Advances and Applications in Mathematical Sciences
Volume 21, Issue 2, December 2021, Pages 1029-1036


# BILINEAR TRANSFORMATION

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## Abstract

A formula is derived and demonstrated that is capable of directly generating digital filter coefficient from an Analog filter prototype using the bilinear transformation. This formula obviates the need for any algebraic manipulation of the Analog prototype filter and ideal for use in embedded systems that must be take in any general Analog filter specification and dynamically generate digital filter coefficient directly usable in difference equations.

## Introduction

The bilinear transformation is also known as Tustin's method is used in digital signal processing and discrete-time control theory to transform continuous-time system representations to discrete-time and vice-versa.

A function $f : C \to C$ can be thought of as a transformation from one complex to another complex plane. Hence the nature of complex function can be described the manner which it maps regions and curves from one complex to another complex plane. In this chapter we shall discuss bilinear transformation and see how various regions are transformed by the transformations.

**Definition.** A Bilinear transformation is defined as

2020 Mathematics Subject Classification: 15A04.
Keywords: Bilinear transformation, Elementary transformation, Translation.
Received November 2, 2021; Accepted November 15, 2021

$$Z = \frac{a + bz}{c + dz}$$

Where $a$, $b$, $c$, $d$ are constants (complex in general) and $z$ is an independent complex variables being mapped into dependent complex variable $Z$ as illustrated.

**Definition 1.1.** Let $a, b, c, d$ €C $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0.$ We define a bilinear transformation or a mobius transformation $T : C_\infty \to C_\infty$ as:

For $c \neq 0$

$$T(z) = \left\{ \frac{az + b}{cz + d}, \quad z \in C \left\{ -\frac{d}{c} \right\} \right.$$

$$T(z) = \left\{ \frac{a}{c}, \; z = \infty \right\}$$

$$T(z) = \left\{ \infty, \; z = -\frac{d}{c} \right\},$$

and for $c = 0$

$$T(z) = \left\{ \frac{az + b}{d}, \; z \in C \right.$$

$\{\infty, \; z = \infty.$

In general we will write a bilinear transformation $T : C_\infty \to C_\infty$ as $w = T(z) = \frac{az + b}{cz + d}, \; ad - bc \neq 0.$

Without any ambiguity.

**Example.**

$$T(Z) = \frac{2z + 3i}{iz + 5}, \; S(Z) = \frac{iz - 6}{3z}, \; T_1(Z) = 2z + 3, \; T_2(Z) = 2z,$$

$$S_1(Z) = \frac{3}{z} \text{ etc.,}$$

**Definition 1.2.** (Elementary Bilinear Transformation.)

**1. Translation.** We define a translation as $T(z) = z + a$, where a is a finite complex number, i.e. a € C. since $T(z) = \frac{1z + a}{0z + 1}$ and1. 1-a. $0 = 1 \neq 0$, so $T$ is bilinear transformation.

**2. Inversion.** We define inverse $T(z) = \frac{1}{z}$ since $T(z) = \frac{0z + 1}{1z + 0}$ and $0 \cdot 0 - 1 \cdot 1 = -1 \neq 0$, so $T$ is a bilinear transformation.

**3. Rotation.** We define a rotation as $T(z) = e^{i\theta}z$, $\theta \epsilon R\{0\}$. Since $T(z) = \frac{e^{i\theta}z + 0}{0z + 1}$ and $e^{i\theta} \neq 0$, so $T$ is bilinear transformation.

**4. Magnification.** We define a magnification $T(z) = rz$, $r$ € $R^{+}$. Since $T(z) = \frac{rz + 0}{0z + 1}$ and $r \cdot 1 - 0 \cdot 0 = r \neq 0$, so $T$ is bilinear transformation.

**Theorem 1.1.** *Every bilinear transformation is a composition of elementary bilinear transformation, i.e. composition of translation, inversion and dilation (one or two of them may be missing).*

**Proof.** Let us consider a bilinear transformation $T : C_\infty \to C_\infty$ defined as $T(Z) = \frac{az + b}{cz + d}$ where $a, b, c, d$ £C and $ad - bc \neq 0$

**Case.** $Ic = 0$

Hence $ad \neq 0$, i.e. $a \neq 0 \neq d$ and $T(z) = \frac{az + b}{d}$

$$= \frac{a}{d}z + \frac{b}{d}$$

$= T1(z) + \frac{b}{d}$ where $T1(z) = \frac{a}{d}z$ and $\frac{a}{d} \neq 0T1$ is a dilation.

$T2(T1(z))$, where $T2(z) = z + \frac{b}{d}a$ translation

Hence $T = T2 \circ T1$.

**Case II.** $c \neq 0$

Now $T(z) = \frac{az+d}{cz+d} - \frac{a}{c} + \frac{a}{c}$

$= \frac{(bc-ad)}{c^2} \frac{1}{z+\frac{d}{c}} + \frac{a}{c}$ here $\frac{bc-ad}{c^2} \neq 0$

$= \left(\frac{bc-ad}{c^2}\right)\frac{1}{T1(z)} + \frac{a}{c}$ where $T1(z) = z + \frac{d}{c}a$ translation

$= \left(\frac{(bc-ad)}{c^2}\right) T2(T1(z)) + \frac{a}{c}$ where $T2(z) = \frac{1}{z}$ is the inversion

$= T3(T2(T1(z))) + \frac{a}{c}$ where $T3(z) = \frac{bc-ad}{c^2} z$ is dilation

$= T4(T3(T2(T1(z))))$, where $T4(z) = z + \frac{a}{c}$ is translation.

