Advances and Applications in Mathematical Sciences
Volume 21, Issue 8, June 2022, Pages 4475-4482

CERTAIN FIXED-POINT THEOREMS FOR SELF MAPPINGS IN BANACH SPACES

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Abstract

Fixed point theorems for a brace of tone-mappings defined in a unrestricted subset in Banach spaces are considered in this composition. Our outcome ameliorates the earlier results due to Fisher (6). Results have been handed to support the legality of the work.

1. Introduction

Banach [1] established the theorem known as the Banach contraction principle, which supports in uniqueness and also existence theories. The results of the theorem do not engross most of the topological mechanism.

2020 Mathematics Subject Classification: 47H10, 54H25.
Keywords: Fixed point theorem, Unique fixed point, Self-mappings, Banach spaces.
Received March 24, 2022; Accepted April 19, 2022

Browder and Petryshyn identified the solution for asymptotically regular and the non-expansive mapping in the Banach spaces. Many authors defined the fixed point theorems for the single-valued and the multi-valued self-mappings of the closed subset of the Banach space. Nevertheless, in many areas, the mapping under deliberation is not the self-mapping of the closed set. The group of the non-expansive mappings incorporates retrenchment mappings and it is accurately contained in the group of all continuous mappings. Browder [2], Gohde [7] and Kirk [13] individually obtained the fixed point theorem for the non-expansive mappings and it is given on a bounded, convex and closed subset of the regularly convex Banach space and in the space with more affluent structure.

Recently a number of generalizations of non-expansive mappings have been discussed by many authors. For Example, Goebel [8], Goebel-Kirk-Shimi [10], Goebel-Zlotkiewicz [9], Massa-Roux [15], Dotson ([4], [5]), etc. A inclusive examination concerning the fixed point theorems for the non-expansive and associated mappings can be established in Kirk [14]. Further, mapping which convince circumstances analogous to non-expansiveness and acquire a unique fixed point. The mapping which could not be considered as the generalizations of non-expansive mapping. 2 Similar instances happen recently in Gregus [11] and Rhoades [16].

In this article, we have analyzed the fixed-point theorems for the pair of the self-mappings which is considered in a closed subset in the Banach space. The results obtained advance the previous results obtained by Fisher [6]. Outcomes to validate and support the hypothesis are also given.

2. Common Fixed Point for Self-Mappings in Banach Spaces

Definition 2.1 (Engl [5])**.** Let the metric space be (X, d). Sequence $\{a_n\}$ is considered in X is considered to be asymptotically regular if $\lim_{n\to\infty} d(a_n, Ba_n) = 0$.

Definition 2.2 (Browder-Petryshyn [12])**.** Let the metric space be (X, d). Mapping M considered in X into itself will be asymptotically regular in X at a point a if $\lim_{n\to\infty} d(B^n x, B^{n+1} x) = 0$.

The theorem studied by Sessa [16], may be authorized as follows:

Definition 2.3. Let Q and P be the 2 self-mappings of the Banach space in $X \cdot \{P, Q\}$ can be weakly commuting as $\| PQa - QPa \| \leq \| Pa - Qa \|$, for all $a \in X$.

Let X be the Banach space, A be the closed convex subset and the self-mapping of K is considered as B. By Gregus [11]:

$$\| Bx - By \| \leq a\| x - y \| + b\| x - Bx \| + c\| y - By \| \tag{1}$$

Every $x, y \in K$, such that $0 < a < 1$, $a + b + c = 1$.

It is evident that unique fixed point of B is in A.

If B fulfilled (1), B also fulfill $\| Bx - By \| \leq a\| x - y \| + p(\| x - Bx \| + \| y - By \|)$

Every $x, y \in A$, such that $0 < a < l$, $p > 0$, $a + 2p = l$, $p = (b + c)/2$.

