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THE CYCLIC CONNECTIVITY NUMBER OF AN ARITHMETIC GRAPH $G = V_n$

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Abstract

A set S is a cyclic vertex cut if $G - S$ is disconnected and at least two of its components contain cycle. The minimum cardinality of cyclic vertex cut is called the cyclic connectivity number and it is denoted by $\kappa_c(G)$. In this article, the cyclic connectivity of an arithmetic graph is studied. We catogorised arithmetic graphs which are cyclically separable and not cyclically separable. Also, it is shown that $\kappa_c(G) = \infty$ where the number of primes in n does not exceed two.

1. Introduction

For notations and graph theory terminologies, we follow [3]. Plummer studied the cyclic connectivity of planar graphs in [8]. The definition of the cyclic connectivity is from [8]. The definition of an arithmetic graph is studied from [10]. The arithmetic graph V_n is defined as a graph with its vertex set is the set consists of the divisors of n (excluding 1) where n is a positive integer and $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$ where p_i's are distinct primes and $a_i's \geq 1$ and two distinct vertices a, b which are not of the same parity are adjacent in this

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graph if $(a, b) = p_i$, for some $i, 1 \leq i \leq r$. The vertices a and b are said to be of the same parity if both a and b are the powers of the same prime, for instance $a = p^3, b = p^4$. In [5], the connectivity number of an arithmetic graph is studied by L. Mary Jenitha and S. Sujitha. Later, the various parameters of connectivity of an arithmetic graph are studied by the same authors in [4] [6] [7]. Also, various authors studied different parameters of an arithmetic graph. In this paper, we investigated the cyclic connectivity concepts for an arithmetic graph $G = V_n$. The following theorems and definitions are used in sequel.

Definition 1.1. [1] The common neighbourhood (CN-neighborhood) of a vertex $v \in V(G)$ denoted by $N_{cn}(v)$ is defined as $N_{cn}(v) = \{u \in V(G) : uv \in E(G)$ and $|\Gamma(u, v)| \geq 1\}$, where $|\Gamma(u, v)|$ is the number of common neighbourhood between the vertices u and v.

Definition 1.2. [8] The vertex set S is a cyclic vertex cut if $G - S$ is disconnected and at least two of its component contains cycle. The minimum cardinality of cyclic vertex cut is called cyclic connectivity number and it is denoted by $\kappa_c(G)$.

Theorem 1.3. [9] *For an arithmetic graph* $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$, *then the number of vertices of G is* $|V| = \prod_{i=1}^{r} (a_i + 1) - 1$.

Theorem 1.4. [6] *For an arithmetic graph* $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2}$ *where* p_1 *and* p_2 *are distinct primes,* $a_1, a_2 \geq 1$ *then* $\epsilon = 4a_1a_2 - a_1 - a_2$, *where* ϵ *is the size of the graph G.*

Theorem 1.5. [6] *For an arithmetic graph* $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2}$ *where* p_1 *and* p_2 *are distinct primes,* $a_1, a_2 \geq 1$ *then G is a bipartite graph.*

Theorem 1.6. [6] *Let* $G = V_n$, *be an arithmetic graph where* $n = p_1^{a_1} \times p_2^{a_2}$ *then* $\Delta(G) = \begin{cases} [a_j \prod_{i=1,\, i \neq j}^{r} (a_i + 1) - 1] - |a_j - 1| \\ a_i + a_j, \text{ for } a_j > a_i = 1 \end{cases}$ *for*

$a_j \geq a_i \geq 2$ $\delta(G) = 2$.

Theorem 1.7. [6] *Let* $G = V_n$ *an arithmetic graph* $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$, *for any vertex* $u = \prod_{i \in B} P_i^{\alpha_i}$ *where* $B \subseteq 1, 2, 3, \ldots, r, 1 \leq \alpha_i \leq a_i$ $\forall i \in B$. (1) *If* $u = p_j$ *where* $j \in 1, 2, 3, \ldots, r$, *then* $\deg(u) = [a_j \prod_{i=1,\, i \neq j}^{r} (a_i + 1) - 1] - | a_j - 1 |$.

(2) *If* $u = p_i^{\alpha_i} 1 < \alpha_i \leq a_i \forall i \in B$, *then* $deg(u) = [\prod_{i=1,\, i \notin B}^{r} (a_i + 1)] - 1$

(3) *If* $u = \prod_{i \in B,} p_i^{\alpha_i}, |B| \geq 2, 1 < \alpha_i \leq a_i, \forall i \in B$ *then* $deg(u) = |B| \prod_{i=1,\, i \notin B}^{r} (a_i + 1)$

(4) *If* $u = \prod_{i \in B} p_i^{\alpha_i}, \alpha_i = 1$ *for some* $i \in B' \subseteq B$, *then* $deg(u) = [| B - B' | + \sum_{i \in B'} a_i] \prod_{i=1,\, i \notin B}^{r} (a_i + 1)$ *where* B *is the number of primes product in* u, B' *is the number of primes having power* 1 *in chosen vertex* u.

Theorem 1.8. [6] *Let* $G = V_n$ *an arithmetic graph* $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$, $r > 2$ *such that at least one of* $a_i, i \in \{1, 2, \ldots, r\}$ *does not equal one. Then* $\Delta(G) = [a_j \prod_{j=1,\, i \neq j}^{r} (a_i + 1) - 1] - | a_j - 1 |$ *where* a_j *is the maximum exponent of* p_i, $i \in \{1, 2, \ldots, r\}$, $\delta(G) = r$.

Theorem 1.9. [9] *Let* G *be a* V_n *arithmetic graph, where* $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$, *such that at least one of* a_i, $i \in \{1, 2, 3, \ldots, r\}$ *does not equal one. Then,* (1) $\Delta(G) = a_j \prod_{j=1,\, i \neq j}^{r} (a_i + 1) - 1$ *where* a_j is *the maximum exponent of* p_i, $i \in \{1, 2, \ldots, r\}$, (2) $\delta(G) = r$.

2. Infinite Cyclic Connectivity of an Arithmetic Graph $G = V_n$

In this section, we identify the graphs which are not cyclically separable and their connectivity number $\kappa_c(G) = \infty$. We observe that, the cyclic connectivity number $\kappa_c(G) = \infty$, if the number of primes in n doesn't exceed two.

Theorem 2.1. *For an arithmetic graph* $G = V_n$, $n = p_1 \times p_2 \times p_3 \times \ldots \times p_r$, *for* $r \leq 4$, *has no cyclic vertex cut. In particular*, G *is infinite cyclic connectivity*.

Proof. Case (i) $r = 2$

In this case, the arithmetic graph $G = V_n$, $n = p_i \times p_j$ is a tree. Hence no cyclic vertex cut.

