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NOTE ON STRONGLY *P*-REGULAR NEAR-RINGS

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Abstract

In the paper, we introduce the notion of strongly P-regular near-rings. We have displayed that a near-ring N is strongly P-regular if and only if it is also regular. A near-ring N is called left (right) strongly P-regular if for every 'a' there is a 'n' in N such that $a = na^2 + p\,(a = a^2n + p)$ and $a = ana,$ position P is an arbitrary ideal. We specify some new concepts and justify them with suitable examples. And also we discuss some of the theorems related to it.

1. Introduction

In mathematics, a near-ring (also near ring (or) nearing) is an algebraic structure similar to a ring but satisfying fewer axioms. Near-rings arise naturally from functions on the group. The antiquity of the concept of near-ring is eminent influenced by the knowledge of ring theory. A near-ring is a ring (not undoubtedly with unity) if and only if addition is commutative and multiplication is also distributive on both sides is ample, and commutative of

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addition follows unquestionably. The book of Pilz uses right near-ring, while the book of clay uses left near-ring. The thought of a regular near-ring was introduced in 1968 by J. C. Beidleman and later S. Leigh and H. E. Healtherly etc. S. J. Choi extended P-regularity of a ring to the P-regularity of a near-ring. Regular (von-Neumann regular) ring plays a vital role in the structure theory of rings which was first introduced by Von-Newmann. The generalization of rings (near-rings) plays a vital role in the development of mathematics. A lot of mathematicians studied and established various types of near-rings such as Boolean near-rings, IFP near-ring, etc.

2. Preliminaries

Definition 1. A near-ring N is an algebraic system with two binary operations addition and multiplication with the following properties,

(i) $(N, +)$ is a group (not necessarily abelian)

(ii) $(N, \cdot)$ is associative

(iii) $(x + y) \cdot z = x \cdot z + y \cdot z \forall x, y, z \in N$

(iv) $x \cdot (y + z) = x \cdot y + x \cdot z \forall x, y, z \in N$.

Example 1. Every ring is a Near-ring.

Example 2. Let $Z_6 = \{0, 1, 2, 3, 4, 5\}$ is a group under addition modulo 6 and it is a semi group under multiplication modulo 6. And also satisfies the distributive laws,

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

•	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Clearly $(Z_6, +)$ is a near-ring.

Example 3. $(Z_2, +)$ be the set $Z_2 = \{0, 1\}$ is near-ring.

Definition 2. A subgroup M of a near-ring N with $M \bullet M \subseteq M$ is called a sub-near-ring of N.

Definition 3. Given a near-ring N, $N_0 = \{n \in N;\ n0 = 0\}$ which is called the zero-symmetric part of N, $N_c = \{n \in N;\ n0 = n\} = \{n \in N;\ nn' = n\}$ for every n' in N is called the constant part of N.

Note: N_0 and N_c are sub-near-ring of N.

Definition 4. An element e in N is said to be an idempotent if $e^2 = e$.

Example 4. Let $Z_6 = \{0, 1, 2, 3, 4, 5\}$ be the near-ring under addition modulo 6 and multiplication modulo 6. And $4 \in Z_6$ is an idempotent.

Since $4 \cdot 4 = 4$ (module 6).

Definition 5. A near-ring N is said to be regular if given $a \in N$ there exists an element $n \in N$ such that $a = ana$.

Example 5. Let $N = M_2(N)$, the near-ring of all 2×2 matrices, It is well known N is regular. If $N = \left\{\begin{bmatrix} m & n \\ o & p \end{bmatrix};\ m, n, o, p \in Z\right\}$ is a regular.

In particular, let take $a = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \in M_2(N)$ and $n = \begin{bmatrix} 4 & -2 \\ 0 & 1 \end{bmatrix} \in M_2(N)$, then it is regular.

Definition 6. Let P be ideal in N. Then N is said to be P-regular if $a \in N$ there exists $n \in N$ such that $a = ana + p$ for some $p \in P$. And N is said to be strongly P-regular if for every there exists such that $a = na^2 + p$.

Example 6. Let $M_2(N) = \left\{\begin{bmatrix} u & v \\ w & x \end{bmatrix}, u, v, w, x \in Z\right\}$, the near-ring of all 2×2 matrices, if it is P-regular and also strongly P-regular. In particular, Let if take $a = \begin{bmatrix} 1 & 2 \\ \\ 0 & 4 \end{bmatrix} \in M_2(N)$ and $n = \begin{bmatrix} 4 & -2 \\ \\ 0 & 1 \end{bmatrix} \in M_2(N)$, arbitrary ideal $p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(N)$. Now $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ it is P-regular. And also Let $a = na^2 + p$ if take $a = \begin{bmatrix} 1 & 2 \\ \\ 0 & 4 \end{bmatrix} \in M_2(N)$ and

$n = \begin{bmatrix} 4 & -2 \\ 0 & 1 \end{bmatrix} \in M_2(N)$, arbitrary ideal $p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(N)$. Now

$\begin{bmatrix} 4 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 10 \\ 0 & 16 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 0 & 16 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \subseteq \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$.

Therefore it is strongly P-regular near-ring.

Example 7. Let $M_2(N) = \left\{\begin{bmatrix} 0 & v \\ 0 & x \end{bmatrix}; v, x \in N\right\}$, the near-ring of all 2×2 matrices. If it is not P-regular and strongly P-regular.

