Advances and Applications in Mathematical Sciences
Volume 20, Issue 6, April 2021, Pages 975-1001

ANALYSIS OF A MULTI TYPE SERVICE OF A NON-MARKOVIAN QUEUE WITH BREAKDOWN, DELAY TIME AND OPTIONAL VACATION

P. MANOHARAN and K. SANKARA SASI

Department of Mathematics
Annamalai University
Annamalainagar-608 002, India
E-mail: manomaths.hari@gmail.com
sankarasasi90@gmail.com

Abstract

We consider an $M/G/1$ queuing system with k-types of service, random breakdowns, delay times for repairs to start and a second optional vacation. All arriving customers may choose either of type j services with probability p_j where $\sum_{j=1}^{k} p_j = 1,\ j \in \{1, 2, 3, \ldots, k\}$. When the system becomes empty, the server goes for regular vacation and at the end of the first vacation the server may take a second optional vacation with $pro\ (1-\theta)$. bability θ, otherwise he remains in the system with probability. The system may breakdown at random, its repairs do not start immediately and there is a delay time. The delay times and the repair times follow a general distribution. Using supplementary variable technique, we derive the probability generating function for the number of customers in the system, the average number of customers in the system and the average waiting time of customers in the system. Particular case is deduced to check the validity of the present model with already existing models. Numerical examples are also provided.

1. Introduction

In many examples such as production system, bank services, computer and communication networks, besides feedback the system have vacation. Vacation queues with different vacation policies including Bernoulli schedules, assuming a single vacation policy or multiple vacation policy have

2010 Mathematics Subject Classification: 05C78.
Keywords: breakdown, delay time, optional vacation.
Received December 10, 2019; Accepted May 15 2020

been studied by many researchers. Levy and Yechiali [12], Fuhrman [9], Doshi [7] and [8], Keilson and Servi [11], Baba [1], Cramer [6], Borthakur and Chaudhury [3], Madan [13], [14] and [15], Choi and Park [5], Takagi [18] and [19], Rosenberg and Yechiali [17] ,Chaudhury [4], Badamchi Zadeh and Shankar [2] and many others have studied vacation queues with different P. Manoharan and K. Sankara Sasi vacation policies. Madan and Chaudhury [16] have studied a single server queue with two phase of heterogeneous service under Bernoulli schedule and a general vacation time. In this system, without feedback, the server after completing the service can take vacation with probability θ or remain in the system with probability $(1 - \theta)$ Madan and Anabosi [15] have studied a single server queue with optional server vacations based on Bernoulli schedules and a single vacation policy. In this system, without feedback, the server provides two types of heterogeneous exponential service and a customer may choose either type of service. Moreover, the server after completing the service can take vacation with probability θ or remain in the system with probability $(1 - \theta)$.

In this paper, we consider an $M/G/1$ queuing system with k-types of service, random breakdowns, delay times for repairs to start and a second optional vacation. Using supplementary variable technique, we derive the probability generating function for the number of customers in the system, the average number of customers in the system and the average waiting time of customers in the system. The paper is organaized as follows. In section two the model is described. In section three the distribution of the system is obtained. In section four the performance measures are calculated. In section five a particular case is discussed. In section six numerical illustrations are presented.

2. Model Description

The arrival follows Poisson distribution with mean arrival rate $\lambda > (0)$. The server provides k-types of service to all arriving customers. Customer may choose either of type j services with probability p_j where $\sum_{j=1}^{k} p_j = 1$. The service times follows a general distribution, with distribution function $B_j(x)$ and density function $b_j(x)$ for $j \in \{1, 2, \ldots, k\}$. Further it is assumed that $\mu_j(x)dx$ is the conditional probability of completion of the j^{th} service given that the elapsed service time is x, so that

$$\mu_j(x)dx = \frac{b_j(x)}{[1 - B_j(x)]} \tag{1}$$

and therefore $b_j(x) = \mu_j(x) e^{-\int_0^x \mu_j(t)dt}$, $j \in \{1, 2, \ldots, k\}$.

We assume that the services are mutually independent of each other. Let $B_j^*(s)$, $E(B_j^c)$, $(c \geq 1)$, $j \in \{1, 2, \ldots, k\}$ denote the Laplace-Stieltjes Transform (LST) and finite moments of service times respectively. Thus the total time required by the server to complete a service cycle which may be called as modified service period is given by

$$B = \begin{cases} B_1 & \text{with probability } p_1 \\ B_2 & \text{with probability } p_2 \\ & \vdots \\ B_k & \text{with probability } p_k. \end{cases}$$

The system may breaks down at random, and the breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breaks down, the customer whose service is interrupted, comes back to the head of the queue. Once the system breaks down, its repairs do not start immediately and there is a delay time. The delay times follow a general (arbitrary) distribution with distribution function $S(x)$ and density function $s(x)$. Let $v(x)d(x)$ be the conditional probability of a completion of the delay process given that the delay time is x, so that

$$v(x)dx = \frac{s(x)}{[1 - S(x)]} \tag{2}$$

and hence $s(x) = v(x)e^{-\int_0^x v(t)dt}$.

Further, the repair times follow a general distribution with distribution function $G(x)$ and density function $g(x)$. Further it is assumed that $\eta(x)dx$ is the conditional probability of the completion of the repair process given that the elapsed repair time is x, so that

$$\eta(x)\,dx = \frac{g(x)}{[1 - G(x)]} \tag{3}$$

and hence $g(x) = \eta(x)e^{-\int_0^x \eta(t)\,dt}$.

Whenever the system becomes empty, the server goes for a first phase of regular vacation (FRV) of random length V_1. Let $V_1(x)$ and $v_1(x)$ respectively denote the distribution function and density function of the first vacation time. At the end of FRV, the server may take a second optional vacation (SOV) with probability θ, otherwise he remains in the system with probability $(1 - \theta)$ until a new customer arrives. Let $V_2(x)$ and $v_2(x)$ respectively denote the distribution function and density function for the SOV time. Further it is assumed that $v_i(x)dx$ is the conditional probability of the completion of the i^{th} vacation given that the elapsed vacation time is x, so that

$$v_i(x)\,dx = \frac{v_i(x)}{[1 - V_i(x)]} \tag{4}$$

and hence $v_i(x) = v_i(x)e^{-\int_0^x v_i(t)\,dt}$; $i \in \{1, 2\}$.

