EXPLICIT RUNGE KUTTA METHOD IN SOLVING FUZZY INITIAL VALUE PROBLEM

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Abstract

In this paper, the explicit Runge-Kutta method of order four with Butcher [2] table is used to solve the fuzzy initial value problems, the co-efficient of the Runge Kutta method are taken from the Butcher's [2] table. The efficiency and accuracy of the proposed method is examined with a numerical example.

1. Introduction

In 1972, Chang and Zadeh [3] first presented the Fuzzy functions and its derivatives. In continuing, the principle approach was extended to solve the Fuzzy differential equations by Dubois and Prade [4]. Kaleva et al., [7] solve the fuzzy differential equation with initial values. The numerical method to solve the fuzzy initial value problems are introduced by various researchers like Ma et al. [8] studied classical Euler method and Abbas bandy et al., [1] introduced Taylor method. In this paper, the method of solving the fuzzy initial value problem through Explicit Runge Kutta method with Butcher's coefficients is studied.
2. Preliminaries

Definition 2.1. Trapezoidal fuzzy number is a four tuples $u = (a, b, c, d)$ such that $a < b < c < d$, with base is the interval $[a, d]$ and vertex $x = b, x = c$, and its membership function is given by

$$u(x) = \begin{cases} 
0, & x < 0 \\
\frac{x - a}{b - a}, & a \leq x \leq b \\
\frac{b - a}{1}, & b \leq x \leq c \\
\frac{c - x}{e - x}, & c \leq x \leq d.
\end{cases}$$

And have,

1. $u > 0$ if $a > 0$; (2) $u \geq 0$ if $b > 0$;
2. $u > 0$ if $c > 0$; (4) $u > 0$ if $d > 0$.

Definition 2.2. A fuzzy number $\tilde{u} = \{u: u : R \to [0, 1]\}$ and satisfies the following

1. $\tilde{u}$ is upper semi-continuous.
2. $\tilde{u}$ is fuzzy convex, if $u(\lambda x + (1 - \lambda)y) \geq \min \{u(x), u(y)\} \forall x, y \in R, 0 \leq \lambda \leq 1.$
3. $\tilde{u}$ is normal, $\exists x_0 \in R$ for which $u(x_0) = 1$
4. Closure of the set $\{x \in R, u(x) > 0\}$ is compact.

Definition 2.3. The parametric form of a fuzzy number $\tilde{u}$ is represented as a pair $(u, \tilde{u})$ of maps $(\underline{u}(\delta), \overline{u}(\delta)), 0 \leq \delta \leq 1$, such that

1. $\underline{u}(\delta)$ is a left continuous, bounded and monotonic increasing map.
2. $\overline{u}(\delta)$ is a left continuous, bounded and monotonic decreasing map
3. $\underline{u}(\delta) \leq \overline{u}(\delta)$, for $\delta \in (0, 1]$.

Definition (Fuzzy Arithmetic) 2.4. Let $\tilde{u} = (\underline{u}(\delta), \overline{u}(\delta)), \tilde{v} = (v(\delta), \overline{v}(\delta)), 0 \leq \delta \leq 1$ be arbitrary Fuzzy numbers and let $k \in R$, the arithmetic operations on fuzzy numbers are defined by

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\[
\tilde{u} + \tilde{v} = (\bar{u}(\delta) + \bar{v}(\delta), \underline{u}(\delta) + \underline{v}(\delta))
\]

\[
\tilde{u} - \tilde{v} = (\bar{u}(\delta) + \underline{v}(\delta), \underline{u}(\delta) + \bar{v}(\delta))
\]

\[
\tilde{u} \cdot \tilde{v} = (\min \{\underline{u}(\delta)\underline{v}(\delta), \underline{u}(\delta)\bar{v}(\delta), \bar{u}(\delta)\underline{v}(\delta), \bar{u}(\delta)\bar{v}(\delta)\},\max \{\underline{u}(\delta)\underline{v}(\delta), \underline{u}(\delta)\bar{v}(\delta), \bar{u}(\delta)\underline{v}(\delta), \bar{u}(\delta)\bar{v}(\delta)\})
\]

\[
C \tilde{u} = \begin{cases} (C \bar{u}(\delta), C \underline{u}(\delta)), & \text{if } C \geq 0 \\ (C \underline{u}(\delta), C \bar{u}(\delta)), & \text{if } C < 0. \end{cases}
\]