Case (ii) $r = 3$

Here $n = p_i \times p_j \times p_k$, then the vertex set $V(G) = \{p_i, p_j, p_k, p_i \times p_j, p_i \times p_k, p_j \times p_k, p_i \times p_j \times p_k\}$ and $|V(G)| = 7$. In this arithmetic graph, the possible cycles are C_3 and C_4 that will be discussed in the following sub cases.

Subcase (i) Choose a cycle of length three. Let $V(G) = \{p_i, p_i \times p_j, p_i \times p_k\}$. Clearly $V(G) - V(C) = \{p_j, p_k, p_j \times p_k, p_i \times p_j \times p_k\}$ are adjacent to at least one vertex of the cycle. Hence $V(G) - N[C] = \Phi$.

Subcase (ii) Suppose $V(G) = \{p_i, \times p_j, p_j \times p_k, p_i \times p_k\}$, then by the definition of an arithmetic graph the set of vertices $\{p_i, p_j, p_k\}$ are adjacent to at least one vertex of the cycle C. Hence $V(G) - N[C] = \{p_i \times p_j \times p_k\}$ an isolated vertex.

Subcase (iii) Suppose we choose any cycle of length four. Let $V(C) = \{p_i, p_i \times p_k, p_k, p_i \times p_i \times p_k\}$, then the set of vertices say $\{p_j, p_i \times p_j, p_j \times p_k\}$ are adjacent to at least one vertex of the cycle C. Hence $V(G) - N[C] = \Phi$. Thus there is no cyclic vertex cut for $r = 3$.

Case (iii) $r = 4$

In this case $n = p_i \times p_j \times p_k \times p_l$, and the vertex set $V(G) = \{p_i, p_j, p_k, \times p_l, p_i \times p_j \times p_i \times p_k \times p_i \times p_l, p_j \times p_k, p_j \times p_{l,} p_k \times p_l, p_i \times p_j \times p_k, p_i \times p_j \times p_l, p_j \times p_k \times p_l, p_i \times p_k \times p_l, p_i \times p_j \times p_k \times p_l\}$.

Subcase (i) Choose a cycle C of length three, whose vertices say $V(C) = \{p_i \times p_j, p_i \times p_k, p_j \times p_k\}$. These vertices are not adjacent to vertices such as $\{p_l, p_i \times p_j \times p_k, p_i \times p_j \times p_k \times p_l\}$. The induced graph of $V(G) - N[C]$ is an edge and an isolated vertex and hence there is no cyclic vertex cut.

Subcase (ii) Choose a cycle C of the form say $V(C) = \{p_i, p_i \times p_j, p_i \times p_k \times p_l\}$. Here $V(G) - N[C] = \{p_k \times p_l, p_i \times p_j \times p_k \times p_l\}$. The induced graph of $V(G) - N[C]$ is two isolated vertices and hence there is no cyclic vertex cut.

Subcase (iii) Choose a cycle C of length four for this subcase, which is of the form say $V(C) = \{p_i, p_i \times p_k \times p_l, p_j \times p_k, p_i \times p_j \times p_l\}$. The vertex set of $V(G) - N[C] = \{p_j \times p_k \times p_l\}$. Hence there is no cyclic vertex cut.

Subcase (iv) Choose a cycle C of length four. Here $V(C) = \{p_i, p_j, p_i \times p_j \times p_l, p_i \times p_j \times p_k \times p_l\}$. The vertex set of $V(G) - N[C] = \{p_k \times p_l\}$. Thus, for the above four sub cases we can identify that there is no cyclic vertex cut. Also, these are the only possibilities for choosing the cycle. Hence $\kappa_c(G) = \infty$.

Theorem 2.2. *For an arithmetic graph* $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2}$, $a_i \geq 1$, *for* $i = 1, 2$ *has no cyclic vertex cut and hence* $\kappa_c(G) = \infty$.

Proof. By Theorem 1.5, the given graph G is a bipartite graph with two partition A and B. Let A be the set of prime and prime power vertices of G and B be the product of primes and product of prime power vertices of G.

Case (i) $a_i = 1$, for $i = 1, 2$

The proof follows from case (i) of Theorem 2.1 and hence $\kappa_c(G) = \infty$.

Case (ii) $a_1 > 1$, $a_2 = 1$

Here $V(G) = \{p_1, p_1^2, p_1^3, p_1^4 \ldots, p_1^{a_1}, p_2, p_1 \times p_2, p_1^2 \times p_2, \ldots, p_1^{a_1} \times p_2\}$. In this case $A = \{p_1, p_1^2, p_1^3, p_1^4, \ldots, p_1^{a_1}, p_2\}$ and $B = \{p_1 \times p_2, p_1^2 \times p_2, \ldots, p_1^{a_1} \times p_2\}$. Since the graph is a bipartite graph there is no possibility for having cycle of length 3. Also, we can observe that, in the partition A, all the vertices other than p_1 and p_2 are pendent vertices. So there is no possibility for having two vertex disjoint cycles and hence $\kappa_c(G) = \infty$.

Case (iii) $a_1 = a_2 = 2$,

Let $V(G) = \{p_1, p_1^2, p_2, p_2^2, p_1 \times p_2, p_1^2 \times p_2, p_1 \times p_2^2, p_1^2 \times p_2^2\}$. Clearly G has the partition $A = \{p_1, p_1^2, p_2, p_2^2\}$ and $B = \{p_1^2 \times p_2, p_1 \times p_2^2, p_1^2 \times p_2^2\}$. By proof of Theorem 1.4 in the first partition two vertices have only one common adjacent vertex in the second partition and hence there is no possibility for having vertex disjoint cycles. Therefore $\kappa_c(G) = \infty$.

Case (iv) $a_1 > 2$, $a_2 \geq 2$,

The vertex set of G be $V(G) = \{p_1, p_1^2, p_1^3, p_1^4, \ldots, p_1^{a_1}, p_2, p_2^2, p_2^3, p_2^4, \ldots, p_2^{a_2}, p_1 \times p_2, p_1^2 \times p_2, \ldots, p_1^{a_1} \times p_2, p_2^2 \times p_1, p_2^3 \times p_1, p_2^4 \times p_1, \ldots, p_2^{a_2} \times p_1, \ldots, p_1^{a_1} \times p_2^{a_2}\}$. Let us choose any cycle of length four say $V(C) = \{p_1^{a_1}, p_1 \times p_2^{a_2}, p_2, p_1 \times p_2\}$. Since it is necessary to choose prime, prime power vertices from the first partition to form a cycle C we have $V(G) - N[C] = \Phi$.

Example 2.3. Consider the graph $G = V_{210}$, $210 = 2 \times 3 \times 5 \times 7$. The vertex set $V(G) = \{2, 3, 5, 7, 2 \times 3, 2 \times 5, 2 \times 7, 3 \times 5, 3 \times 7, 5 \times 7, 2 \times 3 \times 5, 2 \times 3 \times 7, 3 \times 5 \times 7, 2 \times 5 \times 7, 2 \times 3 \times 5 \times 7\}$. Let $V(C) = \{2 \times 3, 2 \times 5, 3 \times 5\}$ be the vertices of the cycle of length 3 having degree sum minimum. The induced graph of $V(G) - N[C]$ has an edge with vertices 7 and $2 \times 3 \times 5 \times 7$ and an isolated vertex $2 \times 3 \times 5$. Thus there is no cyclic vertex cut. Hence $\kappa_c(G) = \infty$.

