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MULTIPLE INTRUDER LOCATING DOMINATION IN CHORDAL AND BIPARTITE GRAPHS

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Abstract

For a graph $G = (V, E)$ and a subset $S \subseteq V$, a vertex u is called a codeword if $u \in S$, else it is called a non-codeword. The set S is called a Multiple Intruder Locating Dominating (MILD) set if every non-codeword v is adjacent to a codeword u which is not adjacent to any other non-codeword other than v. The cardinal of a minimum MILD set of graph G is called its MILD number, denoted by $\gamma_{ml}(G)$. The problem of finding the MILD number of a graph is known to be NP-complete for arbitrary graphs. In this paper, we prove that the problem remains NP-complete even when restricted to chordal graphs, split graphs and bipartite graphs. We then establish the MILD numbers of some classes of bipartite graphs.

1. Introduction

A vertex v of a graph $G = (V, E)$ is said to dominate another vertex u if v is adjacent to u. A subset D of V is said to be a dominating set if every vertex not in D is dominated by some vertex in D. The cardinality of a minimum dominating set in G is called the domination number of G, denoted by $\gamma(G)$.

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The problem of finding a minimum dominating set is known to be NP-complete for arbitrary graphs [5]. Further, the problem is proved to be NP-complete even when restricted to classes of graphs like bipartite, split and chordal graphs [1]. A wide variety of domination problems have been well documented in the books [9, 8]. A locating dominating set is a subset S of V such that every vertex not in S has non-empty distinct neighborhood with S [11]. The cardinality of a smallest locating dominating set in a graph G is called the locating domination number of G, denoted by $\gamma_{ml}(G)$. A vertex v is called a codeword if it is in S, else it is called a non-codeword. The complexity of locating domination problem for different classes of graphs is discussed in [2, 3, 4]. For some variations of the locating domination parameter, refer [6, 7, 10].

In a graph $G = (V, E)$, if a codeword u is adjacent to only one non-codeword v, then it is said to be a devout dominator of v, and v, a secure non-codeword. Together, $u - v$ are said to form a code pair. For a subset S of V, if every non-codeword has a devout dominator, then S is called a Multiple Intruder Locating Dominating set of G. The cardinality of a Multiple Intruder Locating Dominating set in a graph G is called the Multiple Intruder Locating domination number of G and it is denoted by $\gamma_{ml}(G)$. Originally defined in [12], the problem is proven to be NP-complete for arbitrary graphs in [13]. A MILD set S is said to be a connected MILD set if the subgraph induced by S is connected.

The cardinality of a minimum connected MILD set of a graph G is called its connected MILD number. In this paper, we investigate into the problem of finding the MILD number of some classes of graphs, and wherever possible, check if the results hold for the problem of connected MILD number as well.

2. MILD Problem in Chordal Graphs

We show that the problem of finding MILD number remains NP-complete even for chordal graphs.

Definition 1. For a cycle C in a graph, an edge which is not in C but the end points are the vertices of C is called a chord. A simple graph that has a chord for every cycle of length 4 or more is called a chordal graph.

We reduce the MILD problem in chordal graphs to the dominating set problem in general graphs.

Dominating set

INSTANCE: Graph $G = (V, E)$ and a positive integer $k \leq |V|$

QUESTION: Does G have a dominating set of size $\leq k$?

MILD-chordal

INSTANCE: Chordal graph $G' = (V', E')$ and positive integer $k' \leq |V'|$

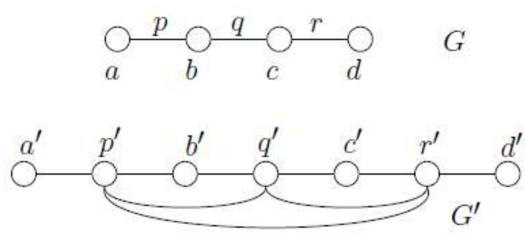

Figure 1. Construction of G' from G in MILD-chordal.

QUESTION: Does G' have a MILD set of size $\leq k'$?