Let \( D : \tilde{u} \times \tilde{u} \to \mathbb{R}^+ \cup \{0\}, \)

\[
D(u, v) = \sup_{\delta \in [0, 1]} \max \left\{ \left| \bar{u}(\delta) - \bar{v}(\delta) \right|, \left| \underline{u}(\delta) - \underline{v}(\delta) \right| \right\}, \quad \text{be Hausdorff distance between fuzzy numbers, where} \quad \tilde{u} = (\bar{u}(\delta) - \bar{v}(\delta)), \quad \tilde{v} = (\bar{v}(\delta) - \underline{v}(\delta)).
\]

The following properties are well known:

\[
D(u + w, v + w) = D(u, v), \forall u, v, w \in \tilde{u},
\]

\[
D(ku, kv) = \left| k \right| D(u, v), \forall k \in \mathbb{R}, u, v \in \tilde{u},
\]

\[
D(u + v, w + e) = D(u, w) + D(v, e), \forall u, v, w, e \in \tilde{u}.
\]

And \((\tilde{u}, D)\) is a complete metric space.

**Definition 2.5.** Let \( F \) be the set of all fuzzy numbers, the \( \delta \)-level set of fuzzy number \( \tilde{u} \in F, 0 \leq \delta \leq 1 \), is defined by \([u]_\delta = \{x \in \mathbb{R} / u(x) \geq \delta \text{ if } 0 \leq \delta, 1\}.\)

The \( \delta \)-level set \([u]_\delta = (\bar{u}(\delta), \underline{u}(\delta))\) is closed and bounded.

**Lemma 2.1** [8]. If the sequence of positive numbers \( \{W_n\}_{n=0}^N \) satisfy

\[
\left| W_{n+1} \right| \leq A \left| W_n \right| + B, \quad 0 \leq n \leq N - 1,
\]

for the given \( A, B \in \mathbb{Z}^+ \), \( \left| W_n \right| \leq A^n \left| W_0 \right| + B \frac{A^n - 1}{A - 1}, \quad 0 \leq n \leq N - 1.\)

**Lemma 2.2** [8]. If the sequence of positive numbers \( \{W_n\}_{n=0}^N, \{V_n\}_{n=0}^N \) satisfy

\[
\left| W_{n+1} \right| \leq \left| W_n \right| + A \max \left\{ \left| W_n \right|, \left| V_n \right| \right\} + B,
\]

\[
\left| V_{n+1} \right| \leq \left| V_n \right| + A \max \left\{ \left| W_n \right|, \left| V_n \right| \right\} + B.
\]
for the given $A, B \in \mathbb{Z}^+, U_n = |W_n| + |V_n|$, $0 \leq n \leq N$, then
$U_n \leq A^n U_0 + B \frac{A^n - 1}{A - 1}$, $0 \leq n \leq N$, where $A = 1 + 2A$ and $B = 2B$.

**Theorem 2.1** [8]. Let $F(t, u, v)$ and $G(t, u, v)$ be in $C^1(K)$ and its partial derivatives are bounded above $K$ then, for random fixed $\delta, 0 \leq \delta \leq 1$, the approximate solutions $\underline{y}(t_{n+1}; \delta)$ and $\overline{y}(t_{n+1}; \delta)$ meet the exact solutions $\underline{y}(t; \delta)$ and $\overline{y}(t; \delta)$ regularly in $t$.

**Theorem 2.2** [8]. Let $F(t, u, v)$ and $G(t, u, v)$ be in $C^1(K)$ and its partial derivatives are bounded above $K$, $2lh < t$, then, for random fixed $\delta, 0 \leq \delta \leq 1$, the solutions $\underline{y}^i(t_n; \delta)$ and $\overline{y}^i(t_n; \delta)$, $i = 1, 2, \ldots$ not diverge to the algebraic solutions $\underline{y}(t_n; \delta)$ and $\overline{y}(t_n; \delta)$ in $t_0 \leq t_n \leq t_N$, when $i \to \infty$.

3. Fuzzy Initial Value Problems (FIVP)

Consider the fuzzy initial value differential equation has the form:

$$
\begin{align*}
{y}'(t) &= f(t, y(t)); \quad t \in [t_0, l] \\
y(t_0) &= y_0,
\end{align*}
$$

(1)

here $y$ is a fuzzy map in $t$, $f(t, y)$ is a fuzzy map of $t$ and fuzzy variable $y$, the derivative of $y$ is denoted by $y'$ and $y(t_0) = y_0$ is a fuzzy number (in triangular shaped).