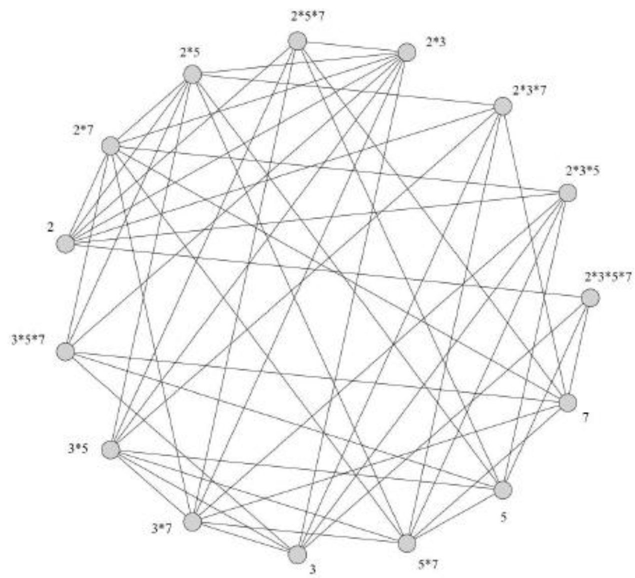

Figure 1. Arithmetic Graph $G = V_{210}$.

Theorem 2.4. *For an arithmetic graph* $G = V_n$, $n = p_i^{a_i} \times p_j^{aj} \times p_k^{a_k}$ *has no cyclic vertex cut if* (i) $a_i > 1$ *for exactly one* i

(ii) $a_i = a_j = 2$ *and* $a_k = 1$

(iii) $a_i = a_j = a_k = 2$

(iv) $a_i = 1$, $a_j = 2$, $a_k = 3$.

Proof. Case (i) $a_i > 1$ for exactly one i

In this case $n = p_i^{a_i} \times p_j \times p_k$ and the vertex set of G be $V(G) = \{p_i, p_i^2, p_i^3, p_i^4, \ldots, p_i^{a_i}, p_j, p_k, p_i \times p_j, p_j \times p_k, p_i^2 \times p_j, \ldots, p_i^{a_i} \times p_j,$ $p_i \times p_k, p_i^2 \times p_k, \ldots, p_i^{a_i} \times p_k, p_i \times p_j \times p_k, p_i^2 \times p_j \times p_k, \ldots, p_i^{a_i} \times p_j \times p_k\}$. Let us choose any cycle C such that $\sum_{v \in C} d(v) =$ mimimum. By the proof of Theorem 1.9 we can easily observe that the possible cycles whose degree sum is minimum are $V(X) = \{p_i^{a_i}, p_i \times p_j, p_i \times p_k\}$, $V(Y) = \{p_i^{a_i} \times p_k, p_j \times p_k,$

$p_k\}$. Here $N(X) = \{p_j, p_k, p_i \times p_j, p_i \times p_k, p_i^{a_i}; 1 \leq a_i \leq \infty\}$ and $N(Y) = \{p_i, p_j, p_k, p_j \times p_k, p_i \times p_j \times p_k, p_i^{a_i} \times p_j; 1 \leq a_i \leq \infty, p_i^{a_i} \times p_k; 1 \leq a_i \leq \infty\}$.

By the definition of an arithmetic graph $V(G) - N[X] = \begin{cases} p_i^{a_i} \times p_j; 2 \leq a_i < \infty \\ p_i^{a_i} \times p_k; 2 \leq a_i < \infty \\ p_i^{a_i} \times p_j \times p_k; 2 \leq a_i \leq \infty \end{cases}$ and $V(G) - N[Y] = \begin{cases} p_i^{a_i}; 2 \leq a_i < \infty \\ p_i^{a_i} \times p_j \times p_k; 2 \leq a_i < \infty \end{cases}$

which are not adjacent to any one of the vertices of X and Y respectively. Moreover they are non adjacent to themselves. Hence no cyclic vertex cut.

Case (ii) $a_i = a_j = 2$ and $a_k = 1$

Here $n = p_i^{a_i} \times p_j^{a_j} \times p_k$ and the vertex set of G be $V(G) = \{p_i, p_i^2, p_j, p_j^2, p_k, p_i \times p_j, p_i \times p_j^2, p_i \times p_k, p_i^2 \times p_j, p_i^2 \times p_j^2, p_i^2 \times p_k, p_i \times p_j \times p_k, p_i \times p_j^2 \times p_k, p_i \times p_j^2 \times p_k, p_i^2 \times p_j \times p_k, p_i^2 \times p_j^2 \times p_k\}$. In this case we have exactly two cycles of length three whose degree sum is minimum say X and Y. The vertices of the cycles are $V(X) = \{p_i^2, p_i \times p_j^2, p_i \times p_k\}$ and $V(Y) = \{p_k, p_j^2 \times p_k, p_i^2 \times p_k\}$. By the definition of an arithmetic graph $V(G) - N[X] = \{p_j^2, p_i^2 \times p_j \times p_k, p_i^2 \times p_j^2 \times p_k\}$ and $V(G) - N[Y] = \{p_i^2, p_j^2, p_i^2 \times p_j^2\}$. The induced graph of $V(G) - N[X]$ is a disconnected graph with three vertices. Suppose if it is connected, then there exists a path between any two vertices of $[V(G) - N[X]]$, this implies that there exists at least two edges between three vertices. Therefore $p_j^{a_j} p_i^{a_i} \times p_j^2 \times p_k$ or $p_i^{a_i} \times p_j \times p_k p_i^{a_i} \times p_j^{a_j} \times p_k$ is an edge which contradicts the definition of an arithmetic graph. Similar as for $[V(G) - N[Y]]$. Hence the result.