It is also assumed that the vacation times V_1 and V_2 are mutually independent of each other having LSTs $V_i^*(s)$ and finite moments, $E(V_i^k)$, $(k \geq 1)$, $i \in \{1, 2\}$. Thus the total time required to complete the vacation cycle, which may be called as modified vacation period is given by

$$V = \begin{cases} V_1 + V_2 \text{ with probability } \theta \\ V_1 \text{ with probability } (1 - \theta). \end{cases}$$

3. System size Distribution

We set up the steady state equations for the stationary queue size distribution by treating elapsed service time, delay time, repair time, FRV time and SOV time as supplementary variables. Using these equations, we derive the probability generating functions, assuming that the system is in steady state condition. Let $N(t)$ be the system size (including one being served, if any), $B_j^{(0)}(t)$ be the elapsed service time at t for type j, $S^{(0)}(t)$ be the elapsed delay time at t, $R^{(0)}(t)$ be the elapsed repair time at t, $V_1^{(0)}(t)$ be the elapsed vacation time at t for the FRV, $V_2^{(0)}(t)$ be the elapsed vacation time at t for the SOV. For further development of this model, introduce the random variable $Y(t)$ as follows.

$$Y(t) = \begin{cases} 0 & \text{if the server is on FRV at time } t \\ 1 & \text{if the server is on SOV at time } t \\ 2 & \text{if the server is busy providing type } j \text{ service at time } t \\ 3 & \text{if the system is under delay for repair at time } t \\ 4 & \text{if the system is under repair at time } t \end{cases}$$

The supplementary variables $V_1^0(t),\ V_2^0(t),\ B_j^0(t);\ j \in \{1, 2, \ldots, k\},\ S^{(0)}(t)$ and $R^{(0)}(t)$ are introduced in order to obtain a bivariate Markov process $\{(N(t),\ \partial(t));\ t \geq 0\}$ where

$$\partial(t) = \begin{cases} V_1^0(t) & \text{if } Y(t) = 0 \\ V_2^0(t) & \text{if } Y(t) = 1 \\ B_j^0(t) & \text{if } Y(t) = 2 \\ S^{(0)}(t) & \text{if } Y(t) = 3 \\ R^{(0)}(t) & \text{if } Y(t) = 4 \end{cases}$$

and we define the limiting probability as follows.

$$Q_{1,n}(x)dx = \lim_{t \to \infty} \Pr\{N(t) = n,\ \partial(t) = V_1^{(0)}(t),\ x < V_1^{(0)}(t) \leq x + dx\},\ n \geq 0,\ x > 0$$

$$Q_{2,n}(x)dx = \lim_{t \to \infty} \Pr\{N(t) = n,\ \partial(t) = V_2^{(0)}(t),\ x < V_2^{(0)}(t) \leq x + dx\},\ n \geq 0,\ x > 0$$

$$P_{j,n}(x)dx = \lim_{t \to \infty} \Pr\{N(t) = n,\ \partial(t) = B_j^{(0)}(t),\ x < B_j^{(0)}(t) \leq x + dx\},\ n \geq 0,\ x > 0$$

$$D_n(x)dx = \lim_{t\to\infty} \Pr\{N(t) = n, \partial(t) = S^{(0)}(t), x < S^{(0)}(t) \le x + dx\}, n \ge 0, x > 0$$

$$R_n(x)dx = \lim_{t\to\infty} \Pr\{N(t) = n, \partial(t) = R^{(0)}(t), x < R^{(0)}(t) \le x + dx\}, n \ge 0, x > 0.$$

Further it is assumed that $B_j(0) = 0, B_j(\infty) = 1; S(0) = 0, S(\infty) = 1, R(0) = 0, R(\infty) = 1$ and are continuous at $x = 0$, where $V_i(0) = 0, V_i(\infty) = 1$ are distribution functions for $j \in \{1, 2, \dots, k\}$ and $i \in \{1, 2\}$ so that,

$$\mu_j(x)dx = \frac{dB_j(x)}{[1 - B_j(x)]}; v(x)dx = \frac{dS(x)}{[1 - S(x)]} \eta(x)dx = \frac{dG(x)}{[1 - G(x)]} \text{ and}$$

$$v_i(x)dx = \frac{dV_i(x)}{[1 - V_i(x)]}.$$

The differential-difference equations governing the system are

$$\frac{d}{dx} P_{j,n}(x) + (\lambda + \mu_j(x) + \alpha)P_{j,n}(x) = \lambda P_{j,n-1}(x), n \ge 1, j = 1, 2, \dots, k \tag{5}$$

$$\frac{d}{dx} P_{j,0}(x) + (\lambda + \mu_j(x) + \alpha)P_{j,n}(x) = 0, j = 1, 2, \dots, k \tag{6}$$

$$\frac{d}{dx} D_n(x) + (\lambda + v(x) + \alpha)D_n(x) = \lambda D_{n-1}(x), n \ge 1 \tag{7}$$

$$\frac{d}{dx} D_0(x) = 0, x > 0 \tag{8}$$

$$\frac{d}{dx} R_n(x) + (\lambda + \eta(x))R_n(x) = \lambda R_{n-1}(x), n \ge 1 \tag{9}$$

$$\frac{d}{dx} R_0(x) + (\lambda + \eta(x))R_0(x) = 0, \tag{10}$$

$$\frac{d}{dx} Q_{1,n}(x) + (\lambda + v_1(x))Q_{1,n}(x) = \lambda Q_{1,n-1}(x), n \ge 1 \tag{11}$$

$$\frac{d}{dx} Q_{1,0}(x) + (\lambda + v_1(x))Q_{1,0}(x) = 0 \tag{12}$$

$$\frac{d}{dx} Q_{2,n}(x) + (\lambda + v_2(x))Q_{2,n}(x) = \lambda Q_{2,n-1}(x), n \ge 1 \tag{13}$$

$$\frac{d}{dx} Q_{2,0}(x) + (\lambda + v_2(x)) Q_{2,0}(x) = 0, \tag{14}$$

$$\lambda Q_{1,0} = \sum_{m=1}^{k} \int_0^{\infty} P_{m,0}(x) \mu_m(x) dx + \int_0^{\infty} R_0(x) \eta(x) dx + (1 - \theta)$$

$$\int_0^{\infty} Q_{1,0}(x) v_1(x) dx + \int_0^{\infty} Q_{2,0}(x) v_2(x) dx \tag{15}$$

where

$$Q_{1,0} = \int_0^{\infty} Q_{1,0}(x) dx.$$

The boundary conditions are

$$Q_{1,0}(0) = \lambda Q_{1,0} \tag{16}$$

$$Q_{1,n}(0) = 0, \; n \geq 1 \tag{17}$$

$$Q_{2,n}(0) = \theta \int_0^{\infty} Q_{1,n}(x) v_1(x) dx, \; n \geq 0 \tag{18}$$

$$P_{j,0}(0) = p_j \sum_{m=1}^{k} \int_0^{\infty} P_{m,1}(x) \mu_m(x) dx + p_j \int_0^{\infty} R_1(x) \eta(x) dx$$

$$+ p_j p_j (1 - \theta) \int_0^{\infty} Q_{1,1}(x) v_1(x) dx$$

$$+ p_j \int_0^{\infty} Q_{2,1}(x) v_1(x) dx \quad j = 1, 2, \ldots, k \tag{19}$$