The exact solution of the problem in (1) $[Y(t)]_\delta = [\underline{y}(t; \delta), \overline{y}(t; \delta)]$ be approximated by some $[y(t)]_\delta = [\underline{y}(t; \delta), \overline{y}(t; \delta)]$.$$

\begin{align*}
[y(t_0)]_\delta &= [\underline{y}(t_0; \delta), \overline{y}(t_0; \delta)], \quad \delta \in (0, 1]
\end{align*}

we write $f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$ and $\underline{f}(t, y) = F[t, y, \underline{y}], \overline{f}(t, y) = G[t, y, \overline{y}]$.

Because of $y' = f(t, y)$ we have

$$
\underline{f}(t, y(t); \delta) = F[t, y(t); \delta, \overline{y}(t, \delta)]
$$

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\[ f(t, y(t); \delta) = G[t, y(t); \delta, \tilde{y}(t, \delta)] \]

The extension principle gives the membership map as

\[ f(t, y(t))(s) = \sup \{y(t)(\tau) | s = f(t, \tau), s \in R\} \]

so fuzzy number \( f(t, y(t)) \). From this it follows that

\[ [f(t, y(t))]_\delta = [\tilde{f}(t, y(t)); \delta], \delta \in (0, 1], \]

where

\[ \tilde{f}(t, y(t); \delta) = \min \{f(t, u) | u \in [y(t)]_\delta\} \]

\[ f(t, y(t); \delta) = \max \{f(t, u) | u \in [y(t)]_\delta\}. \]

**Theorem 3.1** [8]. If a function \( f \) satisfy the following

\[ |f(t, u) - f(t, u')| \leq g(t, |u - u'|), t \geq 0, u, u' \in R, \]

where \( g : R^+ \rightarrow R^+ \) is a continuous function and \( \delta \rightarrow g(t, \delta) \) is increasing, the initial value problem \( u'(t) = g(t, u(t)), u(0) = u_0 \), has a solution on \( R^+ \) for \( u_0 = 0 \). Then the FVIP (1) has a unique fuzzy solution.

4. Explicit Runge-Kutta Method

The family of explicit Runge-Kutta methods is a generalization of the Runge Kutta method. It is given by

\[ y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i \]

where,

\[ k_1 = f(t_n, y_n), \]

\[ k_2 = f(t_n + hc_2, y_n + h(a_{21} k_1)), \]

\[ k_3 = f(t_n + hc_3, y_n + h(a_{31} k_1 + a_{32} k_2)), \]

\[ \vdots \]

\[ k_s = f(t_n + hc_s, y_n + h(a_{s1} k_1 + a_{s2} k_2 + \ldots + a_{s,s-1} k_{s-1})). \]
The integer $s$ (the number of stages), and the coefficients $a_{ij} (1 \leq j < i \leq s)$, $b_i (i = 1, 2, \ldots, s)$ and $c_i (i = 2, 3, \ldots, s)$. The matrix $[a_{ij}]$ is called the Runge Kutta matrix, while the $b_i$ and $c_i$ are known as the weights and the nodes. These data are usually arranged in a mnemonic device, known as a Butcher tableau [2]:

$$
\begin{array}{cccc}
0 & & & \\
& c_2 & a_{21} & \\
& c_3 & a_{31} & a_{32} & \\
& & \vdots & \vdots & \\
& & \vdots & \vdots & \\
& c_s & a_{s1} & a_{s2} & \ldots & a_{s,s-1} & \\
& & b_1 & b_2 & \ldots & b_{s-1} & b_s
\end{array}
$$

The Runge-Kutta method is consistent if $\sum_{j=1}^{i-1} a_{ij} = c_i$, $i = 2, 3, \ldots, s$.

There are also accompanying requirements if one requires the method to have a certain order $p$, meaning that the local truncation error is $O(h^{p+1})$, these can be derived from the definition of the truncation error itself.

In general, if an explicit $s$-stage Runge-Kutta method has order $p$, then it can be proven that the number of stages must satisfy $s \geq p$, and if $p \geq 5$, then $s \geq p + 1$. However, it is not known whether these bounds are sharp in all cases; for example, all known methods of order 8 have at least 11 stages, though it is possible that there are methods with fewer stages. Indeed, it is an open problem what the precise minimum number of stages is for an explicit Runge-Kutta method to have order $p$ in those cases where no methods have yet been discovered that satisfy the bounds above with equality. Some values which are known are:

<table>
<thead>
<tr>
<th>$p$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>mins</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
</tbody>
</table>
The provable bounds above then imply that we cannot find methods of orders $p = 1, 2, \ldots, 6$ that require fewer stages than the methods we already know for these orders. However, it is conceivable that we might find a method of order $p = 7$ that has only 8 stages, whereas the only ones known today have at least 9 stages as shown in the table.