Case (iii) $a_i = a_j = a_k = 2$

Here $n = p_i^{a_i} \times p_j^{a_j} \times p_k^{a_k}$ the vertex set of G be $V(G) = \{p_i, p_i^2, p_j, p_j^2, p_k, p_k^2, p_i \times p_j, p_i \times p_j^2, p_i \times p_k, p_i \times p_k^2, p_i^2 \times p_j, p_i^2 \times p_j^2, p_i^2 \times p_k, p_i^2 \times p_k^2, p_j \times p_k, p_j \times p_k^2, p_j^2 \times p_k, p_j^2 \times p_k^2, p_i \times p_j \times p_k, p_i \times p_j^2 \times p_k, p_i^2 \times p_j \times p_k, p_i^2 \times p_k \times p_i^2, p_j \times p_k^2, p_i \times p_j \times p_k^2, p_i \times p_k^2, p_i \times p_j^2 \times p_k^2, p_i^2 \times p_j^2 \times p_k^2\}$. Choose any cycle C such that $\sum_{v \in C} d(v)$ is minimum. We can easily observe that $C = \{p_i^{a_i}, p_i \times p_j^{a_j}, p_i \times p_k^{a_k}\}$ is the only cycle whose degree sum is minimum. Now $V(G) - N[C] = \{p_j^{a_j}, p_k^{a_k}, p_j^{a_j} \times p_k^{a_k}, p_i^{a_i} \times p_j \times p_k, p_i^{a_i} \times p_j^{a_j} \times p_k, p_i^{a_i} \times p_j^{a_j} \times p_k^{a_k}\}$. The induced graph of $V(G) - N[C]$ is a disconnected graph with three components G_1, G_2, G_3 where G_1, G_2 are two isolated vertices and G_3 is a connected graph. The connected graph G_3 consists of four vertices say $V(G_3) = \{p_j^{a_j}, p_k^{a_k}, p_i^{a_i} \times p_j \times p_k, p_i^{a_i} \times p_j^{a_j} \times p_k\}$ which is not a cycle. Suppose G_3 is a cycle, then at least three vertices is of degree greater than or equal to two but in G_3 only two vertices say $p_k^{a_k}, p_i^{a_i} \times p_j \times p_k$ have degree 2, which is a contradiction. Therefore, there is no cycle in G_3.

Case (iv) $a_i = 3, a_j = 2, a_k = 1$.

In this case also we choose a cycle C such that $\sum_{v \in C} d(v) =$ minimun. Let the vertices of cycles be $V(X) = \{p_i^{a_i}, p_i \times p_j^2, p_i \times p_k\}; 2 \le a_i \le 3$ and $V(Y) = \{p_j^2 \times p_k, p_i^{a_i} \times p_k, p_k\}; 2 \le a_i \le 3$. Similar as case (iii), $V(G) - N[X] = \{p_j^2, p_i^{a_i} \times p_j^{a_j} \times p_k; 2 \le a_i \le 3, 1 \le a_j \le 2\}$. Also the induced graph of $V(G) - N[X]$ is a disconnected graph with two isolated vertices $p_i^2 \times p_j^2 \times p_k$ and $p_i^2 \times p_j^2 \times p_k$ and a connected graph. The connected graph consists of three vertices. Since $\gcd(p_i^2 \times p_j \times p_k, p_i^3 \times p_j \times p_k) = p_i \times p_j \times p_k$, there is no edge between these two vertices hence it is not a closed path. Similarly

$[V(G) - N(Y)]$ is a disconnected graph with three components and each component is an isolated vertex. Thus, there is no cyclic vertex cut.

Theorem 2.5. *For an arithmetic graph* $G = V_n$, $n = p_i^{a_i} \times p_j^{a_j} \times p_k^{a_k} \times p_l^{a_l}$, $a_i > 1$, *for exactly one i has no cyclic vertex cut. The cyclic vertex cut number* $\kappa_c(G) = \infty$.

Proof. The proof is similar to case (i) of Theorem 2.4.

3. Finite Cyclic Connectivity of an Arithmetic Graph $G = V_n$

In the study of a cyclic connectivity of an arithmetic graph $G = V_n$, we found that for every $G = V_n$ other than $n = p_1 \times p_2$ have cycles. Also, the number of cycles in $G = V_n$ depends on the number of primes in n. If the number of primes increases the number of cycles are also increases. As we have more cycles in $G = V_n$, we need to choose a cycle C whose $\sum_{v \in C} d(v)$ is minimum. The required cycle can be choosen in a particular way, which is discussed in the corresponding theorems. Moreover, for finding the cyclic connectivity number $\kappa_c(G)$ we need to eliminate the number of common neighbours of each vertices $v \in V$ (The common neighbours can be accounted once). By Definition 1.1 let $|W| = |\Gamma(v_i, v_j)| + |\Gamma(v_j, v_k)| + |\Gamma(v_k, v_i)|$ be the cardinality of the common neighbouhood vertices of the cycle C_3.

Theorem 3.1. *For an arithmetic graph* $G = V_n$, $n = p_i^{a_i} \times p_j^{a_j} \times p_k^{a_k}$ *has cyclic vertex cut if*

(i) $a_i \geq 3$ *for any two i*

(ii) $a_i \geq 3$; a_j, $a_k \geq 2$.

Proof. Case (i) $a_i \geq 3$ for any two i,

Let a_i, $a_j \geq 3$; $a_k = 1$. The vertices of $G = V_n$ where $n = p_i^{a_i} \times p_j^{a_j} \times p_k$ be $V(G) = \{p_i, p_i^2, p_i^3, \ldots, p_i^{a_i}, p_j, p_j^2, p_j^3, \ldots, p_j^{a_j}, p_k, p_i \times p_j, p_i \times p_j^2, p_i,$

$\ldots, p_i \times p_j^{a_j}, p_i \times p_k, p_i^2 \times p_j, \ldots, p_i^2 \times p_k, p_i^3 \times p_j, \ldots, p_i^3 \times p_k, p_i \times p_j \times p_k,$ $\ldots, p_i \times p_j^{a_j} \times p_k, \ldots, p_i^{a_i} \times p_j^{a_j} \times p_k$. We choose the cycle C whose degree sum is minimum in the following way. (1) As the prime vertex $p_i, i \in \{1, 2, \ldots, r\}$ has the greatest degree among the primes and prime power vertices of G choose the first vertex of the cycle as $p_i^{a_i}, a_i \geq a_j$.

(2) Since $n = p_i^{a_i} \times p_j^{a_j} \times p_k$ and by proof of Theorem 1.9 vertices with more prime products have less degree the second vertex must be $p_i \times p_j^{a_j}$.

(3) Also the only vertex which is adjacent to both $p_i^{a_i}, a_i \geq a_j$ and $p_i \times p_j^{a_j}$ is $p_i \times p_k$ the third vertex is $p_i \times p_k$.

Hence the vertices of cycle $V(C)$ is $\{p_i^{a_i}, p_i \times p_j^{a_j}, p_i \times p_k\}$.

Now the set of vertices which are not adjacent to C are $\{p_j^2, p_j^3, \ldots, p_j^{a_j},$ $p_i^2 \times p_j \times p_k, \ldots, p_i^2 \times p_j^{a_j} \times p_k, \ldots, p_i^{a_i} \times p_j \times p_k, p_i^{a_i} \times p_j^{a_j} \times p_k\}$. The induced graph of $V(G) - N(C)$ contains at least one cycle which is of length four say $p_j^2 p_i^2 \times p_j \times p_k p_j^3 p_i^3 \times p_j \times p_k p_j^2$ which satisfy the definition of cyclic connectivity. Since the cycle with $\sum_{v \in C} d(v)$ is minimum, we have $\kappa_c(G) = (a_j + 1)(2a_k + a_i + 1) + (a_i + 1)(a_k + 1) - 4 - |W|$.