$$P_{j,n}(0) = p_j \sum_{m=1}^{k} \int_0^{\infty} P_{m,n+1}(x) \mu_m(x) dx + p_j \int_0^{\infty} R_{n+1}(x) \eta(x) dx$$

$$+ p_j (1 - \theta) \int_0^{\infty} Q_{1,n+1}(x) v_1(x) dx$$

$$+ p_j \int_0^{\infty} Q_{2,n+1}(x) v_1(x) dx \quad j = 1, 2, \ldots, k \tag{20}$$

$$D_n(0) = \alpha \sum_{m=1}^{k} \int_0^{\infty} P_{m,n+1}(x)\,dx, \; n \geq 1 \tag{21}$$

$$D_0(0) = 0 \tag{22}$$

$$R_n(0) = \int_0^{\infty} D_n(x)\theta(x)\,dx, \; n \geq 1 \tag{23}$$

$$R_0(0) = 0 \tag{24}$$

and the normalizing condition is

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^{\infty} P_{m,n}(x)dx + \sum_{n=1}^{\infty} \int_0^{\infty} R_n(x)dx$$

$$+ \sum_{n=1}^{\infty} \int_0^{\infty} D_n(x)dx + \sum_{n=0}^{\infty} \sum_{i=1}^{2} \int_0^{\infty} Q_{i,n}(x)dx = 1. \tag{25}$$

For $x > 0; |z| \leq 1, j \in \{1, 2, \ldots, k\}; i \in \{1, 2\}$, we define the following Probability Generating Functions

$$P_j(x, z) = \sum_{n=0}^{\infty} z^n P_{j,n}(x); \; P_j(0, z) = \sum_{n=0}^{\infty} z^n P_{j,n}(x); \; P_j(z) = \int_0^{\infty} P_j(x, z)dx$$

$$D(x, z) = \sum_{n=0}^{\infty} z^n D_n(x); \; D(0, z) = \sum_{n=0}^{\infty} z^n D_n(0); \; D(z) = \int_0^{\infty} D(x, z)dx$$

$$R(x, z) = \sum_{n=0}^{\infty} z^n R_n(x); \; R(0, z) = \sum_{n=0}^{\infty} z^n R_n(x); \; R(z) = \int_0^{\infty} R(x, z)dx$$

$$Q_i(x, z) = \sum_{n=0}^{\infty} z^n Q_{i,n}(x); \; Q_i(0, z) = \sum_{n=0}^{\infty} z^n Q_{i,n}(0); \; Q_i(z) = \int_0^{\infty} Q_i(x, z)dx.$$

Multiplying equation (5) by z^n and summing from $n = 1$ to ∞ and adding the resultant with equation (6), we get

$$\frac{d}{dx}P_j(x, z) + (\lambda + \mu_j(x) + \alpha)P_j(x, z) = \lambda z P_j(x, z)$$

$$\frac{\frac{d}{dx}P_j(x, z)}{P_j(x, z)} = -(\lambda - \lambda z + \mu_j(x) + \alpha).$$

Integrating the above equation with respect to x between 0 and x, we get

$$\frac{P_j(x, z)}{P_j(0, z)} = e^{-(\lambda - \lambda z + \alpha)x - \int_0^x \mu_j(t)dt}.$$

From equation (1)

$$-\mu_j(t)dt = -\frac{b_j(t)}{[1 - B_j(t)]}, \ j = \{1, 2, \dots, k\}.$$

Integrating the above equation with respect to x between 0 and x, we get

$$e^{-\int_0^x \mu_j(t)dt} = (1 - B_j(x)), \ j = \{1, 2, \dots, k\}. \tag{26}$$

Using equation (26) in (25), we get

$$P_j(x, z) = P_j(0, z)[1 - B_j(x)]e^{-(\lambda - \lambda z + \alpha)x}; \ j = \{1, 2, \dots, k\}. \tag{27}$$

Multiplying equation (7) by z^n and summing from $n = 1$ to ∞ and adding the resultant with equation (8), we get

$$\frac{d}{dx}D(x, z) + (\lambda + v(x))D(x, z) = \lambda z D(x, z)$$

$$\frac{\frac{d}{dx}D(x, z)}{D(x, z)} = -(\lambda - \lambda z + v(x))$$

Integrating the above equation with respect to x between 0 and x, we get

$$\frac{D(x, z)}{D(0, z)} = e^{-(\lambda - \lambda x)x - \int_0^x v(t)dt}. \tag{28}$$

From equation (2)

$$-v(t)dt = -\frac{s(t)}{[1 - s(t)]}.$$

Integrating the above equation with respect to x between 0 and x, we get

$$e^{-\int_0^x v(t)dt} = (1 - S(x)). \tag{29}$$

Using equation (29) in (28), we get

$$D(x, z) = D(0, z)[1 - S(x)]e^{-(\lambda - \lambda z)x}. \tag{30}$$

Multiplying equation (9) by z^n and summing from $n = 1$ to ∞ and adding the resultant with equation (10), we get

$$\frac{d}{dx} R(x, z) + (\lambda + \eta(x))R(x, z) = \lambda z R(x, z)$$

$$\frac{\frac{d}{dx} R(x, z)}{R(x, z)} = -(\lambda - \lambda z + \eta(x)).$$

Integrating the above equation with respect to x between 0 and x, we get

$$\frac{R(x, z)}{R(0, z)} = e^{-(\lambda - \lambda x)x - \int_0^x \eta(t)dt}. \tag{31}$$

From equation (3)

$$-\eta(t)dt = -\frac{g(t)}{[1 - G(t)]}.$$

Integrating the above equation with respect to x between 0 and x, we get

$$e^{-\int_0^x \eta(t)dt} = (1 - G(x)). \tag{32}$$

Using equation (32) in (31), we get

$$R(x, z) = R(0, z)[1 - G(x)]e^{-(\lambda - \lambda z)x}. \tag{33}$$

Multiplying equation (11) by z^n and summing from $n = 1$ to ∞ and adding the resultant with equation (12), we get

$$Q_1(x, z) = Q_2(0, z)[1 - V_1(x)]e^{-(\lambda - \lambda x)x}. \tag{34}$$

Multiplying equation (13) by z^n and summing from $n = 1$ to ∞ and adding the resultant with equation (14), we get

$$Q_2(x, z) = Q_2(0, z)[1 - V_2(x)]e^{-(\lambda - \lambda z)x}. \tag{35}$$

Multiplying equation (17) by z^n and summing from $n = 1$ to ∞ and using equation (16), we get

$$Q_1(0, z) = \lambda Q_{1,0}. \tag{36}$$

Multiplying equation (18) by z^n and summing from $n = 0$ to ∞, we get

$$Q_2(0, z) = \theta Q_1(0, z)V_1^*(\lambda - \lambda z). \tag{37}$$

Multiplying equation (20) by z^{n+1} and summing from $n = 1$ to ∞ and adding with z times equation (19), we get

$$zP_j(0, z) = p_j \sum_{m=1}^{k} \int_0^\infty P_m(x, z)\mu_m(x)dx + p_j \int_0^\infty R(x, z)\eta(x)dx$$

$$+ p_j(1 - \theta)\int_0^\infty Q_1(x, z)v_1(x)dx + \int_0^\infty Q_2(x, z)v_2(x)dx - p_j \lambda Q_{1,0}. \tag{38}$$