Hence $T = T4 \circ T3 \circ T2 \circ T1$. Hence the proof.

**Definition.** Let $X$ be a nonempty set, and $f : X \to X$, a €X is said to be fixed point of $f$ if $f(a) = 0$.

**Example.** 1. If $f : R \to R$ be the mapping defined as $f(x) = x^2$, 0 and 1 are fixed points of $f$.

2. If $f : R \to R$ be the mapping defined as $f(x) = x^3$, 0 and ± 1 are fixed points off.

3. If $f : R \to R$ be the mapping defined as $f(x) = \sin(x)$, 0 is only fixed points of $f$.

Note: for $f : X \to X$, the fixed points can be find outs by solving the equation $f(x) = x$ on $X$.

**Example.** Find all fixed points of the bilinear transformation $T(z) = \frac{3z+2}{z-5}$

**Solution.** Here $T(\infty) = \frac{3}{1} = 3 \neq \infty$, $\infty$ is not fixed point of $T$.

So all possible fixed points are finite complex numbers and they are given by the solution of equation $Z = T(z)$ in $C$, i.e.

$$z = \frac{3z + 2}{z - 5}$$

$$\text{i.e. } z^2 - 8z - 2 = 0$$

And the solution are $4 \pm 3\sqrt{2}$.

Hence the fixed points of $T$ are $4 \pm 3\sqrt{2}$.

**Theorem.** *Any bilinear transformation has at most two fixed points unless it is the identity transformation.*

**Proof.** Let us consider a bilinear transformation $T : C_\infty \to C_\infty$ defined as $T(Z) = \dfrac{az + b}{cz + d}$ where $a, b, c, d \in C$ and $ad - bc \neq 0$.

**Case I.** $c = 0$.

Hence $ad \neq 0$, i.e. $a \neq 0 \neq d$, and

$$T(z) = \frac{az + b}{d} = \frac{a}{d} z + \frac{b}{d}.$$

Here $\dfrac{a}{b} \neq 0$ a finite complex number.

So $T(\infty) = \infty$, i.e. $\infty$ is a fixed point in this case.

The other possible fixed points of $T$ are given by the solution of the equation $z = T(z)$, i.e.

$$z = \frac{a}{d} z + \frac{b}{d} \text{ in } C.$$

Now three subcases arise here.

Subcases $IA : a = d, b = 0,$

Subcases $IB : a = d, b \neq 0,$

Subcases $IC : a \neq d.$

Subcases $IA : a = d, b = 0,$

In this subcases $T = I,$ the identity transformation, i.e. every points of $C$ is also its fixed points.

Hence in this subcases every points of $C\infty$ is a fixed points of $T$.

Subcases $IB : a = d, b \neq 0,$

Here equation $z = T(z)$ has no solution in $C$.

Hence inthis subcases there is only one fixed points of $T$ which is $\infty$.

Subcases $IC : a \neq d.$

Here $z = \frac{b}{d-a}$ is the only fixed points of $T$ in $C$.

Hence in this subcases there are two fixed points and fixed points of $T$ are $\infty$ and $\frac{b}{d-a}$.

**Cases II** $c \neq 0.$

Now $T(\infty) = \frac{a}{c} \neq \infty.$ So $\infty$ is not a fixed point of $T$.

So all possible fixed points of $T$ are in $C$ and they are given by the solution of the equation $Z = T(z)$ i.e. $\frac{az+b}{cz+d}$ i.e. $cx^2 + (d-a)z - b = 0$ in C … (i)

Fundamental theorem of algebra says this equation has at most two solutions in $C$, i.e.

In this case $T$ has at most two fixed points and the are given by the solutions of the equation (i) In $C$.

Thus every bilinear transformation has most two fixed points unless it is the identity transformation.

**Corollary.** *If a bilinear transformation have three fixed points* (*or more than two fixed points*) *Then it the identity transformation*.

**Problem.** Show that $w = \dfrac{z-1}{z+1}$ maps the imaginary axis in the $z$-plane onto the circle $|w| = 1$. What are portion of the $z$-plane corresponds to interior of the circle $|w| = 1$.

**Solution.**

$|w| = 1 \leftrightarrow \left| \dfrac{z-1}{z+1} \right| = 1$

$\leftrightarrow |z-1| = |z+1|$

$\leftrightarrow |x + iy - 1| = |x + iy + 1|$

$\leftrightarrow |(x-1)^2 + y^2| = (x+1)^2 + y^2$

$\leftrightarrow x = 0.$

Hence the trans formation $w = \dfrac{z-1}{z+1}$ maps the imaginary axis $x = 0$ onto the circles $|w| = 1$.

Alsosince the points $z = 1$ is mapped to $w = 0$ it follows that half plane $x > 0$ is mapped onto the interior of the $|w| = 1$.

## Conclusion

The bilinear coefficient formula was presented derived and demonstrated. This formula shown to be powerful tool for filter generation. Once the design consideration are understood there is real benefits to be gained by these formula in the systems.

The system must be able to deal with the potential numerical problems resulting from high order, non-sectioned filters, usually though the use of high precisions/ range floating point Arithmetic and number format.

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