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COMMON FIXED POINT FOR SELF-MAPPINGS WITH $(E.A)$ AND (CLR_{ST}) PROPERTIES

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Abstract

The purpose of this paper is to prove common fixed point theorems by using the property $(E.A)$ and the common limit range property (CLR_{ST}) for pairs of weakly compatible mappings satisfying a weak contraction involving cubic terms of distance functions in metric space. Our results generalize and extend the result of Kumar et al. [11].

1. Introduction and Preliminaries

Fixed point theory is one of the most effective research area in

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Mathematics. It has enormous applications in various fields such as Economics, Game theory, applied science etc. The basic tool to study fixed point theory is Banach contraction principle [12] which states that every contraction mapping on a complete metric space has a unique fixed point. This result has been extended, generalized and unified by various authors in various diverse abstract spaces. Jungck [3] obtained one of interesting characterizations of Banachs contraction principle for pairs of mappings using the notion of commutative mappings.

In 1982, Sessa [13] initiated to relax commutative condition using the notion of weak commutative of mappings. It is seen that common fixed point theorems for contractive type mappings generally involve a commutativity type condition, continuity of one or more mappings, a condition on containment of range spaces.

Further, in 1986, Jungck [4] weakened the notion of commutative and weak commutative mappings to compatible mappings and defined as follows:

Definition 1.1. Let P and Q be two mappings of a metric space (X, d) into itself. Then the mappings P and Q are said to be compatible if

$$\lim_{n \to \infty} d(PQu_n, QPu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in X such that $\lim_{n \to \infty} Pu_n = \lim Qu_n = t$ for some $t \in X$.

In 1996, Jungck [5] introduced the notion of weakly compatible mappings and showed that compatible mappings are weakly compatible but converse may not be true.

Definition 1.2. A pair of self-mappings P and Q on a metric space (X, d) is called weakly compatible if the mappings commute at their coincidence points i.e., if $Pt = Qt$ for some $t \in X$ implies $PQt = QPt$.

In the general setting, the notion of property $(E.A)$, which requires the closedness of the range subspace, was introduced by Aamri and El-Moutawakil [7] as follows:

Definition 1.3. A pair of self-mappings P and Q on a metric space (X, d)

is said to satisfy property $(E.A)$ if there exists a sequence $\{u_n\} \in X$ such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = t$ for some $t \in X$.

Remark 1.4. One can note that weak compatibility and property $(E.A)$ are independent concepts.

The common limit range property (CLR), which is an analogue to $(E.A)$ property, is introduced by Sintunavarat and Kumam [14] as follows:

Definition 1.5. Two self mappings P and Q of a metric space (X, d) are said to satisfy the common limit in the range of Q property, denoted by (CLR_Q), if there exists a sequence $\{u_n\} \in X$ such that

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = Qt \text{ for some } t \in X.$$

Imdad [8] extended the (CLR) property as follows:

Definition 1.6. Two pairs (P, S) and (Q, T) of a metric space (X, d) are said to satisfy the common limit range property with respect to the mappings S and T, denoted by (CLR_{ST}), if there exists a sequence $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Su_n = \lim_{n\to\infty} Qv_n = \lim_{n\to\infty} Tv_n = t, \text{ for some } t \in S(X) \cap T(X).$$

It is observed that the properties $(E.A)$ and (CLR) relax the continuity hypothesis of all the mappings under consideration and also relax the containment condition of the range subspace of the mapping into the range subspaces of the other mappings. The property $(E.A)$ allows replacing the condition of completeness of the space (or the range subspaces of the mappings involved) with a condition of closedness of the range subspace. Common limit range property (CLR) ensures that the requirement of the closedness of the subspaces for the existence of fixed point can be relaxed entirely. In this paper, we prove common fixed point theorems for pairs of mappings using the notion of weakly compatibility along with property $(E.A)$ and (CLR_{ST}) that satisfy a weak contraction condition involving the cubic terms of distance function.

2. Main Results

In 2021, Kumar et al. [11] introduced a new weak contraction that involves cubic terms of distance function and proved common fixed point theorems for compatible mappings and its variants.