Multiplying equation (21) by z^n and summing from $n = 1$ to ∞, we get

$$D(0, z) = \alpha z \sum_{m=1}^{k} P_m(z). \tag{39}$$

Multiplying equation (23) by z^n and summing from $n = 1$ to ∞, we get

$$R(0, z) = \int_0^\infty D(x, z)v(x)dx \tag{40}$$

Multiplying equation (27) by $\mu_j(x)$ and integrating over 0 to ∞, we get

$$\int_0^\infty P_j(x, z)\mu_j(x)dx = P_j(0, z)B_j^*(\lambda - \lambda z + \alpha),\ j \in \{1, 2, 3, \ldots, k\}. \tag{41}$$

Multiplying equation (30) by $v(x)$ and integrating over 0 to ∞, we get

$$\int_0^\infty D(x, z)v(x)dx = D(0, z)S^*(\lambda - \lambda z). \tag{42}$$

Multiplying equation (33) by $\eta(x)$ and integrating over 0 to ∞, we get

$$\int_0^\infty R(x, z)\eta(x)dx = R(0, z)G^*(\lambda - \lambda z). \tag{43}$$

Multiplying equation (34) by $v_1(x)$ and integrating over 0 to ∞, we get

$$\int_0^\infty Q_1(x, z)v_1(x)dx = Q_1(0, z)V_1^*(\lambda - \lambda z).$$

Multiplying equation (35) by $v_2(x)$ and integrating over 0 to ∞, we get

$$\int_0^\infty Q_2(x, z)v_2(x)dx = Q_2(0, z)V_2^*(\lambda - \lambda z). \tag{44}$$

Integrating equation (27) between 0 and ∞, we get

$$P_j(z) = P_j(0, z)\left[\frac{1 - B_j^*(\lambda - \lambda z + \alpha)}{(\lambda - \lambda + \alpha)}\right], \; j \in \{1, 2, 3, \ldots, k\}. \tag{45}$$

Integrating equation (9.30) between 0 and ∞, we get

$$D(z) = D(0, z)\left[\frac{1 - S^*(\lambda - \lambda z)}{(\lambda - \lambda z)}\right]. \tag{46}$$

Integrating equation (33) between 0 and ∞, we get

$$R(z) = R(0, z)\left[\frac{1 - G^*(\lambda - \lambda z)}{(\lambda - \lambda z)}\right]. \tag{47}$$

Integrating equation (34) between 0 and ∞, we get

$$Q_1(z) = Q_1(0, z)\left[\frac{1 - V_1^*(\lambda - \lambda z)}{(\lambda - \lambda z)}\right]. \tag{48}$$

Integrating equation (35) between 0 and ∞, we get

$$Q_1(z) = Q_1(0, z)\left[\frac{\theta V_1^*(\lambda - \lambda z)[1 - V_2^*(\lambda - \lambda z)}{(\lambda - \lambda z)}\right]. \tag{49}$$

Using equations (41) to (44) in equation (38), we get

$$zP_j(0, z) = p_j \sum_{m=1}^{k} P_m(0, z) B_m^*(\lambda - \lambda z + \alpha) + p_j \alpha z \sum_{m=1}^{k} P_m(z) S^*(\lambda - \lambda z) G^*(\lambda - \lambda z).$$

$$+ p_j \{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda z) - 1\} \lambda Q_{1,0}. \quad (50)$$

Put $j = 1$ and $k = 1$ in equation (50), we get

$$P_1(0, z) = \frac{p_1(\lambda - \lambda z + \alpha)\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda z) - 1\} \lambda Q_{1,0}}{D_1(z)} \quad (51)$$

where

$$D_1(z) = z(\lambda - \lambda z + \alpha) - p_1\{B_1^*(\lambda - \lambda z + \alpha)[(\lambda - \lambda z + \alpha)$$

$$-\alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)] + \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)\}.$$

Putting $j = 1$ and $k = 2$ in equation (50), we get

$$P_1(0, z) A_1(z) = p_1 P_2(0, z) A_2(z) + p_1 A_3(z)(\lambda - \lambda z + \alpha) \lambda Q_{1,0} \quad (52)$$

where

$$A_1(z) = z(\lambda - \lambda z + \alpha) - p_1\{B_1^*(\lambda - \lambda z + \alpha)[(\lambda - \lambda z + \alpha)$$

$$-\alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)] + \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)\}$$

$$A_2(z) = B_2^*(\lambda - \lambda z + \alpha)[(\lambda - \lambda z + \alpha) - \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)]$$

$$+ \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)$$

$$A_3(z) = \{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda z) - 1\}.$$

Putting $j = 2$ and $k = 2$ in equation (50), we get

$$P_2(0, z) B_1(z) = p_2 P_1(0, z) B_2(z) + p_2 B_3(\lambda - \lambda z + \alpha) \lambda Q_{1,0} \quad (53)$$

where

$$B_1(z) = z(\lambda - \lambda z + \alpha) - p_2\{B_2^*(\lambda - \lambda z + \alpha)[(\lambda - \lambda z + \alpha)$$

$$-\alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z) + \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)\}]$$

$$B_2(z) = B_1^*(\lambda - \lambda z + \alpha)[(\lambda - \lambda z + \alpha) - \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)]$$

$$+ \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)$$

$$B_3(z) = \{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda z) - 1\}.$$

Using equation (53) in (52), we get

$$P_1(0, z) = \frac{p_1(\lambda - \lambda z + \alpha)\lambda Q_{1,0}[p_2 A_2(z) B_3(z) + A_3(z) B_1(z)]}{A_1(z) B_1(z) - p_1 p_2 A_2(z) B_2(z)} \tag{54}$$

where

$$p_2 A_2(z) B_3(z) + A_3(z) B_1(z) = z(\lambda - \lambda z + \alpha)$$

$$\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda z) - 1\} \tag{55}$$

$$A_1(z) B_1(z) - p_1 p_2 A_2 B_2(z) = z(\lambda - \lambda z + \alpha)\{z(\lambda - \lambda z + \alpha)$$

$$- p_1\{B_1^*(\lambda - \lambda z + \alpha)[(\lambda - \lambda z + \alpha)$$

$$- \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)] + \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)\}$$

$$- p_2\{B_2^*(\lambda - \lambda z + \alpha)[(\lambda - \lambda z + \alpha)$$

$$- \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)] + \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)\}\}. \tag{56}$$

