-1}^1 and S_{n-1}^2. Also $\{1, 3\}$ is in both S_{n-1}^1 and S_{n-1}^3 and $\{2, 3\}$ is in both S_{n-1}^2 and S_{n-1}^3. Instead of $(122, \ldots, 2\{2, 3\}$ and $211, \ldots, 1\{1, 3\})$; $(133, \ldots, 3\{2, 3\}$ and $311, \ldots, 1\{1, 2\})$; $(233, \ldots, 3\{1, 3\}$ and $322, \ldots, 2\{1, 2\})$ we take $\{1, 2\}, \{1, 3\}$ and $\{2, 3\}$ respectively. Thus $\gamma_L(S_n) = 3(\gamma_L(S_{n-1})) - 3$.

Theorem 4.10. *Let G be a Sierpiński gasket graph* S_n. *Then* $\gamma_{CL}(S_n) \leq 3(\gamma_L(S_n)) + (2n - 6)$ *for* $n \geq 4$.

5. Conclusion

This paper provides the LDS and connected LDS for triangular snake graphs, double triangular snake graphs, Sierpiński triangle graphs and Sierpiński gasket graphs. We extend these results for more classes of graphs in our future work.

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