Theorem 2.1 [11]. *Let* f, g, S *and* T *be four mappings of a complete metric space* (X, d) *into itself satisfying the following conditions*:

(C1) $f(X) \subset T(X), g(X) \subset S(X)$,

(C2) $d^3(fx, gy) \leq p \max\{\frac{1}{2}[d^2(Sx, fx)d(Ty, gy) + d(Sx, fx)d^2(Ty, gy)],$

$d(Sx, fx)d(Sx, gy)d(Ty, fx), d(Sx, gy)d(Ty, fx)d(Ty, gy)\}$

$-\phi(m(Sx, Ty))$,

for all $x, y \in X$, where

$$m(Sx, Ty) = \max\{d^2(Sx, Ty), d(Sx, fx)d(Ty, gy), d(Sx, gy)d(Ty, fx),$$

$$\frac{1}{2}[d(Sx, fx)d(Sx, gy) + d(Ty, fx)d(Ty, gy)]\}$$

and p is a real number satisfying $0 < p < 1$ and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$;

(C3) one of the mappings f, g, S, T is continuous.

Assume that the pairs (f, S) and (g, T) are compatible or compatible of type (A) or compatible of type (B) or compatible of type (C) or compatible of type (P), then f, g, S and T have a unique common fixed point in X.

Now, we extend and generalize the theorem 2.1 for three pairs of mappings using the notion of weakly compatible mappings along with property $(E \cdot A)$.

Theorem 2.2. *Let* f, g, S, T, P *and* Q *be six mappings of a metric space* (X, d) *into itself satisfying*:

(C4) *The pairs* (f, PQ) *and* (g, ST) *share the common property* $(E \cdot A)$,

(C5) $PQ(X)$ *and* $ST(X)$ *are closed subsets of* X,

(C6) $fQ = Qf$, $PQ = QP$, $gT = Tg$ *and* $ST = TS$,

(C7) $d^3(fx, gy) \leq p \max\{\frac{1}{2}[d^2(PQx, fx)d(STy, gy) + d(PQx, fx)d^2(STy, gy)],$

$d(PQx, fx)d(PQx, gy)d(STy, fx), d(PQx, gy)d(STy, fx)d(STy, gy)\}$

$-\phi(m(PQx, STy)),$

for all $x, y \in X$, *where*

$$m(PQx, STy) = \max\{d^2(PQx, STy), d(PQx, fx)d(STy, gy),$$

$$d(PQx, gy)d(STy, fx),$$

$$\frac{1}{2}[d(PQx, fx)d(PQx, gy) + d(STy, fx)d(STy, gy)]\}$$

and p *is a real number satisfying* $0 < p < 1$ *and* $\phi : [0, \infty) \to [0, \infty)$ *is a continuous function with* $\phi(0) = 0$ *and* $\phi(t) > 0$ *for* $t > 0$.

Then f, g, S, T, P *and* Q *have a unique common fixed point in* X *provided that the pairs* (f, PQ) *and* (g, ST) *are weakly compatible.*

Proof. Since the pairs (f, PQ) and (g, ST) share the common property $(E \cdot A)$, there exists two sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} PQu_n = \lim_{n\to\infty} gv_n = \lim_{n\to\infty} STv_n = w,$$

for some $w \in X$. Since $PQ(X)$ is a closed subset of X, we have $\lim_{n\to\infty} PQu_n = w \in PQ(X)$. Therefore, there exists $u \in X$ such that $PQu = w$. Now we show that $fu = w$. On putting $x = u$ and $y = v_n$ in (C7), we have

$d^3(fu, gv_n)$

$$\leq p \max \{\frac{1}{2}[d^2(PQu, fu)d(STv_n, gv_n) + d(PQu, fu)d^2(STv_n, gv_n)],$$

$$d(PQu, fu)d(PQu, gv_n)d(STv_n, fu), d(PQu, gv_n)d(STv_n, fu)$$

$$d(STv_n, gv_n)\} - \phi(m(PQu, STv_n)),$$

where

$$m(PQu, STv_n) = \max \{d^2(PQu, STv_n), d(PQu, fu)d(STv_n, gv_n),$$

$$d(PQu, gv_n)d(STv_n, fu),$$

$$\frac{1}{2}[d(PQu, fu)d(PQu, gv_n) + d(STv_n, fu)d(STv_n, gv_n)]\}.$$

Taking limits as $n \to \infty$, it follows that

$$d^3(fu, w) \leq p \max \{\frac{1}{2}[d^2(w, fu)d(w, w) + d(w, fu)d^2(w, w)],$$

$$d(w, fu)d(w, w)d(w, fu), d(w, w)d(w, fu)d(w, w)\}$$

$$- \phi(m(PQu, w)),$$

where

$$m(PQu, w) = \max \{d^2(w, w), d(w, fu)d(w, w), d(w, w)d(w, fu),$$

$$\frac{1}{2}[d(w, fu)d(w, w) + d(w, fu)d(w, w)]\} = 0,$$

which implies that $d^3(fu, w) \leq 0$ and hence $fu = w = PQu$.