Using equations (55) and (56) in equation (54), we get

$$P_1(0, z) = \frac{p_1(\lambda - \lambda z + \alpha)\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda x) - 1\}\lambda Q_{1,0}}{D_2(z)} \tag{57}$$

where

$$D_2(z) = z(\lambda - \lambda z + \alpha) - [p_1 B_1^*(\lambda - \lambda z + \alpha) + p_2 B_2^*(\lambda - \lambda z + \alpha)][(\lambda - \lambda z + \alpha)$$

$$- \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)] - \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z).$$

Using equation (52) in (53), we get

$$P_2(0, z) = \frac{p_2(\lambda - \lambda z + \alpha)\lambda Q_{1,0}\ [p_1 B_2(z) A_3(z) + B_3(z) A_1(z)]}{A_1(x) B_1(z) - p_1 p_2 A_2(z) B_2(z)} \tag{58}$$

$$p_1 B_2(z) A_3(z) + B_3(z) A_1(z) = z(\lambda - \lambda z + \alpha)\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda z) - 1\}. \quad (59)$$

Using equations (56) and (59) in equation (58), we get

$$P_2(0, z) = \frac{p_2(\lambda - \lambda z + \alpha)\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda x) - 1\}\lambda Q_{1,0}}{D_2(z)}. \quad (60)$$

From equations (51), (54) and (60), we get

$$P_j(0, z) = \frac{p_j(\lambda - \lambda z + \alpha)\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda x) - 1\}\lambda Q_{1,0}}{D_2(z)},$$

$$j = 1, 2, \ldots, k \quad (61)$$

where

$$D_3(z) = z(\lambda - \lambda z + \alpha) - [(\lambda - \lambda z + \alpha) - \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)]$$

$$\times \sum_{m=1}^{k} \{p_m B_m^*(\lambda - \lambda z + \alpha) - \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)\}$$

$$P_j(z) = \frac{[p_j(\lambda - \lambda z + \alpha)\{(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda z) - 1\} \, [1 - B_j^*(\lambda - \lambda z + \alpha)]\lambda Q_{1,0}]}{(\lambda - \lambda z + \alpha) D_3(z)}$$

$$= \frac{p_j\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda z) - 1\} [1 - B_j^*(\lambda - \lambda z + \alpha)]\lambda Q_{1,0}]}{D_3(z)},$$

$$j = 1, 2, \ldots, k. \quad (62)$$

Using equations (39) and (62) in equation (46), we get

$$D(z) = \frac{D(0, z)(1 - S^*(\lambda - \lambda z))}{(\lambda - \lambda z)} = \alpha z \sum_{m=1}^{k} p_m(z) \frac{(1 - S^*(\lambda - \lambda z))}{(\lambda - \lambda z)}$$

$$= \frac{\alpha z(1 - S^*(\lambda - \lambda z))\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)] V_1^*(\lambda - \lambda z) - 1\}}{(\lambda - \lambda z) D_3(z)}$$

$$\sum_{m=1}^{k} p_m [1 - B_m^*(\lambda - \lambda z + \alpha)] \lambda Q_{1,0}. \tag{63}$$

Using equations (39), (40), (42) and (62) in equation (47), we get

$$R(z) = \frac{\alpha z(1 - G^*(\lambda - \lambda z))S^*(\lambda - \lambda z)\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)]V_1^*(\lambda - \lambda z) - 1\}}{(\lambda - \lambda z)D_3(z)}$$

$$\sum_{m=1}^{k} p_m [1 - B_m^*(\lambda - \lambda z + \alpha)] \lambda Q_{1,0}. \tag{64}$$

Using equation (36) in (48), we get

$$Q_1(z) = \left[\frac{1 - V_1^*(\lambda - \lambda z)}{(1 - z)}\right] Q_{1,0}. \tag{65}$$

Using equation (36) in (49), we get

$$Q_2(z) = \left[\frac{\theta V_1^*(\lambda - \lambda z)[1 - V_2^*(\lambda - \lambda z)]}{(1 - z)}\right] Q_{1,0}. \tag{66}$$

Adding equations (62) to (64), we get

$$\sum_{m=1}^{k} P_m(z) + D(z) + R(z) + R(z) = \frac{N_1(z)}{D_4(z)} \lambda Q_{1,0} \tag{67}$$

where

$$N_1(z) = u_1(z) u_2(z) u_3(z)$$

$$u_1(z) = [(1 - \theta) + \theta V_2^*(\lambda - \lambda z)]V_1^*(\lambda - \lambda z) - 1$$

$$u_2(z) = (\lambda - \lambda z) + \alpha z(1 - G^*(\lambda - \lambda z)S^*(\lambda - \lambda z)$$

$$u_3(z) = \sum_{m=1}^{k} p_m [1 - B_m^*(\lambda - \lambda z + \alpha)]$$

$$D_4(z) = (\lambda - \lambda z)D_3(z).$$

At $z = 1$, $u_1(z)$ becomes

$$u_1(1) = [(1-\theta) + \theta V_2^*(0)]V_1^*(0) - 1 = 0$$

$$u_2(1) - = \lambda - \lambda z + \alpha(1 - G^*(0)S^*(0)) = 0$$

$$u_3(1) = \sum_{m=1}^{k} p_m [1 - B_m^*(\alpha)] = 1 - \sum_{m=1}^{k} p_m B_m^*(\alpha)$$

$$N_1(1) = u_1(1)u_2(1)u_3(1) = 0. \tag{68}$$

Differentiating $N_1(z)$ with respect to z, we get $N_{1'}(1)$ and $N_{1''}(1)$ as

$$N_{1'}(1) = u_1(1)[u_{2'}(1)u_3(1) + u_2(1)u_{3,}(1)] + u_{1'}(1)u_2(1)u_3(1) = 0$$

$$N_{1''}(1) = u_1(1)[u_{2''}(1)u_3(1) + 2u_{2'}(1)u_{3'}(1) + u_2(1)u_{3''}(1)]$$

$$+2u_{1'}(1)[u_{2'}(1)u_3(1) + u_2(1)u_{3'}(1)] + u_{1''}(1)u_2(1)u_3(1)$$

$$= 2u_{1'}(1)u_{2'}(1)u_3(1). \tag{69}$$

Differentiating $u_1(z)$ with respect to z, we get $u_{1'}(1)$ as

$$u_{1'}(1) = (-\lambda)\{[(1-\theta) + \theta V_2^*(0)]V_1^{*'}(0) + \theta V_2^{*'}(0)V_1^*(0)\}$$

$$= \lambda[E(V_1) + \theta E(V_2)]. \tag{70}$$

Differentiating $u_2(z)$ with respect to z, we get $u_{2'}(1)$ as

$$u_{2'}(1) = -\lambda + \alpha(1 - G^*(0)S^*(0)) + \lambda\alpha\{G^{*'}(0)S^*(0) + G^*(0)S^{*'}(0)\}$$

$$= -\lambda[1 + \alpha(E(G) + E(S))]. \tag{71}$$

Using equations (68), (70) and (71) in equation (69), we get

$$N_{1''}(1) = -2\lambda^2[E(V_1) + \theta E(V_2)][1 + \alpha(E(G) + E(S))]\left[1 - \sum_{m=1}^{k} p_m B_m^*(\alpha)\right].$$