Theorem 2. *Problem MILD − chordal is NP-complete.*

Proof. Given a subset of V' it can be easily checked if it forms a MILD set, and hence MILD − chordal $\in NP$. For a given graph G with n vertices and m edges, construct a graph G' as follows:

- For each vertex $v \in V$ and each edge $p \in E$, create a vertex v' and a vertex p' respectively and, add them to the sets X and Y respectively. Thus, $V' = X \cup Y$.
- If edge p joins the vertices, say, a and b in G, then add an edge from each of the vertices a' and b' to the vertex p' in G'.
- Add an edge between every two vertices in Y.

An example of this construction is shown in Figure 1. Consider any induced subgraph P_3 in G with vertices u, v, w and edges $p = uv$ and $q = vw$. The corresponding part in G' has the vertices p' and q' joined by an edge forming a triangle (i.e., K^3) $p'vq'$. Thus G' is a chordal graph. Further, $|V'| = m + n$ and $|E'| = 2m + \frac{m}{2}(m-1)$. Thus, G' can be constructed from G in polynomial time. Now, we show that G has a dominating set of cardinality k if and only if G' contains a MILD set of cardinality $k' = m + k$.

Given a dominating set of size k in G, make the corresponding vertices in X and all the vertices in Y codewords. Now in G, any vertex should dominate another vertex through an edge. Since only one edge joins two vertices in G, every vertex in Y is adjacent to only two vertices of X. Hence in G, if a vertex a dominates a vertex b through an edge p, then in G', the vertex a' helps the vertex p' to devout dominate b'. Thus if a vertex is dominated in G then the corresponding vertex is devout dominated in G'. In other words, if G has a dominating set of size k, then G' has a MILD set of size $m + k$.

Conversely, consider a graph G' with a MILD set of cardinality $m + k$. Since every two vertices are adjacent in Y, one of the following two cases should happen:

Case (i) if one of the vertices in Y is a devout dominator, then all vertices in it must be codewords and every vertex in X would be either a secure noncodeword or a non devout dominating codeword.

Case (ii) if one of the vertices in Y is a non-codeword, then none of them can be devout dominators and thus, they would be either secure non-codewords or non-devout dominating codewords. Also, all the vertices in X would be codewords.

Consider case (ii). All vertices in Y would be either a secure non-codeword or non-devout dominating codeword. Pair every such secure non-codeword with a devout dominator which will be in X. Now, in each of those pairs, interchange codeword and non-codeword. Then, all vertices in Y (which are m in number) would be codewords and every vertex in X would be either a secure non-codeword or a non-devout dominating codeword. Thus, if a graph is given as in the case (ii), then it can be transformed into case (i) without altering the MILD number.

Suppose G' is as in the case (i), then whichever vertices in X are codewords (and non-codewords), put the corresponding vertices in G into S (and $V - S$).

Thus, if G' has a MILD set of size $m + k$, then G has a dominating set of size k. This completes the proof.

Definition 3. A graph that has vertices which can be partitioned into an independent set and a clique is called a split graph.

Theorem 4. *Problem MILD − split is NP-complete.*

Proof. In the graph G' of MILD − chordal, the set Y happens to be a clique and X is an independent set. This makes the constructed graph G' a split graph and hence the proof.

Note that the MILD set in the construction of MILD − chordal induces a connected subgraph.

Corollary 2.1. *The problem of finding the minimum connected MILD set of a graph is NP-complete for split graphs.*

3. MILD Problem in Bipartite Graphs

First, we prove that the problem of finding the MILD number of an arbitrary bipartite graph is NP-complete. This is done by reducing the Dominating set problem to MILD.