If $ST(X)$ is a closed subset of X, then there exists a point v in X such that $STv = w$. Now we show that $gv = w$. On putting $x = u_n$ and $y = v$ in (C7), we obtain

$$d^3(fu_n, gv)$$

$$\leq p \max \{\frac{1}{2}[d^2(PQu_n, fu_n)d(STv, gv) + d(PQu_n, fu_n)d^2(STv, gv)],$$

$$d(PQu_n, fu_n)d(PQu_n, gv)d(STv, fu_n), d(PQu_n, gv)d(STv_n, fu_n)$$

$$d(STv, gv)\} - \phi(m(PQu_n, STv)),$$

where

$$m(PQu_n, STv) = \max\{d^2(PQu_n, STv), d(PQu_n, fu_n)d(STv, gv),$$

$$d(PQu_n, gv)d(STv, fu_n),$$

$$\frac{1}{2}[d(PQu_n, fu_n)d(PQu_n, gv) + d(STv, fu_n)d(STv, gv)]\}.$$

Taking limits as $n \to \infty$ and on simplification, we have $d^3(w, gv) \leq 0$ and hence $gv = w = STv$.

Hence $fPQu = PQfu$ i.e., $fw = PQw$ and $gSTv = STgv$ i.e., $gw = STw$. (2.1)

Now we show that $fw = w$. Suppose that $fw \neq w$. On putting $x = w$ and $y = v$ in (C7), we have

$$d^3(fw, gv) \leq p\max\{\frac{1}{2}[d^2(PQw, fw)d(STv, gv) + d(PQw, fw)d^2(STv, gv)],$$

$$d(PQw, fw)d(PQw, gv)d(STv, fw), d(PQw, gv)d(STv, fw)d(STv, gy)\}$$

$$-\phi(m(PQw, STv)),$$

for all $x, y \in X$, where

$$m(PQw, STv) = \max\{d^2(PQw, STv), d(PQw, fw)d(STv, gv),$$

$$d(PQw, gv)d(STv, fw),$$

$$\frac{1}{2}[d(PQw, fw)d(PQw, gv) + d(STv, fw)d(STv, gv)]\}.$$

On simplification, we have $d^3(fw, w) \leq -\phi(d^3(fw, w))$, a contradiction and hence $fw = w = PQw$. We show that $gw = w$. Suppose that $gw \neq w$. On putting $x = u$ and $y = w$ in (C7), we have

$$d^3(fu, gw) = d^3(w, gw) \leq p\max\{\frac{1}{2}[d^2(w, w)d(gw, gw) + d(w, w)d^2(gw, gw)],$$

$$d(w, w)d(w, gw)d(gw, w), d(w, gw)d(gw, w)d(gw, gw)\}$$

$$-\phi(m(w, gw)),$$

where

$$m(w, gw) = \max\{d^2(w, gw), d(w, w)d(gw, gw),$$

$$d(w, gw)d(gw, w),$$

$$\frac{1}{2}[d(w, w)d(w, gw) + d(gw, w)d(gw, gw)]\},$$

which implies that $d^3(w, gw) \leq -\phi(d^3(gw, w))$, a contradiction and hence $gw = w = STw$. Therefore, $fw = gw = PQw = STw = w$.

