Advances and Applications in Mathematical Sciences
Volume 22, Issue 10, August 2023, Pages 2125-2140

COMMON FIXED POINT THEOREM IN b-METRIC SPACES SATISFYING CLR_{ST}-PROPERTY WITH APPLICATIONS

RAKESH TIWARI and SHASHI THAKUR*

Department of Mathematics
Government V. Y. T. Post-Graduate
Autonomous, College, Durg 491001
Chhattisgarh, India
E-mail: rtiwari@govtsciencecollegedurg.ac.in

Department of Mathematics
C.L.C. Government Arts and Commerce College
Dhamdha, Chhattisgarh, India

Abstract

In this paper, we establish the existence and uniqueness of common fixed point for quadruple self mappings satisfying the CLR_{ST} -property in b-metric spaces. These mappings are weakly compatible pairwise and satisfying quadratic type contraction condition. An example is also provided to verify the effectiveness and usability of our main result. Our result improves various results appeared in the current literature. As applications, we present a fixed point theorem for four finite families of self mappings which can be utilized to derive common fixed point theorems involving any number of finite mappings and also we provide the existence of solution of system of nonlinear integral equation. Our result is a distinct version of the result of Manoj Kumar et al. [16].

1. Introduction

Metric fixed point theory plays an important role in mathematics. Banach contraction principle is the fundamental result in fixed point theory and is generalized in many ways. Jungck [12] proved a common fixed point theorem for commutings maps as a generalization of the Banach's fixed point theorem.

2020 Mathematics Subject Classification: 47H10, 54H25.
Keywords: fixed point, common fixed point, weakly compatible maps, CLR_{ST} property, b-metric spaces.
*Corresponding author; E-mail: shashithakur89520@gmail.com
Received July 3, 2022; Accepted June 12, 2023

Sessa [18] introduced the concept of weakly commuting mappings, Jungck [13] extended this concept to compatible maps. In 1998, Jungck and Rhoades [14] introduced the notion of weak compatibility and showed that compatible maps are weakly compatible but the converse is not true.

In 1989, Bakhtin [3] introduced the concept of b-metric spaces which is a generalization of metric spaces. After that, Czerwik [6, 7] defined it such as current structure which is consider a generalization of metric spaces. Aydi et al. [2] proved common fixed point results for single-valued and multivalued mappings satisfying a weak φ-contraction in b-metric spaces. Several authors established fixed and common fixed point results in a generalization of b-metric spaces [9, 10]. Further, several interesting results have been obtained about the existence of a fixed point and common fixed point in b-metric spaces [5, 6, 16, 17, 20]. Recently, some authors established common fixed point theorems for generalization of contraction mapping in b-metric spaces [4, 8].

2. Preliminaries

We recall some definitions which will be used in the sequel.

Definition 2.1[3]**.** Let X be a non-empty set and $s \geq 1$ a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if for each $x, y, z \in X,$ the following conditions are satisfied.

1. $d(x, y) = 0$ iff $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

Definition 2.2[5]**.** Let (X, d) be a b-metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we have $x \in Y$.

Definition 2.3[13]**.** Let f and g be given self mappings on a set X. The pair (f, g) is said to be weakly compatible if f and g commute at their coincidence point (i.e. $fgx = gfx$ whenever $fx = gx$).

Definition 2.4[15]**.** A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied.

1. ψ is nondecreasing and continuous,

2. $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.5[1]**.** An ultra altering distance function is a continuous nondecreasing mapping $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0, t > 0$ and $\varphi(0) \geq 0$.

Definition 2.6[19]**.** Two self maps f and S of a metric space (X, d) are said to satisfy common limit range property with respect to S, denoted (CLR_S) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t \text{ where } t \in S(X).$$

Definition 2.7[11]**.** Two pairs (f, S) and (g, T) of self mappings of a metric space (X, d) are said to satisfy common limit range property with respect to S and T, denoted (CLR_{ST}) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = t \text{ where } t \in S(X) \cap T(X).$$

In the present paper, we establish the existence and uniqueness of common fixed point for quadruple self mappings satisfying CLR_{ST}-property in b-metric spaces. These mappings are weakly compatible pairwise and satisfying quadratic type contraction condition. An example is also provided to verify the effectiveness and usability of our main result. Our result improves various results appeared in the current literature. As an application, we present a fixed point theorem for four finite families of self mappings which can be utilized to derive common fixed point theorems involving any number of finite mappings and also we provide the existence of solution of system of nonlinear integral equation.