Fisher [6] extended the Gregus's result showing $B : A \to A$ always have unique fixed point in the A if

$$\| Bx - By \| \leq a\| x - y \| + (1 - a) \max(\| x - Bx \|, \| y - By \|) \tag{2}$$

Every $x, y\ G\ K$ such that $0 < a < 1$. N-Positive integers, R-non-negative reals, and $G = \{g : R^2 \to R\}$ the set of real functions, where

(1) g is considered to be increasing in all variable coordinates,

(2) $g\ (\mu, \mu)' \leq 2p\mu$ for $\mu \in R$, such that $0 < p < 1/2$.

Now we prove the following:

Let $B : A \to A$ and $g \in F$ where

$\| Bx - By \| \leq a\| x - y \| + g(\| x - Bx \|, \| y - By \|) + q(\| x - By \| + \| y - Bx \|)$

Every $x, y \in K$, so that

(i) $0 < a < l$, $p > 0$, $q > 0$;

(ii) $a + 2p + 2q = l$, and

(iii) $q - pq - ap < 0$.

Then in A the fixed of B will be present.

Proof. In A consider an arbitrary point x and Progression $[B^n x]$, where $n \in N$. From triangular property and equation (4)

$$\| B^n x - B^{n-1}x \| \leq \alpha \| B^{n-1}x - B^{n-2}x \| + g(\| B^{n-1}x B^n x \|, \| B^{n-2}x - B^{n-1}x \|)$$
$$+ q(\| B^{n-1}x - B^{n-2}x \| + \| B^{n-1} x - B^n x \| \quad (5)$$

We claim that

$$\| B^{n-1}x - B^{n-1}x \| \leq \| B^{n-1}x - B^{n-2}x \|$$

Suppose that (6) does not hold.

Then (3), (5) and (ii) imply

$$\| B^n x - B^{n-1}x \| \leq \alpha(\| B^n x - B^{n-1}x \|) + p(\| B^n x - B^{n-1}x \|$$
$$+ 2q(\| B^n x - B^{n-1}x \|) = \| B^n x - B^{n-1}x \|$$

which is not true.

Every $x \in X$, $n \in N$, (6) results

$$\| B^n x - B^{n-1}x \| \leq \| Bx - x \|$$

By triangular inequality, (3), (4), (7) and (ii), we also obtain

$$\| B^3 x - Bx \| \leq \alpha(\| B^2 x - Bx \| + \| Bx - x \|) + g(\| B^3 x - B^2 x \|, \| Bx - x \|)$$
$$+ q\| B^2 x - Bx \| + \| B^3 x - B^2 x \| + \| B^2 x - Bx \| + (\| Bx - x \|)$$
$$\leq 2(\alpha + p + 2q)\| Bx - x \|$$
$$= 2(1 - p)\| Bx - x \|$$

A is considered to be convex, $z = 1/2(B^2 x + B^3 x)$ is the midpoint and it is in A.

Therefore (6) implies that

$$2\| z - B^2 x \| = 2\| z - B^3 x \|$$

$$\leq \| B^3x - B^2x \|$$

$$\leq \| Bx - x \|$$

and

$$2\| z - Bx \| \leq \| B^2x - Bx \| + \| B^3x - Bx \|$$

$$\leq \| Bx - x \| + 2(1-p)\| Bx - x \|$$

$$= \| (3 - 2p) \| \| Bx - x \|$$

As triangular inequality, (3), (4) and (7) lead us to the inequality

$$2\| Bz - z \| \leq \| Bx - B^2x \| + \| Bx - B^3x \|$$

$$\leq a\| z - Bx \| + g(\| Bz - z \|, \| B^2x - Bx \|) + q(\| z - B^2x \| + \| z - Bx \|$$

$$+ \| Bz - z \|)$$

$$+ a\| z - B^2x \| + g(\| Bz - z \|, \| B^3x - B^2x \|) + q(\| z - B^2x \| + \| Bz - z \|$$

$$+ \| z - B^2x \|) \leq (a+q) \| z - Bx \| + (a+2q) \| z - B^2x \| + q\| z - B^3x \|$$

$$+ 2g(\| Bz - z \|, \| Bx - x \|) + 2q\| Bz - z \| \| Bz - z \| \leq \| Bx - x \|$$

Also (3), (ii), (iii) and (8), (9), (10) gives

$$4\| Bz - z \| \leq (a+q)(3-p)\| Bz - z \| + (a+q)\| Bz - z \| + q\| Bz - z \|$$

$$+ 8p\| Bz - z \| + 4q\| Bz - z \| = (4a + 10q + 8p - ap - pq)$$

$$\| Bz - z \| = (4 + 2q - 2ap - 2pq)\| Bz - z \|$$

$$< \| Bz - z \|$$

which is not true.