Case (ii) $a_i \geq 3\ a_j, a_k \geq 2$

The proof is similar as case (i) we have $\kappa_c(G) = 2(a_i + a_j + a_k) + a_i a_j$ $+ a_j a_k + a_k a_i - 1 - |W|$.

Theorem 3.2. *For an arithmetic graph* $G = V_n, p_1^{a_1} \times p_2^{a_2} \times p_3^{a_3} \times \ldots \times p_5^{a_5}$. $\kappa_c(G) = \begin{cases} 2^r + 2^{(r-2)} - 4 - |W| & \text{if } a_i = 1\ \forall i \\ 2^{(r-1)} + 10a_1 + 10 - |W| & \text{if } a_i \neq 1 \text{ for } i = 1. \end{cases}$

Proof. By Theorem 1.3, $|V| = 2^r - 1,$ here we have two cases,

Case (i) If $a_i = 1$ for all i

Consider the cycle C_3 in which $\sum_{v \in C} d(v)$ is minimum, $V(C_3) = \{p_1, p_1 \times p_2, p_1 \times p_3 \times p_4 \times p_5\}$. Here the induced graph of $V(G) - N(C_3)$ is disconnected and has at least two cycles whose vertices are $p_1 p_1 \times p_2 p_1 \times p_3 \times p_4 \times p_5 p_1$ and $p_3 \times p_4 p_4 \times p_5 p_5 \times p_3 p_3 \times p_4$. Hence we have $\kappa_c(G) = d(p_1) + d(p_1 \times p_2) + d(p_1 \times p_3 \times p_4 \times p_5) - 3 - |W|$

$= 2^{r-1} + 2.2^{r-2} + 4.2^{r-4} - 4 - |W|$

$= 2^r + 2^{r-2} - 4 - |W|$

Case (ii) If $a_i > 1$ for exactly one i

Here the vertices of the cycle whose degree sum minimum is $V(C) = \{p_1^{a_1}, p_1 \times p_2, p_1 \times p_3 \times p_4 \times p_5\}$. Similar as above case we have $\kappa_c(G) = d(p_1^{a_1}) + d(p_1 \times p_2) + d(p_1 \times p_3 \times p_4 \times p_5) - 3 - |W|$

Example 3.3. Consider a graph $G = V_{1080}, 1080 = 2^2 \times 3^2 \times 5$. The vertex set $V(G) = \{2, 2^2, 2^3, 3, 3^2, 3^3, 5, 2 \times 3, 2 \times 3^2, 2 \times 3^3, 2 \times 5, 2^2 \times 3, 2^2 \times 3^2, 2^2 \times 3^3, 2^2 \times 5, 2^3 \times 3, 2^3 \times 3^2, 2^3 \times 3^3, 2^3 \times 5, 3 \times 5, 3^2 \times 5, 3^3 \times 5, 2 \times 3 \times 5, 2 \times 3^2 \times 5, 2 \times 3^3 \times 5, 2^2 \times 3 \times 5, 2^2 \times 3^2 \times 5, 2^2 \times 3^3 \times 5, 2^3 \times 3 \times 5, \times 2^3 \times 3^2 \times 5, 2^3 \times 3^3 \times 5\}$. consider the cycle of length 3 whose vertices are $\{2^3, 2 \times 3^3, 2 \times 5\}$ and hence the cyclic connectivity set $S = \{N(2^3) \cup N(2 \times 3^3) \cup N(2 \times 5) - \{2^3, 2 \times 3^3, 2 \times 5\}\}$.

$S = \{2, 2^2, 3, 5, 2 \times 3, 2 \times 3^2, 2^2 \times 3, 2^2 \times 3^2, 2^2 \times 3^3, 2^2 \times 5, 2^3 \times 3, 2^3 \times 3^2, 2^3 \times 3^3, 2^3 \times 5, 3 \times 5, 3^2 \times 5, 3^3 \times 5, 2 \times 3 \times 5, 2 \times 3^2 \times 5, 2 \times 3^3 \times 5\}$. Now the induced graph of $V(G) - S$ is disconnected and at least two of its component have cycle, one of the component G_1 is the cycle of length 3 whose vertices are $\{2^3, 2 \times 3^3, 2 \times 5\}$ another component G_2 has cycle of length 4 whose vertices are $\{3^2, 2^2 \times 3 \times 5, 3^3, 2^3 \times 3 \times 5\}$. Thus we have $\kappa_c(V_{1080}) = |S| = 20$.

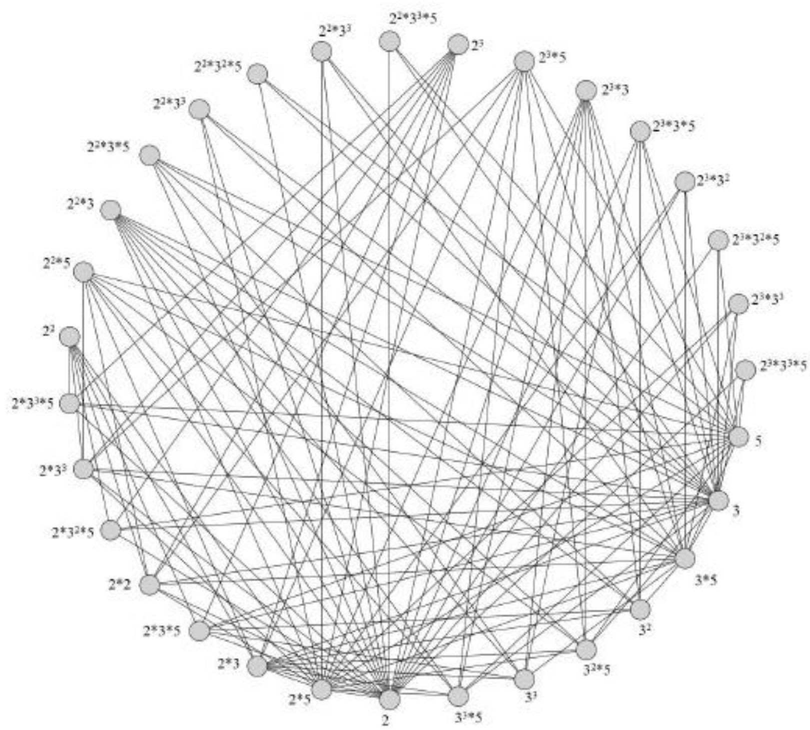

Figure 2. Arithmetic Graph $G = V_{1080}$.