Example 8. $(M(Z_4), +, \cdot)$ is a regular near-ring. But it is not a strongly regular near-ring. In particular, let $Z_4 = \{0, 1, 2, 3\}$ be the group of integer modulo 4, it is regular near-ring.

Lemma 1 [1]**.** *Suppose N is a left strongly regular near-ring if* $ab = 0$ *for some* a, b *in N, Then* $ba = b0$.

Lemma 2 [1]**.** *Suppose N is a left strongly regular near-ring if* $b^2 = b^3$ *for some bN then* $b^2 = b$.

Theorem 1. *Let N be a left strongly P-regular near-ring. If for some* a, n *in N and then it is P-regular.*

Proof. Since N be a left strongly P-regular near-ring $a = na^2 + p$ for $p \in P$. $(a - (na^2 + p))a = 0$ and by Lemma 1, $a(a - (na^2 + p)) = a0$ and $ana(a - (na^2 + p)) = ana0$. Therefore

$$\begin{aligned}(a - (na^2 + p))^2 &= a(a - (na^2 + p)) - (na^2 + p)(a - (na^2 + p)) \\ &= a0 - (ana + p)(a - (na^2 + p)) \\ &= a0 - ana(a - (na^2 + p)) + p(na^2 + p) = a0 - ana0 + p.\end{aligned}$$

Now

$$\begin{aligned}(a - (na^2 + p))^3 &= (a - (na^2 + p))(a - (na^2 + p))^2 = (a - (na^2 + p))((a0 - an)a0 + p) \\ &= a((a0 - an)a0 + p) - ((na^2 + p)(a0 - an)a0 + p)\end{aligned}$$

$$= (a0 - an)a0 + p - (a0a0 - ana0)(na^2 + p) - p(na^2 + p)$$

$$= (a0 - an)a0 + p - (a0 - a0)(na^2 + p) - p(na^2 + p)$$

$$= (a0 - an)a0 + p = (a - (na^2 + p))^2.$$

Hence by Lemma 2, we have $(a - (na^2 + p))^2 = (a - (na^2 + p))$. Consequently,

$$0 = (a - (na^2 + p))a = ((a - (na^2 + p))^2 a$$

$$= ((a0 - an)a0 + p)a = (a0 - an)a0a + pa = (a0 - an)a0 + p$$

$$= (a - (na^2 + p))^2 = (a - (na^2 + p)).$$

Thus $a = na^2 + p$. Hence $a = ana + p$.

Definition 7. We say that a near-ring N has the property $(*)$ if it satisfies:

(i) for any a, b in N, $ab = 0$ implies $ba = b0$

(ii) for any a in N, $a^2 = a^3$ implies $a = a^2$.

Theorem 2. *Let N be a right strongly P-regular near-ring. If $a = a^2n + p$ for some a, b in N and $p \in P$. Then $a = ana + p$ where P is arbitrary ideal.*

Proof. The proof is similar to the above theorem.

Theorem 3. *Let N be a P-regular near-ring if for some a, n in N and there exists $p \in P$ if it is regular.*

Proof. Assume that N is a P-regular near-ring (i.e.) $a = ana + p$ for some a, n in N and $p \in P$. Let take P be an arbitrary ideal (i.e.) $p = 0$. Now $a = ana + p = ana + 0 = ana$. Therefore $a = ana$, N is regular near-ring.

Assume that N is a regular near-ring. We have to prove that N is P-regular near-ring. Since $a = ana$. If take zero in right side only we get, $a = ana + 0$ (since 0 is an arbitrary ideal P). Hence $a = ana + p$.

Theorem 4. *For any idempotent e and any n in* N, $en = ene + p$ *where P be an arbitrary ideal.*

Proof. Let $e^2 = e$ and $n \in N$. Clearly $en = ne^2 + p$. As we have by Lemma 1 $ene(en - (ene + p)) = ene0$. As in the proof of Theorem 1, We can prove that $(en-(ene+p))^2 = en0 = ene0 - p$. In Similar way $(en-(ene+p))^3 = (en-(ene+p))^2$. Hence $(en-(ene+p))e = (en-(ene+p))^2 e = (en0-ene0-p)$ $e = (en-ene)e0 - p = en - ene - p$. Thus $en = ene + p$.

Definition 8. A sub-near-ring B of a near-ring N is called a bi-ideal of N if $B \subseteq BNB$.

Example 9. Let $N = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \in Z^{+}\right\}$. Then N is a near-ring under usual addition and matrix multiplication. Let $B = \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}; x \in Z^{+}\right\}$ it is a bi-ideal.

Example 10. Let $N = \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}; x, y \in Z^{+}\right\}$ it is not a bi-ideal.

Theorem 5. *Bi-ideal of strongly P-regular near-ring is a Bi-ideal near-ring of N.*

Proof. We know that a sub-near-ring B of a near-ring N is called a bi-ideal of N if Since strongly P-regular if $b = nb^2 + p$, for some b, n in N and $p \in P$. Now $BNB = n(BNB)^2 + p = n(BNBBNB) + p = n(B(NBN)B)$ $+p = n(BNB) + p = nB + p = B + p \subseteq B$. Hence $B \subseteq BNB$. Therefore N is Bi-ideal of near-ring.

3. Conclusion

In mathematics, the study on near-rings into an object of the exercise for several bit of research. In this paper, we try to study the concepts of the strongly P-regularity and apply a few ideals in the particular concepts.

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