3. Main Result

The following Lemma will be used in the proof of our main Theorem.

Lemma 3.1. *Let f, g, S and T be continuous self-mappings of a b-metric space (X, d).*

Suppose that

1. *The pairs (f, S) and (g, T) satisfies the (CLR_S) and (CLR_T) properties respectively.*

2. *$fX \subset TX$ and $gX \subset SX$,*

3. *SX and TX are closed in X,*

4. *gy_n converges for each sequence $\{y_n\}$ in X whenever $\{Ty_n\}$ converges (respectively fx_n converges for every sequence $\{x_n\}$ in X whenever $\{Sx_n\}$ converges),*

5. *there exists $\phi \in \varphi$ and $\psi \in \Psi$,*

such that

$$\psi(b^2 d^2(fx, gy)) \leq \psi(M_b(x, y)) - \phi(M_b(x, y)) \tag{3.1}$$

for all $x, y \in X$, where

$$M_b(x, y) = \max\{d^2(fx, Ty), d^2(Sx, gy), d^2(Sx, Ty), \frac{1}{2b}(d(Sx, Ty)d(fx, Sx) + d(fx, Sx)d(gy, Ty))\}$$

Then the pairs (f, S) and (g, T) share the CLR_{ST}-property.

Proof. If the pair (f, S) satisfy the (CLR_{ST}) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = p \text{ where } p \in SX.$$

Now, since $fX \subset TX$, so, for each sequence $\{x_n\}$, there exists a sequence $\{y_n\}$ in X such that $fx_n = Ty_n$. But TX is closed, so $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} fx_n = p$. So that $p \in TX$ and in all $p \in SX \cap TX$. Thus we get $fx_n \to p, Sx_n \to p, Ty_n \to p$ as $n \to \infty$. Let us show that $gy_n \to p$ as $n \to \infty$. On the contrary suppose that $gy_n \to q \neq p$ as $n \to \infty$.

Put $x = x_n$ and $y = y_n$ in (3.1), we get

$$\psi(d^2(fx_n, gy_n)) \leq \psi(b^2 d^2(fx_n, gy_n))$$

$$\leq \psi(M_b(x_n, y_n)) - \phi(M_b(x_n, y_n)),$$

where

$$M_b(x_n, y_n) = \max\{d^2(fx_n, Ty_n), d^2(Sx_n, gy_n), d^2(Sx_n, Ty_n),$$

$$\frac{1}{2b}(d(Sx_n, Ty_n)d(fx_n, Sx_n) + d(fx_n, Sx_n)d(gy_n, Ty_n))\}$$

implies

$$\lim_{n\to\infty} M_b(x_n, y_n) = \max\{0, d^2(p, q), 0, 0\}$$

$$= d^2(p, q),$$

we have

$$\psi(d^2(p, q)) \leq \psi(b^2 d^2(p, q))$$

$$\leq \psi(\lim_{n\to\infty} M_b(x_n, y_n)) - \phi(\lim_{n\to\infty} M_b(x_n, y_n))$$

$$\leq \psi(d^2(p, q)) - \phi(d^2(p, q)).$$

From the definition of φ, we get

$$\psi(d^2(p, q)) \leq \psi(d^2(p, q)) - \phi(d^2(p, q)),$$

implies that $\phi(d^2(p, q)) = 0$. Hence $q = p$, a contradiction. Which shows that the pairs (f, S) and (g, T) share the CLR_{ST} property. This completes the proof.