Theorem 3.4. *For an arithmetic graph* $G = V_n$, $n = p_1 \times p_2 \times p_3 \times \ldots \times p_r$, $r > 5$.

$$\kappa_c(G) = \begin{cases} 2^{(r-1)} + 2\left(\frac{r+1}{2}\right)2^{\left(\frac{r-1}{2}\right)} - 4 - |W| & \text{if } r \text{ odd} \\ 2^{(r-1)} + \left(\frac{r}{2}\right)2^{\left(\frac{r}{2}\right)} + \left(\frac{r}{2}+1\right)2^{\left(\frac{r}{2}-1\right)} - 4 - |W| & \text{if } r \text{ even.} \end{cases}$$

Proof. By Theorem 1.3, $|V| = 2^r - 1$.

Case (i) r is odd and $r > 5$.

Consider the cycle C_3 in which $\sum_{v \in C} d(v)$ is minimum, C_3 can be choose n in the following way (1) One vertex must be a single prime let it be p_i.

(2) Since $n = p_1 \times p_2 \times \ldots \times p_{\frac{r}{2}} \times p_{\frac{r+1}{2}} \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r$ and by the proof of Theorem 1.9, vertices with more prime products have less degree. Therefore the second vertex must be the product of $\frac{r+1}{2}$ primes including p_i.

(3) The third vertex must be the product of remaining $\frac{r-1}{2}$ primes and p_i, otherwise either it violates the adjacency or minimum degree. Since v_i and v_j are adjacent in $G = V_n$ iff the sum of number of primes in v_i and v_j is less than or equal to $r+1$ and $\gcd(v_i, v_j) = p_i$. In this way, we choose a cycle $V(C_3) = \{p_1,\ p_1 \times p_2 \times \ldots p_{\frac{r}{2}} \times p_{\frac{r+1}{2}},\ p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r\}$. Now, let S be the set of vertices which are adjacent to the vertices of C. The induced graph of $V(G) - S$ is a disconnected graph with at least two of its components contain cycles. Thus it satisfies the definition of the cyclic connectivity, hence the set of vertices in S is called cyclic vertex cut. Since the cycle which we have chosen is $\sum_{v \in C} d(v)$ is minimum in G. $|S|$ is called the cyclic connectivity number $\kappa_c(G)$. Now $|S| = \kappa_c(G) = d(p_1) + d(p_1 \times p_2 \times p_3 \times \ldots \times p_{\frac{r+1}{2}}) + d(p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r) - 3 - |W| = 2^{(r-1)} + 2\left(\frac{r+1}{2}\right)2^{\left(\frac{r-1}{2}\right)} -4 - |W|$.

Case (ii) Assume that r is even and $r > 5$ Similar as case (i) choose a cycle C such that the sum of the degree is minimum. Let the vertices of the cycle be $V(C) = \{p_1,\ p_1 \times p_2 \times \ldots \times p_{\frac{r}{2}},\ p_1 \times p_{\left(\frac{r}{2}+1\right)} \times \ldots \times p_r\}$. Here p_1 is the only prime which is common to all the three vertices of C. Similar as case (i) here also we get $\kappa_c(G) = d(p_1) - 1 + d(p_1 \times p_2 \times p_3 \times \ldots \times p_{\left(\frac{r}{2}\right)}) - 1 + d(p_1$

$\times\, p_{\left(\frac{r}{2}+1\right)} \times \ldots \times p_r) - 1 - |W| = 2^{(r-1)} + \frac{r}{2} 2^{\frac{r}{2}} + \left(\frac{r}{2}+1\right) 2^{\left(\frac{r}{2}-1\right)} - 4 - |W|.$

Theorem 3.5. *For an arithmetic graph* $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$, *if* $r > 5$, $a_i > 1$ *for exactly one* i. *The cyclic vertex cut number*

$$\kappa_c(G) = \begin{cases} 2^{(r-1)} + 2\left[a_1 + \dfrac{r-1}{2}\right] 2^{\frac{r-1}{2}} - 4 - |W| & \text{if } r \text{ odd} \\ 2^{(r-1)} + [2a_1 + r - 2] 2^{\left(\frac{r}{2}-1\right)} + [2a_1 + r] 2^{\left(\frac{r}{2}-2\right)} - 4 - |W| & \text{if } r \text{ even.} \end{cases}$$

Proof. Case (i) r is odd

Let $G = V_n$ be an arithmetic graph where $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$, where exactly one $a_i \neq 1$, the vertex set $V(G) = \{p_1, p_1^2, \ldots, p_1^{a_1}, p_2, p_3, \ldots, p_r, p_1 \times p_2, p_1^2 \times p_2, \ldots p_1^{a_1} \times p_2, p_2 \times p_3, \ldots, p_1^{a_1} \times p_r, p_1 \times p_2 \times p_3, \ldots, p_1 \times p_2 \times p_3 \times \ldots \times p_r, \ldots, p_1^{a_1} \times p_2 \times p_3 \times \ldots \times p_r\}$. The cycle can be chosen in the following way. Arrange the primes of n in such a way that the greatest power of p_i in the first position. (i) Since the prime vertex p_i, $i \in \{1, 2, \ldots, r\}$ has the greatest degree among the primes and prime power vertices of G, choose the first vertex of the cycle as $p_1^{a_1}$, $a_1 \neq 1$.

(ii) Since the vertices with more prime products will has less degree, second vertex consists of first $\frac{r+1}{2}$ primes including p_1,

(iii) Third vertex must be the product of $r - \frac{r+1}{2}$ primes and p_1. The primes in the third vertex other than p_1 should not be in the second vertex, otherwise it violates the adjacency of the second and the third vertex in C. Now the vertices of the cycle, $V(C) = \{p_1^{a_1}, p_1 \times p_2 \times \ldots \times p_{\frac{r+1}{2}}, p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r\}$. Let S be the set of all vertices adjacent to the vertices of C other than $\{p_1^{a_1}, p_1 \times p_2 \times \ldots \times p_{\frac{r+1}{2}}, p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r\}$. The induced graph of $V(G) - S$ is a disconnected graph with at least two component contains a cycle. Thus it satisfies the definition of the cyclic vertex cut and $\sum_{v \in C} d(v)$ is minimum, the cyclic connectivity of G is $|S|$. By Theorem 1.7, we have $\kappa_c(G) = d(p_1^{a_1}) + d(p_1 \times p_2 \times \ldots \times p_{\left(\frac{r+1}{2}\right)}) + d(p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r) - 3 - |W|$.

$$= 2^{(r-1)} + 2\left[a_1 + \frac{r-1}{2}\right] 2^{\left(\frac{r-1}{2}\right)} - 4 - |W|.$$

Case (ii) r is even

Let $G = V_n$ be an arithmetic graph where $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$ where exactly one $a_i \neq 1, r$ is even and $r > 4$, then the vertex set $V(G) = \{p_1, p_1^2, \ldots, p_1^{a_1}, p_2, p_3, \ldots, p_r, p_1 \times p_2, p_1^2 \times p_2, \ldots, p_1^{a_1} \times p_2, p_2 \times p_3, \ldots, p_1^{a_1} \times p_r, p_1^{a_1} \times p_2 \times p_3 \times \ldots \times p_r\}$. Arrange n in such a way that power of p_1 is greater than 1. The vertices of the cycle, $V(C) = (p_1^{a_1}, p_1 \times p_2 \times \ldots \times p_{\frac{r}{2}}, p_1 \times p_{\left(\frac{r}{2}+1\right)} \times \ldots \times p_r\}$. Similar as above case (i) we get $\kappa_c(G) = d(p_1^{a_1}) + d(p_1 \times p_2 \ldots \times p_{\frac{r}{2}}) + d(p_1 \times p_{\left(\frac{r}{2}+1\right)} \ldots \times p_r)$ $- 3 - |W| = 2^{(r-1)} + [2a_1 + r - 2] 2^{\left(\frac{r}{2}-1\right)} + [2a_1 + r] 2^{\left(\frac{r}{2}-2\right)} - 4 - |W|$.