A slight variation of “the” Runge-Kutta method is also due to Kutta in 1901 and is called the 3/8-rule. The primary advantage this method has is that almost all of the error coefficients are smaller than in the popular method, but it requires slightly more FLOPs (floating-point operations) per time step. Its Butcher [3] tableau is

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{3} & \frac{1}{3} & & \\
\frac{2}{3} & -\frac{1}{3} & & 1 \\
\frac{1}{3} & 1 & -1 & 1 \\
\frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\
\end{array}
$$

5. Explicit Runge-Kutta Fourth Order

The explicit Runge Kutta methods is to prompt the variance among the standards of $y$ at $t_{n+1}$ and $t_n$ as $y_{n+1} - y_n = h \sum_{i=0}^{s} b_i k_i$. Where $b_i$'s are persistent for all $i$ and $k_i = f(t_n + h c_i, y_n + h \sum_{j=1}^{i-1} \lambda_{ij} k_j)$ with $h = t_{n+1} - t_n$ and $y_0 = t_0$.

$$
k_1 = f(t_1, y_1) \\
k_2 = f(t_1 + h c_2, y_1 + h (\lambda_{21} k_1)) \\
k_3 = f(t_1 + h c_3, y_1 + h (\lambda_{31} k_1 + \lambda_{32} k_2)) \\
k_4 = f(t_1 + h c_4, y_1 + h (\lambda_{41} k_1 + \lambda_{42} k_2 + \lambda_{43} k_3)).
$$
From Butcher table,

<table>
<thead>
<tr>
<th></th>
<th>( c_1 = 0 )</th>
<th>( c_2 = \frac{1}{3} )</th>
<th>( c_3 = \frac{2}{3} )</th>
<th>( c_4 = 1 )</th>
<th>( \lambda_4 = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_1 = \frac{1}{8} )</td>
<td>( b_2 = \frac{3}{8} )</td>
<td>( b_3 = \frac{3}{8} )</td>
<td>( b_4 = \frac{1}{8} )</td>
<td>( \lambda_{41} = 1 )</td>
<td></td>
</tr>
<tr>
<td>( \lambda_{21} = \frac{1}{3} )</td>
<td>( \lambda_{31} = -\frac{1}{3} )</td>
<td>( \lambda_{32} = 1 )</td>
<td>( \lambda_{41} = 1 )</td>
<td>( \lambda_{42} = -1 )</td>
<td></td>
</tr>
</tbody>
</table>

Hence,

\[
y_{n+1} = y_n + h \left[ \frac{1}{8} (k_1 + k_4) + \frac{3}{8} (k_2 + k_3) \right],
\]

\[
k_1 = f(t_i, y_i),
\]

\[
k_2 = f\left(t_i + \frac{h}{3}, y_i + \frac{h k_1}{3}\right),
\]

\[
k_3 = f\left(t_i + \frac{2h}{3}, y_i + h \left( -\frac{1}{3} k_1 + k_2 \right) \right),
\]

\[
k_4 = f(t_i + h, y_i + h (k_1 - k_2 + k_3)).
\]


The exact solution of the problem in (1) \([Y(t)]_0 = [\underline{Y}(t; \delta), \overline{Y}(t; \delta)]\) is estimated by some \([y(t)]_0 = [\underline{y}(t; \delta), \overline{y}(t; \delta)]\). The grating points are

\[
h = \frac{T - t_0}{N}, \quad t_1 = t_0 + i h; \quad 0 \leq i \leq N.
\]

Now, by the equations (3) and (4), we define

\[
y_{n+1} = y_n + h \left[ \frac{1}{8} (k_1 + k_4) + \frac{3}{8} (k_2 + k_3) \right]
\]

\[
\underline{Y}(t_{n+1}; \delta) = \underline{Y}(t_n; \delta) + \left[ \frac{h}{8} (k_1 y(t_1; \delta)) + k_2 (y(t; \delta)) \right] + \frac{3h}{8} (k_3 (y(t; \delta)) + k_4 (y(t; \delta)))
\]