Now, we prove our main result as follows:

Theorem 3.2. *Let f, g, S and T be continuous self-mappings of a b-metric space (X, d) satisfying the inequality* (3.1). *If the pairs (f, S) and (g, T) satisfy the (CLR_{ST})-property, then (f, S) and (g, T) have a point of coincidence. Moreover f, g, S and T have a unique common fixed point provided both the pairs (f, S) and (g, T) are weakly compatible.*

Proof. Since the pairs (f, S) and (g, T) satisfying the (CLR_{ST}) property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = p,$$

where $p \in S(X) \cap T(X)$. Since $p \in SX$, so their exists a point $q \in X$ such that $Sq = p$. We show that $fq = Sq$. Putting $x = q$, $y = y_n$ in (3.1), we get

$$\psi(b^2 d^2(fq, gy_n)) \leq \psi(M_b(q, y_n) - \phi(M_b(q, y_n)), \tag{3.2}$$

where

$$M_b(q, y_n) = \max\{d^2(fq, Ty_n), d^2(Sq, gy_n), d^2(Sq, Ty_n),$$
$$\frac{1}{2b}(d(Sq, Ty_n)d(fq, Sq) + d(fq, Sq)d(gy_n, Ty_n))\}$$

taking limit $n \to \infty$, we get

$$\lim_{n\to\infty} M_b(q, y_n) = \max\{d^2(fq, p), d^2(p, p), d^2(p, p),$$
$$\frac{1}{2b}(d(p, p)d(fq, p) + d(fq, p)d(p, p))\}$$
$$= \max\{d^2(fq, p), 0, 0, 0\}$$
$$= d^2(fq, p).$$

Taking limit $n \to \infty$, in (3.2) and using the definition of φ, we get

$$\psi(d^2(fq, p)) \leq \psi(b^2 d^2(fq, p))$$
$$\leq \psi(d^2(fq, p)) - \phi(d^2(fq, p))$$

implies that $\Phi(d^2(fq, p)) = 0$. Hence $fq = p = Sq$. Therefore q is the coincidence of the pair (f, S). Since the pair (f, S) is weakly compatible and $fq = Sq$. Therefore $fSq = Sfq$, which implies that $fp = Sp$. As $p \in TX$, there exists a point $r \in X$ such that $Tr = p$. We show that $gr = Tr$.

Putting $x = q$, $y = r$ in (3.1), we get

$$\psi(b^2 d^2(fq, gr)) \leq \psi(M_b(q, r)) - \phi(M_b(q, r)), \tag{3.3}$$

where

$$M_b(q, r) = \max\{d^2(fq, Tr), d^2(Sq, gr), d^2(Sq, Tr),$$

$$\frac{1}{2b}(d(Sq, Tr)d(fq, Sq) + d(fq, Sq)d(gr, Tr))\}$$

$$= \max\{d^2(p, p), d^2(p, gr), d^2(p, p),$$

$$\frac{1}{2b}(d(p, p)d(p, p) + d(p, p)d(gr, p))\}$$

$$= \max\{0, d^2(p, gr), 0, 0\}$$

$$= d^2(p, gr).$$

Thus, from (3.3) and using the definition of φ, we get

$$\psi(d^2(p, gr)) \leq \psi(b^2 d^2(p, gr))$$

$$\leq \psi(d^2(p, gr)) - \phi(d^2(p, gr)),$$

which implies that $\phi(d^2(p, gr)) = 0$. Hence $gr = p = Tr$. Therefore r is the point of coincidence of the pair (g, T). Since the pair (g, T) is weakly compatible and $gr = Tr$, therefore $gTr = Tgr$ which implies that $gp = Tp$.

Now we show that p is a common fixed point of the pair (f, S). Putting $x = p$, $y = r$ in (3.1), we get

$$\psi(b^2 d^2(fp, gr)) \leq \psi(M_b(p, r)) - \phi(M_b(p, r)), \tag{3.4}$$

where

$$M_b(p, r) = \max\{d^2(fp, Tr), d^2(Sp, gr), d^2(Sp, Tr),$$

$$\frac{1}{2b}(d(Sp, Tr)d(fp, Sp) + d(fp, Sp)d(gr, Tr))\}$$

$$= \max\{d^2(fp, p), d^2(fp, p), d^2(fp, p),$$

$$\frac{1}{2b}(d(fp, p)d(p, p) + d(Sp, Sp)d(p, p))\}$$

$$= \max\{d^2(fp, p), d^2(fp, p), d^2(fp, p), 0\}$$

$$= d^2(fp, p).$$

Thus, from (3.4) and using the definition of φ, we get

$$\psi(d^2(fp, p)) \leq \psi(b^2 d^2(fp, p))$$

$$\leq \psi(d^2(fp, p)) - \phi(d^2(fp, p)),$$

implies that $\phi(d^2(fp, p)) = 0$. Hence $fp = p = Sp$, which shows that p is a common fixed point of the pair (f, S). Similarly, we can show that $gp = p = Tp$. Hence p is a common fixed point of f, g, S and T.