Theorem 3.6. *For an arithmetic graph* $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$, if $a_1 \geq a_2 > 1$, $r > 3$. The cyclic vertex cut number

$$\kappa_c(G) =$$
$$\begin{cases} (a_2+1)2^{(r-1)} + \left[\dfrac{2a_1 + r - 1}{2}\right][2^{\frac{r-1}{2}} + (a_2+1)2^{\left(\frac{r-3}{2}\right)} - 4 - |W| & \textit{if } r \textit{ odd} \\ (a_2+1)[2^{(r-1)} + (r + 2a_1)2^{\left(\frac{r}{2}-3\right)}] + (2a_1 + r - 2)2^{\left(\frac{r}{2}-1\right)} - 4 - |W| & \textit{if } r \textit{ even.} \end{cases}$$

Proof. Let $G = V_n$ be an arithmetic graph where $n = p_1^{a_1} \times p_2^{a_2} \ldots \times p_r^{a_r}$, if $a_1 \geq a_2 > 1$, and $r > 3$. The vertex set of G be $V(G) = \{p_1, p_1^2, \ldots, p_1^{a_1}, p_2, p_2^2, \ldots, p_2^{a_2}, p_3, p_4, \ldots, p_r, p_1 \times p_2, p_1^2 \times p_2, \ldots, p_1^{a_1} \times p_2, p_2 \times p_3, p_2^2 \times p_3, \ldots, p_2^{a_2} \times p_r, \ldots, p_1^{a_1} \times p_2^{a_2} \times p_3 \times \ldots \times p_r\}$. Arrange p_1 in such a way that $a_1 \geq a_2 > 1$.

Case (i) If r is odd, Here the following way is used to choose the cycle C whose degree sum is minimum. (i) Since the prime vertex $p_i, i \in \{1, 2, \ldots, r\}$ has the greatest degree among the primes and prime power vertices of G, choose the first vertex of the cycle as $p_1^{a_1}, a_1 \neq 1$.

(ii) Since the vertices with more prime products will have less degree, second vertex consists of first $\frac{r+1}{2}$ primes including $p_2^{a_2}$,

(iii)Third vertex must be the product of $r - \frac{r+1}{2}$ primes and p_1. Also the primes in the third vertex other than p_1 is not in the second vertex. Otherwise it violates the adjacency of second and third vertex. Hence the vertices of the cycle $V(C) = \{p_1^{a_1}, p_1 \times p_2^{a_2} \times p_{\frac{r+1}{2}}, p_1 \times p_{\frac{r+1}{2}+1} \times \ldots p_r\}$ similar as above proof we get $\kappa_c(G) = d(p_1^{a_1}) - 1 + d(p_1 \times p_2^{a_2} \times \ldots \times p_{\left(\frac{r+1}{2}\right)}) - 1 + d(p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r) - 1 - |W|$.

$$= (a_2 + 1) 2^{(r-1)} + \left[\frac{2a_1 + r - 2}{2}\right][2^{\frac{r-1}{2}} + (a_2 + 1) 2^{\left(\frac{r-3}{2}\right)}] - 4 - |W|.$$

Case (ii) If r is even, choose the first cycle C as $\{p_1^{a_1}, p_1 \times p_2^{a_2} \times p_{\frac{r}{2}}, p_1 \times p_{\frac{r}{2}+1} \times \ldots \times p_r\}$. Similar as case (i) we get $\kappa_c(G) = d(p_1^{a_1}) - 1 + d(p_1 \times p_2^{a_2} \times \ldots \times p_{\frac{r}{2}}) - 1 + d(p_1 \times p_{\left(\frac{r}{2}+1\right)} \times \ldots \times p_r) - 1 - |W|$.

$$= (a_2 + 1)[2^{(r-1)} + (r + 2a_1) 2^{\left(\frac{r}{2}-3\right)}] + (2a_1 + r - 2) 2^{\left(\frac{r}{2}-1\right)} - 4 - |W|.$$

Theorem 3.7. *For an arithmetic graph* $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$, *if* $a_1 \neq 1$ *for* $i \in \{1, 2, 3\}$, $r > 3$. *The cyclic vertex cut number*

$\kappa_c(G) =$

$$\begin{cases} (a_2+1)(a_3+1)2^{(r-3)} + [2a_1 + r - 1][2^{\left(\frac{r-5}{2}\right)}](a_2+1)(a_3+1) - 4 - |W| & \text{if } r \text{ odd} \\ (a_2+1)(a_3+1)2^{(r-3)} + [2a_1 + r - 1]2^{\left(\frac{r}{2}-1\right)} + [2a_1 + r]2^{\left(\frac{r}{2}-2\right)} - 4 - |W| & \text{if } r \text{ even.} \end{cases}$$

Proof. Case (i) r is odd and $r > 3$

Let $G = V_n$ be an arithmetic graph where $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$, if $a_1 \neq 1$ for $i \in \{1, 2, 3\}$, then the vertex set $V(G) = \{p_1, p_1^2, \ldots, p_1^{a_1}, p_2, p_2^2, \ldots, p_2^{a_2}, p_3, \ldots, p_r, p_1 \times p_2, p_1^2 \times p_2, \ldots, p_1^{a_1} \times p_2, p_2 \times p_3, \ldots, p_1^{a_1} \times p_r, p_1^{a_1} \times p_2 \times p_3 \times \ldots \times p_r\}$. Arrange n in such a way that the power of p_1 and p_2 are greater than 1 and $a_1 \geq a_2$. Also, the prime whose power greater than one will be in the $\left(\frac{r+1}{2} + 1\right)^{th}$ place of n. The vertices of the cycle is

$V(G) = \{p_1^{a_1}, p_1 \times p_2^{a_2} \times \ldots \times p_{\left(\frac{r+1}{2}\right)}, p_1 \times p_{\left(\frac{r+1}{2}+1\right)}^{a\left(\frac{r+1}{2}+1\right)} \times \cdots \times p_r\}$. Similar to Theorem 3.6, we get $\kappa_c(G) = d(p_1^{a_1}) - 1 + d(p_1 \times p_2^{a_2} \times \ldots \times p_{\left(\frac{r+1}{2}\right)}) - 1 + d(p_1 \times p_{\left(\frac{r+1}{2}+1\right)}^{a\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r) - 1 - |W|$.