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where

\[ k_1 = F(t_n, \overline{y}(t; \delta), \overline{\bar{y}}(t; \delta)) \]

\[ k_2 = F(t_n + \frac{h}{3}, \overline{y}(t; \delta) + \frac{hh_1}{3}, \overline{\bar{y}}(t; \delta) + \frac{hh_1}{3}) \]

\[ k_3 = F\left(t_n + \frac{2h}{3}, \overline{y}(t; \delta) + h\left(\frac{-k_1}{3} + k_2\right), \overline{\bar{y}}(t; \delta) + \frac{-k_1}{3} + k_2\right) \]

\[ k_4 = F(t_n + h, \overline{y}(t; \delta) + h(k_1 - k_2 + k_3), \overline{\bar{y}}(t; \delta) + h(k_1 - k_2 + k_3)) \]

And,

\[ \overline{\overline{y}}(t_{n+1}; \delta) = \overline{y}(t_n; \delta) + \frac{h}{8}(k_1(t, y(t; \delta)) + k_4(t, y(t; \delta))) \]

\[ + \frac{3h}{8}(k_2(t, y(t; \delta)) + k_3(t, y(t; \delta))) \]

where

\[ k_1 = G(t_n, \overline{y}(t; \delta), \overline{\bar{y}}(t; \delta)) \]

\[ k_2 = G\left(t_n + \frac{h}{3}, \overline{y}(t; \delta) + \frac{hh_1}{3}, \overline{\bar{y}}(t; \delta) + \frac{hh_1}{3}\right) \]

\[ k_3 = G\left(t_n + \frac{2h}{3}, \overline{y}(t; \delta) + h\left(\frac{-k_1}{3} + k_2\right), \overline{\bar{y}}(t; \delta) + h\left(\frac{-k_1}{3} + k_2\right)\right) \]

\[ k_4 = G(t_n + h, \overline{y}(t; \delta) + h(k_1 - k_2 + k_3), \overline{\bar{y}}(t; \delta) + h(k_1 - k_2 + k_3)) \].

Also we have

\[ y(t_{n+1}; \delta) = y(t_n; \delta) + \frac{h}{8}(k_1(t, y(t; \delta)) + k_4(t, y(t; \delta))) \]

\[ + \frac{3h}{8}(k_2(t, y(t; \delta)) + k_3(t, y(t; \delta))) \]

where

\[ k_1 = F(t_n, y(t; \delta), \overline{\bar{y}}(t; \delta)) \]

\[ k_2 = F\left(t_n + \frac{h}{3}, y(t; \delta) + \frac{hh_1}{3}, \overline{\bar{y}}(t; \delta) + \frac{hh_1}{3}\right) \]
\[ k_3 = F \left( t_n + \frac{2h}{3}, \frac{y(t; \delta)}{3} + h \left( \frac{-k_1}{3} + k_2 \right), \overline{y(t; \delta)} + h \left( \frac{-k_1}{3} + k_2 \right) \right) \]

\[ k_4 = F \left(t_n + h, \frac{y(t; \delta)}{3} + h(k_1 - k_2 + k_3), \overline{y(t; \delta)} + h(k_1 - k_2 + k_3) \right) \]

and,

\[ \overline{y(t_{n+1}; \delta)} = \overline{y(t_n; \delta)} + \frac{h}{8} \left( k_1(t, y(t; \delta)) + k_4(t, y(t; \delta)) \right) \]

\[ + \frac{3h}{8} \left( k_2(t, y(t; \delta)) + k_3(t, y(t; \delta)) \right) \]

Where

\[ k_1 = G \left( t_n, \frac{y(t; \delta)}{3}, \overline{y(t; \delta)} \right) \]

\[ k_2 = G \left( t_n + \frac{h}{3}, \frac{y(t; \delta)}{3} + \frac{hk_1}{3}, \overline{y(t; \delta)} + \frac{hk_1}{3} \right) \]

\[ k_3 = G \left( t_n + \frac{2h}{3}, \frac{y(t; \delta)}{3} + h \left( \frac{-k_1}{3} + k_2 \right), \overline{y(t; \delta)} + h \left( \frac{-k_1}{3} + k_2 \right) \right) \]

\[ k_4 = G \left( t_n + h, \frac{y(t; \delta)}{3} + h(k_1 - k_2 + k_3), \overline{y(t; \delta)} + h(k_1 - k_2 + k_3) \right). \]