Now, we show that p is the unique common fixed point. Let t be another common fixed point of f, g, S and T. Putting $x = p$, $y = t$ in (3.1), we get

$$\psi(b^2 d^2(fp, gt)) \leq \psi(M_b(p, t)) - \phi(M_b(p, t)), \tag{3.5}$$

where

$$M_b(p, t) = \max\{d^2(fp, Tt), d^2(Sp, gt), d^2(Sp, Tt),$$

$$\frac{1}{2b}(d(Sp, Tt)d(fp, Sp) + d(fp, Sp)d(gt, Tt))\}$$

$$= \max\{d^2(p, t), d^2(p, t), d^2(p, t),$$

$$\frac{1}{2b}(d(p, t)d(p, p) + d(p, p)d(t, t))\}$$

$$= \max\{d^2(p, t), d^2(p, t), d^2(p, t), 0\}$$

$$= d^2(p, t).$$

Thus, from (3.5) and using the definition of φ, we get

$$\psi(d^2(p, t)) \leq \psi(b^2 d^2(p, t))$$

$$\leq \psi(d^2(p, t)) - \phi(d^2(p, t)),$$

implies that $\phi(d^2(p, t)) = 0$. Hence $p = t$, which shows that p is the unique common fixed point of f, g, S and T.

Remark 3.3. If we put $g = f$ and $S = T$ in the Theorem 3.2, we get the following result.

Corollary 3.4. *Let f, T be continuous self mappings of a b-metric space satisfying*

$$\psi(b^2 d^2(fx, fy)) \leq \psi(M_b(x, y)) - \phi(M_b(x, y))$$

for all $x, y \in X$, where

$$M_b(x, y) = \max \{d^2(fx, Ty), d^2(Tx, fy), d^2(Tx, Ty),$$
$$\frac{1}{2b}(d(Tx, Ty)d(fx, Tx) + d(fx, Tx)d(fy, Ty))\}.$$

If the pair (f, T) satisfies the (CLR_T) property then the pair (f, T) has a common point of coincidence. Moreover if the pair (f, T) is weakly compatible, then f and T have a unique common fixed point.

Remark 3.5. If we put $\psi(t) = t$ in the Theorem 3.2, we get the following result.

Corollary 3.6. *Let f, g, S and T be continuous self mappings of a b-metric space (X, d) satisfying*

$$b^2 d^2(fx, gy) \leq M_b(x, y) - \phi(M_b(x, y))$$

for all $x, y \in X$, where

$$M_b(x, y) = \max \{d^2(fx, Ty), d^2(Sx, gy), d^2(Sx, Ty),$$
$$\frac{1}{2b}(d(Sx, Ty)d(fx, Sx) + d(fx, Sx)d(gy, Ty))\}.$$

If the pairs (f, S) and (g, T) satisfy the CLR_{ST} -property, then the pairs (f, S) and (g, T) have a point of coincidence. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Now, we illustrate an example to validate our main Theorem 3.2.

Example 3.7. Consider $X = [0, 2]$ and $d(x, y) = \max\{x, y\}$, $\psi(t) = \sqrt{t}$, $\phi(t) = \frac{t}{100}$ and $b = \frac{7}{6}$. Define the mappings f, g, S and T on X such that

$$f(x) = \begin{cases} \frac{x(x+6)}{6}, & \text{if } x \in [0, 1); \\ \frac{x+13}{12}, & \text{if } x \in [1, 2]; \end{cases}$$

$$g(x) = \begin{cases} 0, & \text{if } x \in [0, 1); \\ x - 1, & \text{if } x \in [1, 2]; \end{cases}$$

$$S(x) = \begin{cases} \frac{x(x+2)}{2}, & \text{if } x \in [0, 1); \\ \frac{x+5}{4}, & \text{if } x \in [1, 2]; \end{cases}$$

$$T(x) = \begin{cases} x, & \text{if } x \in [0, 1); \\ \frac{x+2}{3}, & \text{if } x \in [1, 2]; \end{cases}$$