At $z = 1$, $D_3(z)$ and $D_4(z)$ becomes

$$D_3(1) = \alpha - [\alpha - \alpha S^*(0)G^*(0)]\sum_{m=1}^{k} p_m B_m^*(\alpha) - \alpha S^*(0)G^*(0) = 0$$

$$D_4(1) = D_3(1) \times 0 = 0.$$

Differentiating $D_4(z)$ with respect to z, we get $D_{4'}(1)$ as

$$D_4(1) = -\lambda D_3(1)(0) + D_{3'}(1)(0) = 0 \tag{72}$$

Differentiating $D'(z)$ with respect to z, we get $D''(1)$ as

$$D_{4''}(1) = D_{3''}(1)(0) - 2\lambda D_{3'}(1) = -2\lambda D_{3'}(1). \tag{73}$$

Differentiating $D_3(z)$ with respect to z, we get $D_{3'}(1)$ as

$$D_{3'}(1) = \lambda[1 + \alpha(E(S) + E(G))]\left\{\sum_{m=1}^{k} p_m B_m^*(\alpha) - 1\right\} + \alpha \sum_{m=1}^{k} p_m B_m^*(\alpha). \tag{74}$$

Using equation (74) in (73), we get

$$D_{4''}(1) = -2\lambda\{\lambda[1 + \alpha(E(S) + E(G))]\left\{\sum_{m=1}^{k} p_m B_m^*(\alpha) - 1\right\} + \alpha \sum_{m=1}^{k} p_m B_m^*(\alpha)\}.$$

Putting $z = 1$ in equation (67), we get

$$\sum_{m=1}^{k} P_m(1) + D(1) + R(1) = \frac{N_1(1)}{D_4(1)} \lambda Q_{1,0} = \frac{0}{0} \text{ for } m.$$

Using L'Hospital's rule, we get

$$\sum_{m=1}^{k} P_m(1) + D(1) + R(1) = \frac{N_{1'}(1)}{D_{4'}(1)} \lambda Q_{1,0} = \frac{0}{0} \text{ for } m.$$

Again using L'Hospital's rule, we get

$$\sum_{m=1}^{k} P_m(1) + D(1) + R(1) = \frac{N_{1''}(1)}{D_{4''}(1)} \lambda Q_{1,0}$$

$$= \frac{\lambda[E(V_1) + \theta E(V_2)][1 + \alpha(E(G) + E(S))]\left[1 - \sum_{m=1}^{k} p_m B_m^*(\alpha)\right]}{\{\lambda[1 + \alpha(E(G) + E(S))]\left\{\sum_{m=1}^{k} p_m B_m^*(\alpha) - 1\right\} + \alpha \sum_{m=1}^{k} p_m B_m^*(\alpha)\}} \lambda Q_{1,0}.$$

Adding equations (65) and (66), we get

$$Q_1(z) + Q_2(z) = \left[\frac{1 - [(1-\theta) + \theta V_2^*(\lambda - \lambda z)]V_1^*(\lambda - \lambda z)}{(1 - z)}\right] Q_{1,0}$$

$$Q_1 + Q_2(1) = \frac{0}{0} \text{ for } m.$$

So after using L'Hospital's rule, we get

$$Q_1(z) + Q_2(z) = \frac{\lambda[(1-\theta) + \theta V_2^*(\lambda - \lambda z)]V_1^{*'}(\lambda - \lambda z) + \lambda \theta V_2^{*'}(\lambda - \lambda z) V_1^{*'}(\lambda - \lambda z)}{(-1)} Q_{1,0}$$

$$Q_1(1) + Q_2(1) = \lambda[E(V_1) + \theta E(V_2)] Q_{1,0}. \tag{75}$$

Using the fact that $\sum_{m=1}^{k} P_m(1) + D(1) + R(1) + Q_1(1) + Q_1(1) + Q_2(2) = 1$, we get

$$Q_{1,0} = \frac{X(z)}{\lambda[E(V_1) + \theta E(V_2)]} \tag{76}$$

where,

$$X(z) = 1 - \rho \tag{77}$$

where

$$\rho = \frac{\lambda[1 + \alpha(E(S) + E(G))]\left\{1 - \sum_{m=1}^{k} p_m B_m^*(\alpha)\right\}}{\alpha \sum_{m=1}^{k} p_m B_m^*(\alpha)}. \tag{78}$$

Using equation (77) in (76), we get

$$Q_{1,0} = \frac{1 - \rho}{\lambda[E(V_1) + \theta E(V_2)]} \tag{79}$$

and $S^{*'}(0) = -E(S)$ is the mean of delay time, $G^{*'}(0) = -E(S)$ is the mean of repair time, $V_1^{*'}(0) = -E(V_1)$, $V_2^{*'}(0) = -E(V_2)$ are the mean of vacation times of FRV and SOV respectively. $Q_{1,0}$ is the steady state probability that the system is idle due to server's vacation. Also we have $\rho < 1$, which is the stability condition under which steady state solution exists.

$$P(z) = \sum_{m=1}^{k} P_m(z) + D(z) + R(z) + Q_1(z) + Q_2(z) = \frac{N(z)}{D(z)} \lambda Q_{1,0}$$

$$N(z) = \{[(1-\alpha) + \theta V_2^*(\lambda - \lambda z)]V_1^*(\lambda - \lambda z) - 1\}\{\lambda - \lambda z + \alpha z(1 - G^*(\lambda - \lambda z)$$

$$\times S^*(\lambda - \lambda z)\}\{1 - \sum_{m=1}^{k} p_j B_j^*(\lambda - \lambda z + \alpha)]\} - z(\lambda - \lambda z + \alpha)$$

$$+ [\lambda - \lambda z + -\alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)]$$

$$\sum_{m=1}^{k} p_m B_m^*(\lambda - \lambda z + \alpha) + \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)\}$$

$$= \{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)]V_1^*(\lambda - \lambda z) - 1\}\{\lambda - \lambda z + \alpha z - \alpha z G^*(\lambda - \lambda z) S^*(\lambda - \lambda z)$$

$$- \sum_{m=1}^{k} p_m B_m^*(\lambda - \lambda z + \alpha)\{\lambda - \lambda z + \alpha z - \alpha z G^*(\lambda - \lambda z) S^*(\lambda - \lambda z)\}$$

$$- z(\lambda - \lambda z) - \alpha z + \sum_{m=1}^{k} p_m B_m^*(\lambda - \lambda z + \alpha)[\lambda - \lambda z + \alpha - \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)]$$

$$+ \alpha z S^*(\lambda - \lambda z) G^*(\lambda - \lambda z)\}$$

$$= \{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)]V_1^*(\lambda - \lambda z) - 1\}\{\lambda(1-z) + \alpha(1-z)$$

$$\sum_{m=1}^{k} p_m B_m^*(\lambda - \lambda z + \alpha) - z\lambda(1-z)\}$$

$$= u_1(z) u_2(z),$$

where

$$u_1(z) = [(1-\theta) + \theta V_2^*(\lambda - \lambda z)]V_1^*(\lambda - \lambda z) - 1$$

$$u_2(z) = \lambda(1-z) + \alpha \sum_{m=1}^{k} p_m B_m^*(\lambda - \lambda z + \alpha)$$

$$D(z) = \lambda D_3(z)$$

$P(z)$ is the probability generating function for the number of customers in the queue.