Define,

\[ F(t, y(t; \delta)) = ]\frac{h}{8} (k_1(t, y(t; \delta)) + k_4(t, y(t; \delta))) + \frac{h}{8} (k_2(t, y(t; \delta)) + k_3(t, y(t; \delta))) \]

\[ G(t, y(t; \delta)) = ]\frac{h}{8} (k_1(t, y(t; \delta)) + k_4(t, y(t; \delta))) + \frac{3h}{8} (k_2(t, y(t; \delta)) + k_3(t, y(t; \delta))) \]

Thus, \([Y(t_n)]_h = [Y(t_n; \delta), \overline{Y(t_n; \delta)}] \) and \([y(t_n)]_h = [y(t_n; \delta), \overline{y(t_n; \delta)}] \) are the exact and approximate solutions at \( t_n, 0 \leq n \leq N \). The solution at grid points, \( l = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = m \) and \( h = \frac{m - l}{N} = t_{n+1} - t_n \). By above Equations, let \( Y(t_{n+1}; \delta) = Y(t_n; \delta) + F(t_n, Y(t_n; \delta)) \) and \( \overline{Y(t_{n+1}; \delta)} = \overline{Y(t_n; \delta)} + G(t_n, Y(t_n; \delta)) \) and \( y(t_{n+1}; \delta) = y(t_n; \delta) + F(t_n, y(t_n; \delta)) \) and \( \overline{y(t_{n+1}; \delta)} = \overline{y(t_n; \delta)} + G(t_n, y(t_n; \delta)). \)
By Lemma 2.1 and Lemma 2.2,
\[ \lim_{n \to \infty} y(t, \delta) \quad \text{and} \quad \lim_{h \to \infty} \overline{y}(t, \delta) = \overline{Y}(t, \delta). \]
Let \( F(t, u, v) \) and \( G(t, u, v) \) be found by replacing \( [y(t)]_h = [u, v] \)

\[ F(t, u, v) = \left[ \frac{h}{8} \left( k_1(t, u, v) + k_4(t, u, v) \right) + \frac{3h}{8} \left( k_2(t, u, v) + k_3(t, u, v) \right) \right] \]

\[ G(t, u, v) = \left[ \frac{h}{8} \left( k_1(t, u, v) + k_4(t, u, v) \right) + \frac{3h}{8} \left( k_2(t, u, v) + k_3(t, u, v) \right) \right]. \]

The territory where \( F \) and \( G \) are well-defined, therefore
\[ K = \{(t, u, v) | 0 \leq t \leq T, -\infty < v < \infty, -\infty < u \leq v \}. \]

By Theorem 2.1, the approximate solutions \( y(t; r) \) and \( \overline{y}(t; r) \) converges to the precise solution \( Y(t; r) \) and \( \overline{Y}(t; r) \) consistently in \( t \).

### 7. Numerical Example

Consider the FIVP,
\[ y'(t) = y(t), \ t \in [0, 1], \ y(0) = (0.8 + 0.1\delta, 1.1 - 0.1\delta), \ 0 < \delta \leq 1. \]

Now, \( Y(t; \delta) = y(t; \delta)e^t \), \( \overline{Y}(t; \delta) = \overline{y}(t; \delta)e^t \) are the exact solutions, i.e.,
\[ Y(t, \delta) = [Y(t; \delta), \overline{Y}(t; \delta)] = [y(t; \delta)e^t, \overline{y}(t; \delta)e^t] \]

\[ \overline{Y}(t, \delta) = [(0.8 + 0.13\delta)e^t, (1.1 - 0.13\delta)e^t], \ 0 < \delta \leq 1. \]

At \( t = 1 \), we get \( Y(1, \delta) = [(0.8 + 0.13\delta)e, (1.1 - 0.13\delta)e], \ 0 < \delta \leq 1. \)

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact Solution</td>
<td>( Y(1; \delta) )</td>
<td>2.175</td>
<td>2.245</td>
<td>2.315</td>
<td>2.387</td>
<td>2.457</td>
</tr>
<tr>
<td></td>
<td>( \overline{Y}(1; \delta) )</td>
<td>2.990</td>
<td>2.936</td>
<td>2.881</td>
<td>2.827</td>
<td>2.773</td>
</tr>
</tbody>
</table>
8. Conclusion

In this paper, the explicit Runge-Kutta method is used to solve the fuzzy initial valued problem, and used the Butcher table for the coefficient of the Runge Kutta formula. A numerical example is also discussed.

References