By (10) we have

$$\| Bz - z \| < \lambda \| Bx - x \|$$

where $\lambda = 1 - 1/2\, a + 1/2(q - ap - pq)$.

By (iii), $\lambda < 1, (q - ap - pq) > -2$, such that $\lambda > 0$.

If $h = \inf[\| Bx - x \| : x \in A]$. Now $h = 0$ or else,

$X \in A$ where $\| Bx - x \| < h + \in$ for $\in > 0$.

If $0 < \in < (1 - \lambda). hl\lambda$, where

$$\| Bz - z \| \leq \lambda.$$

$$\| Bx - x \| \leq \lambda(h + \in) < h$$

But it is not acceptable.

Hence $h = 0$ and every $A_n = [x \in A : \| Bx - x \| \leq 1 / n], n \in N$,

Cannot be null set, so, $[A_n], n \in N$ is a set which is decreasing.

Let TA_n have a random point x. Hence $\in > 0$, there will be a $y \in A_n$ so that $\| x - By \| < \in$ and $\| By - y \| \leq 1 / n$.

By condition (4) and triangular property, we deduce that

$\| Bx - x \| < \| Bx - By \| + \| x - By \|$

$\leq a(\| x - By \| + \| By - y \|) + g(\| x - Bx \|, \| y - By \|)$

$+ q(\| x - By \| + \| y - By \|) + \in$

$< a + q/n + g(\| B - x \|, 1/n) + (1 + a + q) \in$.

As $\in$ is random, we have $\| Bx - x \| \leq 1/n,\ x \in A_n$

$TA_n \subset A_n$. Hence we have $w \in A_n$ for any $n \in N, \| Bw - w \| \leq 1 /n$ as $n \in N,\ Bw = w$.

Let B have another fixed point w'. Hence (4), implies

$\| w' - w \| = \| Bw' - Bw \|$

$$\leq a\| w' - w \| + g(\| w' - Bw' l \|, \| w - Bw \|) + q(\| w' - Bw \| + \| w - Bw' \|)$$

$$= (1 - 2p) | w' - w |$$

$$< | w' - w |.$$

Hence $w' = w$.

Remark 2.1. The (iii) property is not obtained from (i) and (ii). For example, if $a = \frac{4}{15}$, $p = \frac{1}{6}$, $q = 1/5$

Remark 2.2. If $q = 0$ and $g(u, v) = (1 - a).\max[u, v]$ every $(u, v) \in R^2$ in Theorem 2.1 the property (2) and (4) concur.

Theorem 2.2. *If* $B : A \to A$, $g \in G$, $m \in N$ *be so that*

$$\| B^{,m}x - B^m y \| \leq a\| x - y \| + g(\| x - B^{,m}x \|, \| y - B^m y \|) + q(\| x - B^m y \|$$

$$+ \| y - B^m x \|)$$

every $x, y \in A$, *where* a, p, q *convince* (i), (ii), (iii) *of Theorem* 2.1.

Then K is the unique fixed point of T.

Proof of Theorem 2.2. From the Theorem 2.1, $B^m : A \to A$ will have a unique fixed point w in the set A.

Now $Bw = B(B^m w) = B^m(Bw)$, implies Bw is a fixed point of B^m, and hence $Bw = w$. If B has one more fixed point z, Then $B^m z = z$, hence $z = w$.

3. Conclusions

In this article, fixed point theorems of the pair of self-mappings are considered in a closed subset in Banach space are analyzed. Obtained results enhance the previous results of Fisher [6]. The results obtained advance the previous results obtained by Fisher [6]. Outcomes extend and generalize results of Fisher [6] and Gregus [11].

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