$$= (a_2 + 1)(a_3 + 1) 2^{(r-3)} + [2a_1 + r - 1][2^{\left(\frac{r-5}{2}\right)}](a_2 + 1)(a_3 + 1) - 4$$

Case (ii) r is even and $r > 3$

Let $G = V_n$ be an arithmetic graph where $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$ if $a_1 \neq 1$ for $i \in \{1, 2, 3\}$, $r > 3$, r is even. The vertex set $V(G) = \{p_1, p_1^2, \ldots, p_1^{a_1}, p_2, p_2^2, \ldots, p_2^{a_2}, \ldots, p_2^{a_2}, p_3, \ldots, p_r, p_1 \times p_2, p_1^2 \times p_2, \ldots, p_1^{a_1} \times p_2, p_2 \times p_3 \times \ldots \times p_r\}$. Arrange n in such a way that the power of p_1 and p_2 are greater than 1 and $a_1 \geq a_2$. Also, the prime having power greater than one will be in the $\left(\frac{r}{2}+1\right)^{th}$ place of n. Consider the vertices of the cycle C as

$V(C) = \{p_1^{a_1}, p_1 \times p_2^{a_2} \times \ldots \times p_{\left(\frac{r}{2}\right)}, p_1 \times p_{\left(\frac{r}{2}+1\right)}^{a\left(\frac{r}{2}+1\right)} \times \ldots \times p_r\}$. Similar as case (i)

we get $\kappa_c(G) = d(p_1^{a_1}) - 1 + d(p_1 \times p_2^{a_2} \times \ldots \times p_{\left(\frac{r}{2}\right)}) - 1 + d(p_1 \times p_{\left(\frac{r+1}{2}+1\right)}^{a\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r) - 1 - |W|$.

$$= (a_2+1)(a_3+1)2^{(r-3)} + [2a_1 + r - 1]2^{\left(\frac{r}{2}-1\right)} + [2a_1 + r]2^{\left(\frac{r}{2}-2\right)} - 4 - |W|.$$

Remark 3.8. From the above theorems the authors identified that the cyclic connectivity number of an arithmetic graph $G = V_n$ depends not only on the number of primes in n but also on the powers of prime. The cyclic connectivity number for an arithmetic graph exists if $a_1 \neq 1$ for exactly one i and the number of primes in n must be greater than 4, and if $a_1 \neq 1$ for at least two i $i \in \{1, 2, \cdots, r\}$ and the number of primes must be greater than or equal to 3. The following theorem gives the generalization of the cyclic connectivity number of an arithmetic graph and we have omitted the above discussed cases and starts only from r greater than 3 and $a_1 \neq 1$ for more than three i.

Theorem 3.9. *For an arithmetic graph* $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$, *if* $a_1 \neq 1$, $r > 3$. *The cyclic vertex cut number*

(i) $\kappa_c(G) = [\prod_{i=1,\, i \notin B}^{r}(a_i+1)] + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1,\, i \notin B}^{r}(a_i+1)$

$+ [| B - B' | + \sum_{i \in B'} a_i] \prod_{i=1,\, i \notin B}^{r-1} (a_i + 1) - 4 - | W |$ *if r is odd.*

(ii) $\kappa_c(G) = [\prod_{i=1,\, i \notin B}^{r} (a_i + 1)] + [| B - B' | + \sum_{i \in B'} a_i] \prod_{i=1,\, i \notin B}^{r} (a_i + 1)$

$+ [| B - B' | + \sum_{i \in B'} a_i] \prod_{i=1,\, i \notin B}^{r-1} (a_i + 1) - 4 - | W |$ *if r is even. Here B is the number of primes in the chosen vertex and* B' *is the number of* $a_i \neq 1$ *in the choose n vertex.*

Proof. Case (i) r is odd

Arrange a_i in such a way that $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_r$. Partition the primes of n into two sets say A and B where the set A consists of primes in odd position except p_1, let it be $A = \{p_3^{a_3}, p_5^{a_5}, \ldots, p_r^{a_r}\}$ and the set B consists of even position primes $B = \{p_2^{a_2}, p_4^{a_4}, \ldots, p_{r-1}^{a_r - 1}\}$. Now choose the cycle C such that $V(C) = \{p_1^{a_1}, p_1 \times p_3^{a_3} \times p_5^{a_5} \times \ldots \times p_r^{a_r}, p_1 \times p_2^{a_2} \times p_4^{a_4} \times \ldots \times p_{r-1}^{a_r - 1}\}$. Then similar as above theorems, the cycle which is chosen whose degree sum is minimum. Therefore, we have

$$\kappa_c(G) = [\prod_{i=1,\, i \notin B}^{r} (a_i + 1)] + [| B - B' | + \sum_{i \in B'} a_i] \prod_{i=1,\, i \notin B}^{r} (a_i + 1)$$

$$+ [| B - B' | + \sum_{i \in B'} a_i] \prod_{i=1,\, i \notin B}^{r-1} (a_i + 1) - 3 - | W |.$$

$$= [\prod_{i=1,\, i \notin B}^{r} (a_i + 1)] + [| B - B' | + \sum_{i \in B'} a_i] \prod_{i=1,\, i \notin B}^{r} (a_i + 1)$$

$$+ [| B - B' | + \sum_{i \in B'} a_i] \prod_{i=1,\, i \notin B}^{r-1} (a_i + 1) - 4 - | W |.$$

Case (ii) r is even

Similar as case (i) choose the cycle C whose vertex set $V(C) = \{p_1^{a_1}, p_1 \times p_3^{a_3} \times p_5^{a_5} \times \ldots \times p_{r-1}^{a_r - 1}, p_1 \times p_2^{a_2} \times p_4^{a_4} \times \cdots \times p_r^{a_r}\}$. Hence we have

$$\kappa_c(G) = [\prod_{i=1,\, i \notin B}^{r} (a_i + 1)] - 1 + [| B - B' | + \sum_{i \in B'} a_i]$$

$$\prod_{i=1,\ i\notin B}^{r}(a_i+1)+[|B-B'|+\sum_{i\in B'}a_i]\prod_{i=1,\ i\notin B}^{r-1}(a_i+1)-3-|W|$$

$$=[\prod_{i=1,\ i\notin B}^{r}(a_i+1)]+[|B-B'|+\sum_{i\in B'}a_i]\prod_{i=1,\ i\notin B}^{r}(a_i+1)+[|B-B'|$$

$$+\sum_{i\in B'}a_i]\prod_{i=1,\ i\notin B}^{r-1}(a_i+1)-4-|W|.$$

Conclusion

From the above theorems, we observed that, for an arithmetic graph $G = V_n$, the number of primes in n does not exceed 2 is of infinite cyclic connectivity.

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