4. Performance Measures

Let L_q and L denote the steady state average queue size and system size respectively.

Then $L_q = \left[\frac{d}{dx} P(z)\right]_{z=1} = \frac{d}{dx}\left[\frac{N(z)}{D(z)} Q_{1,0}\right]_{z=1}$

$$u_1(1) = [(1-\theta) + \theta V_2^*(0)]V_1^*(0) - 1 = 0$$

$$u_2(1) = \lambda(1-1) + \alpha \sum_{m=1}^{k} p_m B_m^*(\alpha) = \alpha \sum_{m=1}^{k} p_m B_m^*(\alpha) \tag{80}$$

$$N(1) = u_1(1) u_2(1) = 0$$

$$D_3(1) = \alpha - [\alpha - \alpha S^*(0) G^*(0)] \sum_{m=1}^{k} p_m B_m^*(\alpha) - \alpha S^*(0) G^*(0) = 0$$

$$D(1) = \lambda D_3(1) = 0$$

$$L_q = \frac{d}{dz}\left[\frac{N(z)}{D(z)}\right] Q_{1,0} \text{ at } z = 1$$

$$= \frac{D(z)N'(z) - N(z)D'(z)}{(D(z))^2} Q_{1,0} \text{ at } z = 1$$

$$= \frac{D(1)N'(1) - N(1)D'(1)}{(D(1))^2} Q_{1,0}$$

$$= \frac{D(1)N''(1) - N(1)D''(1)}{(D(1))^2} Q_{1,0}$$

$$= \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} Q_{1,0}. \tag{81}$$

Differentiating $N(z)$ with respect to z, we get $N'(1)$ as

$$N'(1) = u_{1'}(1)u_2(1) + u_1(1)u_{2'}(1)$$

$$= u_{1'}(1)u_2(1). \quad (82)$$

Differentiating $N'(z)$ with respect to z, we get $N''(1)$ as

$$N''(1) = u_{''}(1)u_2(1) + 2u_{1'}(1)u_{2'}(1) + u_1(1)u_{2''}(1)$$

$$= u_{1''}(1)u_2(1) + 2u_{1'}(1)u_{2'}(1). \quad (83)$$

Differentiating $u_1(z)$ with respect to z, we get $u_{1'}(1)$ as

$$u_{1'}(1) = (-\lambda)\{[(1-\theta) + \theta V_1^*(0)]V_1^*(0) + \theta V_2^*(0) + \theta V_2^{*'}(0)V_1^{*'}(0)\}$$

$$= \lambda[E(V_1) + \theta E(V_2)]. \quad (84)$$

Differentiating $u_{1'}(z)$ with respect to z, we get $u_{1''}(1)$ as

$$u_{1''}(1) = \lambda^2[E(V_1^2) + \theta E(V_2^2) + 2\theta E(V_1)E(V_2)]. \quad (85)$$

Differentiating $u_2(z)$ with respect to z, we get $u_{2'}(1)$ as

$$u_{2'}(1) = -\lambda\left\{1 + \alpha\sum_{m=1}^{k} p_m B_m^{*'}(\alpha)\right\} \quad (86)$$

Using equations (80) and (84) in equation (83), we get

$$N'(1) = \lambda\alpha\,[E(V_1) + \theta E(V_2)]\sum_{m=1}^{k} p_m B_m^*(\alpha).$$

Using equations (80), (84), (85) and (86) in equation (83), we get

$$N''(1) = \lambda^2\{\alpha[E(V_1^2) + \theta E(V_2^2) + 2\theta E(V_1)E(V_2)]\sum_{m=1}^{k} p_m B_m^*(\alpha)$$

$$-2\{1 + \alpha\sum_{m=1}^{k} p_m B_m^*(\alpha)\}\,[E(V_1) + \theta E(V_2)]\}.$$

Differentiating $D(z)$ with respect to z, we get $D'(z)$ as

$$D'(z) = \lambda D_{3'}(z)$$

$$D'(1) = \lambda D_{3'}(1) \tag{87}$$

$$D''(z) = \lambda D_{3''}(z)$$

$$D''(1) = \lambda D_{3''}(1) \tag{88}$$

Using equation (74) in (87), we get

$$D'(1) = \lambda^2 [1 + \alpha(E(S) + E(G))] \left\{ \sum_{m=1}^{k} p_m B_m^*(\alpha) - 1 \right\} + \alpha \sum_{m=1}^{k} p_j B_j^*(\alpha).$$

Differentiating $D_{3'}(z)$ with respect to z, we get $D_{3''}(1)$ as

$$D_{3''}(1) = -2\lambda - 2\lambda^2 \sum_{m=1}^{k} p_m B_m^{*'}(\alpha) \{1 + \alpha[E(S) + E(G)]\}$$

$$- 2\alpha\lambda \sum_{m=1}^{k} p_m B_m^*(\alpha) - \lambda^2 \alpha \{1 - \sum_{m=1}^{k} p_m B_m^*(\alpha)\} [E(S^2) + 2E(S)E(G) + E(G^2)]$$

$$+ 2\alpha\lambda \left\{ \sum_{m=1}^{k} p_m B_m^*(\alpha) + 1 \right\} [E(S) + E(G)]. \tag{89}$$

Using equation (89) in (88), we get

$$D''(1) = \lambda^2 \{-2 - 2\lambda \sum_{m=1}^{k} p_m B_m^{*'}(\alpha) \{1 + \alpha[E(S) + E(G)] - 2\alpha \sum_{m=1}^{k} p_m B_m^{*'}(\alpha)$$

$$-\lambda\alpha \left\{ \sum_{m=1}^{k} p_m B_m^*(\alpha) \right\} [E(S^2) + 2E(S)E(G) + E(G^2)]$$

$$+ 2\alpha \left\{ \sum_{m=1}^{k} p_m B_m^*(\alpha) + 1 \right\} [E(S) + E(G)]\}$$

where $E(S^2)$, $E(G^2)$, $E(V_1^2)$, $E(V_2^2)$ are the second moments of delay, repair, FRV and SOV time respectively. L can be obtained using the relation

$L = L_q + \rho$. Using Little's formula, we obtain W_q, the average waiting time in the queue and W, the average waiting time in the system, as $W_q = \frac{L_q}{\lambda}$ and $W = \frac{L}{\lambda}$ respectively.