Since $fQ = Qf$ and $PQ = QP$, so $fQw = Qfw = Qw$ and $PQ(Qw) = QP(Qw) = Qw$. We show that $Qw = w$. Suppose that $Qw \neq w$. On putting $x = Qw$ and $y = w$ in (C7), we have

$$d^3(fQw, gw)$$

$$\leq p\max\{\frac{1}{2}[d^2(PQQw, fQw)d(STw, gw) + d(PQQw, fQw)d^2(STw, gw)],$$

$$d(PQQw, fQw)d(PQQw, gw)d(STw, fQw), d(PQQw, gw)d(STw, fQw)$$

$$d(STw, gw)\} - \phi(m(PQQw, STw)),$$

where

$$m(PQQw, STw) = \max\{d^2(PQQw, STw), d(PQQw, fQw)d(STw, gw),$$

$$d(PQQw, gw)d(STw, fQw),$$

$$\frac{1}{2}[d(PQQw, fQw)d(PQQw, gw) + d(STw, fQw)d(STw, gw)]\}.$$

$$d^3(Qw, gw) \leq p\max\{\frac{1}{2}[d^2(Qw, Qw)d(w, w) + d(Qw, Qw)d^2(w, w)],$$

$$d(Qw, Qw)d(Qw, w)d(w, Qw), d(Qw, w)d(w, Qw)$$

$$d(w, w)\} - \phi(m(Qw, w)),$$

where

$$m(Qw, w) = \max\{d^2(Qw, w),\ d(Qw, Qw)d(w, w),$$

$$d(Qw, gw)d(w, Qw),$$

$$\frac{1}{2}[d(Qw, Qw)d(Qw, w) + d(w, Qw)d(w, w)]\}.$$

which implies that $d^3(Qw, w) \leq -\phi(d^3(Qw, w))$, a contradiction and hence $Qw = w$. Therefore, $w = PQw = Pw$.

Since $gT = Tg$ and $ST = TS$, so $gTw = Tgw = Tw$ and $ST(Tw) = TS(Tw) = Tw$. Next we show that $Tw = w$. Suppose that $Tw \neq w$. On putting $x = w$ and $y = Tw$ in (C7) and on simplification, we get

$$d^3(w, Tw) \leq -\phi(d^3(w, Tw)),$$

a contradiction and hence $Tw = w$. Also, $w = STw = Sw$. Hence w is a common fixed point of self mappings f, g, S, T, P and Q.

Uniqueness. Let $z(\neq w)$ be another common fixed point of self mappings f, g, S, T, P and Q. On putting $x = w$ and $y = z$ in (C7), we obtain

$$d^3(fw, gz) = d^3(w, z) \leq p\max\{\frac{1}{2}[d^2(w, w)d(z, z) + d(w, w)d^2(z, z)],$$

$$d(w, w)d(w, z)d(z, w),\ d(w, z)d(z, w)d(z, z)\}$$

$$-\ \phi(d^2(w, z)),$$

which implies that $w = z$. This completes the proof.

Now we prove our next theorem by using the common limit range property denoted by (CLR_{ST}).

Theorem 2.3. *Let* f, g, S, T, P *and* Q *be six mappings of a metric space* (X, d) *into itself satisfying* (C6), (C7) *and the following*:

(C8) *the pairs* (f, PQ) *and* (g, ST) *share the* $(CLR_{(PQ)(ST)})$ *property*;

Then the mappings f, g, S, T, P *and* Q *have a unique common fixed point provided that the pairs* (f, PQ) *and* (g, ST) *are weakly compatible.*

Proof. Since the pairs (f, PQ) and (g, ST) share the $(CLR_{(PQ)(ST)})$ property, there exists two sequences $\{u_n\}$ and $\{v_n\}$ such that

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} PQu_n = \lim_{n\to\infty} gv_n = \lim_{n\to\infty} STv_n = w, \text{ where}$$

$$w \in PQ(X) \cap ST(X).$$

Since $w \in ST(X)$, there exists $v \in X$ such that $STv = w$. We show that $gv = w$. On putting $x = u_n$ and $y = v$ in (C7), we get

$$d^3(fu_n, gv) \leq p \max\{\frac{1}{2}[d^2(PQu_n, fu_n)d(STv, gv) + d(PQu_n, fu_n)d^2(STv, gv)],$$