We take the sequence $x_n = \{0\}$ and $y_n = \left\{\frac{1}{n}\right\}$. We have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = 0 \in S(X) \cap T(X).$$

Therefore, both pairs (f, S) and (g, T) satisfy CLR_{ST}-property. We see that mappings (f, S) and (g, T) commute at 0 which is coincidence point. Also, $f(x) \subset T(X)$ and $g(x) \subset S(X)$. We can verify the contraction condition (3.1) by a simple calculation for the case $x, y \in [1, 2]$,

$$\psi(b^2 d^2(fx, gy)) = \frac{7}{6}\frac{5}{4} \leq \frac{2751 \times 16}{49 \times 1600} d^2(Sx, gy)$$

$$\leq \frac{2751 \times 16}{49 \times 1600} M_b(x, y) = \psi(M_b(x, y)) - \phi(M_b(x, y)).$$

Thus (3.1) is satisfied. Similarly, we can verify for other cases. Thus all the conditions of Theorem 3.2 are satisfied and 0 is the unique common fixed point of the mappings f, g, S and T.

4. Applications

As an application of Theorem 3.2, we establish a common fixed point theorem for finite families of mappings as follows:

Theorem 4.1. *Let* (X, d) *be a complete* b*-metric space. Let* $\{f_i\}_{i=1}^{m}$, $\{g_j\}_{j=1}^{n}$, $\{S_k\}_{k=1}^{p}$ *and* $\{T_l\}_{l=1}^{q}$ *be four finite families of continuous self mappings on* X *with* $f = f_1 f_2 \ldots f_m$, $g = g_1 g_2 \ldots g_n$, $S = S_1 S_2 \ldots S_p$ *and* $T = T_1 T_2 \ldots T_q$ *satisfying the condition* (3.1). *Suppose that the pairs* (f, S) *and* (g, T) *satisfy the* CLR_{ST}*-property, then* (f, S) *and* (g, T) *have a point of coincidence. Moreover,* $\{f_i\}_{i=1}^{m}, \{g_j\}_{j=1}^{n}, \{S_k\}_{k=1}^{p}$ *and* $\{T_l\}_{l=1}^{q}$ *have a unique common fixed point if the families* $(\{f_i\}, \{g_j\}, \{S_k\}$ *and* $\{T_l\})$ *commute pairwise where* $i \in \{1, 2, \ldots m\}$, $k \in \{1, 2, \ldots p\}$, $j \in \{1, 2, \ldots n\}$ *and* $l \in \{1, 2, \ldots q\}$.

Proof. The conclusions are immediate as f, g, S and T satisfy all the conditions of Theorem 3.2. Now appealing to component wise commutativity of various pairs, one can immediately prove that $fS = Sf$ and $gT = Tg$ and hence, obviously both pairs (f, S) and (g, T) are coincidentally commuting. Note that all the conditions of Theorem 3.2 (for mappings f, g, S and T) are satisfied ensuring the existence of a unique common fixed point, say z. Now, we show that z remains the fixed point of all the component maps. For this, we consider

$$\begin{aligned} f(f_t z) &= ((f_1 f_2 \ldots f_m) f_t) z \\ &= (f_1 f_2 \ldots f_{m-1})((f_m f_t) z) \\ &= (f_1 f_2 \ldots f_{m-1})((f_t f_m) z) \\ &= (f_1 f_2 \ldots f_{m-2})(f_{m-1} f_t (f_m z)) \\ &= (f_1 f_2 \ldots f_{m-2})(f_t f_{m-1} (f_m z)) \\ &= f_1 f_t (f_2 f_3 \ldots f_m z) \\ &= f_t f_1 (f_2 f_3 \ldots f_m z) = f_t (fz) = f_t z. \end{aligned}$$

Similarly, we show that

$$f(S_u z) = S_u(Tz) = S_u z, \quad S(S_u z) = S_u(Sz) = S_u z,$$

$$S(f_t z) = f_t(Sz) = f_t z, \quad g(g_v z) = g_v(gz) = g_v z,$$

$$g(T_w z) = T_w(gz) = T_w z, \quad T(T_w z) = T_w(Tz) = T_w z,$$

$$T(g_v z) = g_v(Tz) = g_v z,$$

which show that (for all t, u, v and w) $f_t z$ and $S_u z$ are other fixed points of the pair (f, S) whereas $g_v z$ and $T_w z$ are other fixed points of the pair (g, T). Now appealing to the uniqueness of common fixed points of both pairs separately, we get

$$z = f_t z = S_u z = g_v z = T_w z,$$

which shows that z is a common fixed point of f_t, S_u, g_v and T_w for all t, u, v and w.