5. Particular case

If there is only one service $(k = 1)$ and no breakdown $(\alpha = 0)$ then, we get

$$P(z) = \frac{(1-\rho)\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)]V_1^*(\lambda - \lambda z) - 1\}}{\{z - B_1^*(\lambda - \lambda z)\}\lambda[E(V_1) + \theta E(V_2)]}$$

which coincides with the probability generating function obtained by Gautam Choudhury irrespective of the notations used.

6. Numerical illustration

To illustrate the effect of some parameters on the system performance measures, we present a numerical example by considering service times and vacation times as exponentially distributed as follows.

Assuming the values $k = 3, p_1 = 0.2, p_2 = 0.5, p_3 = 0.3, \mu_1 = 9, \mu_2 = 12$ $v_1 = 1.5, v_2 = 0.5\eta = 5, v = 5, \alpha = 1$ for the parameters, subject to the stability condition and varying the values of θ from 0.43 to 0.53 in steps of 0.01 and λ from 1 to 2 in steps of 0.1 we have calculated the values of L_q, the corresponding graph is drawn in figure 1. From figure-1, one can notice that the surface displays an upward trend as expected for increasing value of the arrival rate λ and SOV probability θ against the average queue size L_q.

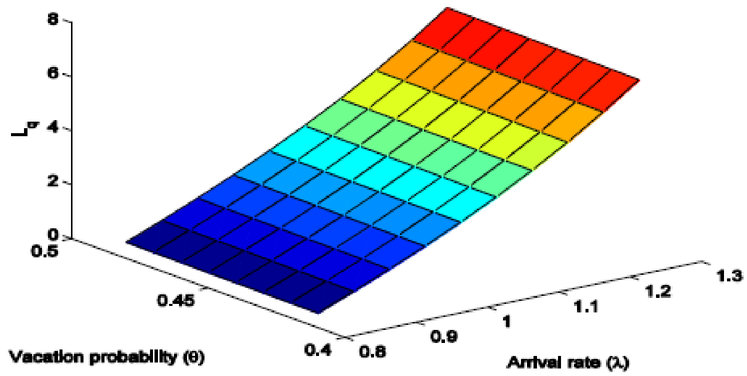

Figure-1.

We consider service times and vacation times are distributed as Erlang-2 distribution. Assuming the values $k = 3, p_1 = 0.2, p_2 = 0.5, p_3 = 0.3, \mu_1 = 7,$ $\mu_2 = 8, \mu_3 = 5, v_1 = 1, v_2 = 1\ \eta = 5, v = 5, \alpha = 1$ for the parameters, subject to the stability condition and varying the values of θ from 0.21 to 0.28 in steps of 0.01 and λ from 1 to 1.14 in steps of 0.02 we have calculated the values of L_q, the corresponding graph is drawn in Figure-2. From Figure-2, we see that the surface displays an upward trend as expected for increasing values of the arrival rate λ and SOV probability θ against the average queue size L_q.

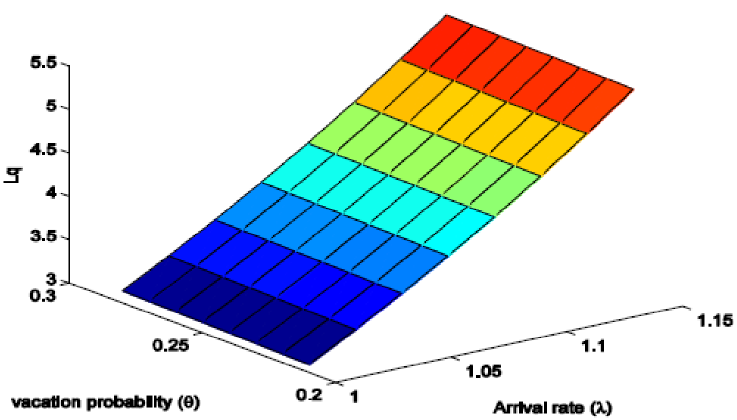

Figure-2.

References

[1] Y. Baba, On the $M^X/G/1$ queue with vacation time, Operation Research Letters 5 (1986), 93-98.

[2] A. Badamchi Zadeh and G. H. Shankar, A two phase queue system with Bernoulli feedback and Bernoulli schedule server vacation, Information and Management Sciences 19 (2008), 329-338.

[3] A. Borthakur and G. Chaudhury, On a batch arrival Possion queue with generalised vacation, Sankhya Ser.B 59 (1997), 369-383.

[4] G. Chaudhury, An $M^X/G/1$ queueing system with a set up period and a vaction period, Questa 36 (2000), 23-28.

[5] B. D. Choi and K. K. Park, The $M/G/1$ retrial queue with Bernoulli schedule, Queueing Systems 7 (1990), 219-228.

[6] M. Cramer, Stationary distributions in a queueing system with vaction times and limited service, Queueing Systems 4 (1989), 57-68.

[7] B. T. Doshi, Queueing systems with vacations-a survey, Queueing Systems 1 (1986), 29-66.

[8] B. T. Doshi, Conditional and unconditional distributions for $M/G/1$ type queues with server vacation, Questa 7 (1990), 229-252.

[9] S. Fuhrmann, A note on the $M/G/1$ queue with server vactions, Questa 31 (1981), 13-68.

[10] B. R. K. Kashyap and M. L. Chaudhry, An introduction to Queueing theory, Kingston, Ontario, 1988.

[11] J. Keilson and L. D. Servi, Oscillating random walk models for $G/G/1$ vacation systems with Bernoulli schedules, Journal of Applied Probability 23 (1986), 790-802.

[12] Y. Levi and U. Yechilai, An $M/M/s$ queue with server vactions, Infor. 14 (1976), 153-163.

[13] K. C. Madan, On a $M^X/M^b/1$ queueing system with general vacation times, International Journal of Information and Management Sciences 2 (1991), 51-61.

[14] K. C. Madan, An $M/G/1$ queue with optional deterministic server vacations, Metron, 7 (1999), 83-95. $M/G/1$ Feedback queue with two types of service 33

[15] K. C. Madan and R. F. Anabosi, A single server queue with two types of service, Bernoulli schedule server vacations and a single vacation policy, Pakistan Journal of Statistics 19 (2003), 331-342.

[16] K. C. Madan and G. Choudhury, A two stage arrival queueing system with a modified Bernoulli schedule vacation under N-policy, Mathematical and Computer Modelling 42 (2005), 71-85.

[17] E. Rosenberg and U. Yechiali, The $M^X/G/1$ queue with single and multiple vacations under LIFO service regime, Operation Research Letters 14 (1993), 171-179.

[18] H. Takagi, Queueing Analysis: A foundation of performance evalution, Vol 1, North Holland, Amsterdam, 1993.

[19] H. Takagi, Time dependent process of $M/G/1$ vacation models with exhaustive service, Journal of Applied Probability 29 (1992), 418-429.