$$d(PQu_n, fu_n)d(PQu_n, gv)d(STv, fu_n), d(PQu_n, gv)d(STv_n, fu_n)$$

$$d(STv, gv)\} - \phi(m(PQu_n, STv)),$$

where

$$m(PQu_n, STv) = \max\{d^2(PQu_n, STv), d(PQu_n, fu_n)d(STv, gv),$$

$$d(PQu_n, gv)d(STv, fu_n),$$

$$\frac{1}{2}[d(PQu_n, fu_n)d(PQu_n, gv) + d(STv, fu_n)d(STv, gv)]\}.$$

Taking limits as $n \to \infty$ and on simplification, we have

$$d^3(w, gv) \leq 0,$$

which implies that $gv = w$ and hence $gv = w = STv$. Since $w \in PQ(X)$, there exists $u \in X$ such that $PQu = w$. We show that $fu = w$. On putting $x = u$ and $y = v_n$ in (C7), we get

$$d^3(fu, gv_n) \leq p \max\{\frac{1}{2}[d^2(PQu, fu)d(STv_n, gv_n) + d(PQu, fu)d^2(STv_n, gv_n)],$$

$$d(PQu, fu)d(PQu, gv_n)d(STv_n, fu), d(PQu, gv_n)d(STv_n, fu)$$

$$d(STv_n, gv_n)\} - \phi(m(PQu, STv_n)),$$

where

$$m(PQu, STv_n) = \max\{d^2(PQu, STv_n), d(PQu, fu)d(STv_n, gv_n),$$

$$d(PQu, gv_n)d(STv_n, fu),$$

$$\frac{1}{2}[d(PQu, fu)d(PQu, gv_n) + d(STv_n, fu)d(STv_n, gv_n)]\}.$$

Taking limits as $n \to \infty$ and on simplification, we have

$$d^3(fu, w) \leq 0,$$

which implies that $fu = w$ and hence $fu = w = PQu$.

Since the pair (f, PQ) is weakly compatible and $fu = PQu = w$, then $fPQu = PQfu$, it implies that $fw = PQw$. Since the pair (g, ST) is weakly compatible and $gv = STv = w$, then $gSTv = STgv$, it implies that $gw = STw$.

Proceeding on the similar lines after equation (2.1) of Theorem 2.2, we see that w is a unique common fixed point of f, g, S, T, P and Q.

If we take $Q = T = I$ (I is identity map in X) in Theorem 2.2, we get

Corollary 2.4. *Let f, g, P and S be self maps of a metric space (X, d) satisfying*

(C9) $d^3(fx, gy) \leq p \max\{\frac{1}{2}[d^2(Px, fx)d(Sy, gy) + d(Px, fx)d^2(Sy, gy)],$

$$d(Px, fx)d(Px, gy)d(Sy, fx), d(Px, gy)d(Sy, fx)$$

$$d(Sy, gy)\} - \phi(m(Px, Sy)),$$

for all $x, y \in X$, where

$$m(Px, Sy) = \max\{d^2(Px, Sy), d(Px, fx)d(Sy, gy),$$

$$d(Px, gy)d(Sy, fx),$$

$$\frac{1}{2}[d(Px, fx)d(Px, gy) + d(Sy, fx)d(Sy, gy)]\}.$$

and p *is a real number satisfying* $0 < p < 1$ *and* $\phi : [0, \infty) \to [0, \infty)$ *is a continuous function with* $\phi(0) = 0$ *and* $\phi(t) > 0$ for $t > 0$;

Suppose that the pairs (f, P) and (g, S) share the common property $(E.A)$.$P(X)$ and $S(X)$ are closed subsets of X. Then the mappings f, g, P and S have a unique common fixed point provided that the pairs (f, P) and (g, S) are weakly compatible.

If we take $Q = T = I$ (I is identity map in X) in Theorem 2.3, we get.

Corollary 2.5. *Let* f, g, P *and* S *be self maps of a metric space* (X, d) *satisfying the condition* (C9). *Suppose that the pairs* (f, P) *and* (g, S) *share the* (CLR_{PS}) *property and are weakly compatible. Then the mappings* f, g, P *and* S *have a unique common fixed point.*

Remark 2.6. Corollary 2.5 is a generalization of the result of Kumar et al. [11] in the sense that the conditions of containment of the range subspace and continuity of the mapping have been relaxed.

3. Conclusion

In this paper, we have proved common fixed point theorems by using the property $(E.A)$ and the common limit range property (CLR_{ST}) for pairs of weakly compatible mappings satisfying a weak contraction involving cubic terms of distance functions in metric space. The results can be further extended and generalized for families of mappings.

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