Remark 4.2. By setting $f = f_1 = f_2 = \ldots = f_m$, $S = S_1 = S_2 = \ldots = S_p$, $g = g_1 = g_2 = \ldots = g_n$ and $T = T_1 = T_2 = \ldots = T_q$, we get the following corollary.

Corollary 4.3. *Let (X, d) be a complete b-metric space. Let f, S, g and T be continuous self mappings such that f^m, S^p, g^n and T^q satisfying the conditions* (3.1). *Also, assume that the pairs (f^m, S^p) and (g^n, T^q) share the $(CLR_{S^pT^q})$-property, where m, p, n and q are fixed positive integers. Then f, S, g and T have a unique common fixed point provided $fS = Sf$ and $gT = Tg$.*

Application to system of integral equation

In this section, we will use Theorem 3.2 to show the solution to the following nonlinear integral equations:

$$x(t) = h(t) + \int_0^1 K_1(t, s, x(s))ds \tag{4.1}$$

$$y(t) = h(t) + \int_0^1 K_2(t,\ s,\ y(s))ds \tag{4.2}$$

where

1. $h(t) : [0, 1] \to \mathbb{R}$ is continuous,

2. $K_i : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R},\ i = 1, 2$ are continuous functions.

Let $X = C[0, 1]$ be a set of all real valued continuous function on $[0, 1]$. Define $d : X \times X \to [0, \infty)$ by:

$$d(x,\ y) = \max_{t \in [0,\ 1]} |\, x(t) - y(t) \,|$$

for all $x,\ y \in X$. Therefore $(X,\ d)$ is a b-metric space with parameter $b = 2$.

Theorem 4.4. *Consider* (4.1) *and* (4.2) *and assume that*

1. $x,\ y \in C[0, 1]$ *and* $K_i : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R},\ i = 1, 2$ *are continuous functions*,

2. *for all* $s, t \in [0, 1]$ *and all* $x,\ y \in C[0, 1]$ *we have*

$$|\, K_1((t,\ s,\ x(s))) - K_2((t,\ s,\ y(s)))\,| \le q(t,\ s) |\, x(s) - y(s)\,|,$$

where $q : [0, 1] \times [0, 1] \to [0, \infty)$ *is a continuous function satisfying*

$$\max_t \int_0^1 |\, q(t,\ s)\,| dt \le \frac{1}{\sqrt{5}} < 1.$$

Then the system (4.1) *and* (4.2) *have a unique common solution in X.*

Proof. Define mappings $f,\ g : X \to X$ by

$$fx(t) = h(t) + \int_0^1 K_1(t,\ s,\ x(s))ds,$$

$$gy(t) = h(t) + \int_0^1 K_2(t,\ s,\ x(s))ds.$$

Let $x,\ y \in X$ from condition (2) for all $t \in [0, 1]$, we have

$$d(fx(t),\ gy(t)) = \max_{t \in [0,1]} |\, fx(t) - gy(t)\,|$$

$$\leq \max_{t \in [0,1]} \int_0^1 | K_1((t, s, x(s))) - K_2((t, s, y(s))) | ds$$

$$\leq \max_{t \in [0,1]} \int_0^1 q(t, s) | x(s) - y(s) | ds$$

$$\leq \max_{t \in [0,1]} \int_0^1 q(t, s) | x(s) - y(s) | ds$$

$$\leq \max_{t \in [0,1]} | x(t) - y(t) | \max_{t \in [0,1]} \int_0^1 q(t, s) ds$$

$$\leq \frac{1}{\sqrt{5}} d(x(t), y(t))$$

which implies

$$2^2 d^2(f(x(t)), g(y(t))) \leq \frac{4}{5} d^2(x(t), y(t))$$

$$\leq \frac{4}{5} M_b(x, y)$$

$$= \psi(M_b(x, y)) - \phi(M_b(x, y)).$$

Therefore $\psi(b^2 d^2(f(x(t)), g(y(t))) \leq \psi(M_b(x, y)) - \phi(M_b(x, y)),$ define $Sx = Tx = x$, $\psi(t) = t$ and $\phi(t) = \frac{t}{5}$. Hence all the conditions of Theorem 3.2 hold then by Theorem 3.2 the system of integral equations (4.1) and (4.2) have a unique common solution in X.

5. Conclusion

From our investigations, we conclude that the self mappings on a b-metric space with CLR_{ST}-property and weak compatibility have a unique common fixed point with certain conditions. Our investigations and results obtained were supported by suitable example and application which provides new path for the researchers in the concerned field.

References

[1] A. H. Ansari, Note on $\varphi - \psi$ -contractive type mappings and related fixed point, The 2nd Regional Conference on mathematics and Applications PNU, 2014, 377-380.

[2] H. Aydi, M. F. Bota, E. Karapinar and S. Moradi, A common fixed point for weak φ-contraction on b-metric spaces, Fixed Point Theory 13 (2012), 337-346.

[3] I. A. Bakhtin, The contraction mapping principal in almost metric spaces, Funct. Anal. 30 (1989), 26-37.

[4] K. Berrah, A. Aliouche and T. Oussaeif, Applications and theorem on common fixed point in complex valued b-metric space, AIMS Mathematics 4(3) (2019), 1019-1033, DOI: 10.3934/math.2019.3.1019.

[5] M. Boriceanu, M. Bota and A. Petrusel, Multivalued fractal in b-metric spaces, Cent. Eur. J. Math. 8(2) (2010), 367-377, https://doi.org/10.2478/s11533-010-0009-4.

[6] S. Czerwik, Contraction mappings in b-metric spaces, Acta Mat. Inf. Uni. Ostraviensis 1 (1993), 5-11.

[7] S. Czerwik, Nonlinear set valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fis. Univ. Modena 46(2) (1998), 263-276.

[8] H. Faraji, D. Savic and S. Radenovic, Fixed point theorems for Geraghty contraction type mappings in b-metric spaces and applications, Axioms 8(34) (2019), 1-12, DOI:10.3390/axioms8010034.

[9] N. Hussain and M. H. Shah, KKM mappings in cone b-metric spaces, Comput. Math. Appl. 62 (2011), 1677-1684, https://doi.org/10.1016/j.camwa.2011.06.004.

[10] N. Hussain, D. Doric, Z. Kadelburg and S. Radenovic, Suzuki type fixed point results in metric type spaces, Fixed Point Theory Appl. 2012, 12 pages.

[11] M. Imdad, S. Chauhan and Z. Kadelburg, Fixed point theorem for mappings with common limit range property satisfying generalized (ψ, φ) weak contractive conditions, Math. Sci. (Springer). 7(16) (2013), 1-8, DOI: 10.1186/2251-7456-7-16.

[12] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly 83 (1976), 261-263, DOI: 10.2307/2318216.

[13] G. Jungck, Compatible mappings and common fixed points, International Journal of Mathematics and Mathematical Sciences 9 (1986), 771-779, https://doi.org/10.1155/S0161171286000935.

[14] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, Indian Journal of Pure and Applied Mathematics 29 (1998), 227-238.

[15] M. S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, Bulletin of the Australian Mathematical Society 30(1) (1984), 1-9.

[16] Manoj Kumar, L. N. Mishra and S. Mishra, Common fixed theorems satisfying (CLR_{ST}) property in b-metric space, Adv. in Dyna. Sys. and Appl. 12(2) (2017), 135-147.

[17] J. R. Roshan, V. Parvaneh, S. Radenovic and M. Rajovic, Some coincidence point results for generalized (ψ, φ)-weakly contractions in ordered *b*-metric spaces, Fixed Point Theory Appl. (2015), 1-21, DOI: 10.1186/s13663-015-0313-6.

[18] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. Soc. 32 (1982), 149-153.

[19] W. Sintunavarat and P. Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math. Article ID 637958, 2011, 14 pages, DOI:10.1155/2011/637958.

[20] V. Ozturk and D. Turkoglu, Common fixed point theorems for mappings satisfying (E:A)-property in *b*-metric spaces, J. of Nonlinear Sci. and Appl. 8 (2015), 1127-1133.