Dominating set

INSTANCE: Graph $G = (V, E)$ and positive integer $k \leq |V|$

QUESTION: Does G have a dominating set of size $\leq k$

MILD-bipartite

INSTANCE: Bipartite graph $G' = (V', E')$ and positive integer $k' \leq |V'|$

QUESTION: Does G' have a MILD set of size $\leq k'$

Theorem 5. *Problem MILD − bipartite is NP-complete.*

Proof. It is easy to check if a given subset of V' forms a MILD set and hence MILD − bipartite $\in NP$. For the given graph G with $|V| = n$ and $|E| = m$, construct the graph G' as follows:

- For every vertex v_i and edge e_i in G, create a vertex v_i' and a vertex e_i' in G'. If two vertices v_i and v_j are adjacent through the edge e_i in G, then in G', make v_i' and v_j' adjacent to the vertex e_i'.
- Whenever edges e_i, e_{i+1} meet at a vertex, say, v_i in G, create two vertices x_i and x_i' adjacent to each other in G' and add an edge from each of the vertices $e_1', e_{i+1}', \ldots, e_p'$ to x_i.

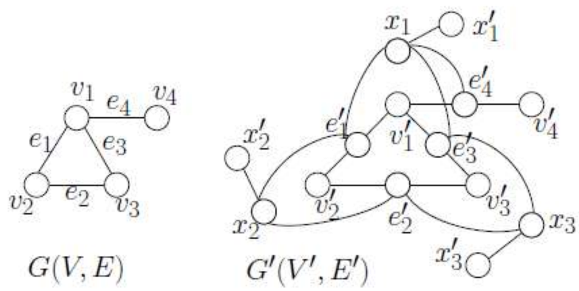

Figure 2. Construction of G' from G in MILD-bipartite.

Every cycle in G' is even making it a bipartite graph. Let $\{v_{i1}, v_{i2}, \ldots, v_{ir}\}$ be the non-pendant vertices in G and let $t = \deg(v_{i1}) + \deg(v_{i2}) + \ldots + \deg(v_{ir})$.

Then $|V'| = n + m + 2r$ and $|E'| = 2m + r + t$, which means G' can be constructed from G in polynomial time. Now, we show that G has a dominating set of size k if and only if G' has a MILD set of size $k' = m + r + k$.

Given a dominating set D of size k in G, make the corresponding vertices in G' as well as all the e_i' and x_i vertices codewords. If a vertex v_i dominates a vertex v_j through an edge, say, e_i in G, then the vertex v_i' enables the vertex e_i' to devout dominate v_i' in G'. Thus, if G has a minating f cardinality k, then G' has a MILD set S of cardinality $k' = m + r + k$.

To prove the converse, suppose given a MILD set S in G' with $|S| = k'$. If all x_i vertices are codewords, then it follows that all the e'_i vertices are codewords and whichever vertices v'_i are codewords, the corresponding vertices v_i in G form the required dominating set.

Suppose some x_i is a non-codeword, then $x'_i \in S$. Also, none of the e'_i vertices which are adjacent to x_i can devout dominate the vertex v'_i, hence $v'_i \in S$. Now, make $x'_i \in S$ and $x'_i \notin S$. Suppose any of the e' vertices, say, e'_i, is a non-codeword, let its other end vertex be v'_i Since v'_j cannot devout dominate, e'_i must have been devout dominated by v'_j. Let us make $e'_i \in S$. and $v'_i \notin S$. Then v'_i being a codeword already, enables e'_i to devout dominate v'_j. If there is another vertex v'_k adjacent to, say, e'_j which is adjacent to v'_j, then e'_i and e'_j would be adjacent to an x vertex, say, x'_j. Since e'_i was a non-codeword before, so would have been x_i, Thus, e'_j would not have been a devout dominator. Hence, taking v'_j out of S will not affect any devout domination by any other vertices in G'. In this way, convert all non-codeword x and e' vertices to codewords and, $|S|$ will still be unaltered.

Now, with all x and e' vertices being codewords, let the number of v' vertices in S be k. A codeword v' vertex enables its adjacent e' vertices to devout dominate. This means, the corresponding k vertices in G dominate the vertices which are adjacent to it, completing the proof.

The MILD set in the construction of MILD − bipartite induces a connected subgraph as well. Hence the following result.

Corollary 3.1. *The problem of finding the minimum connected MILD number of a bipartite graph is NP-complete.*

Finding the MILD number of an arbitrary bipartite graph is proved to be NP-complete. However the MILD number of some classes of bipartite graphs can be determined, which we discuss next. For a comprehensive collection of classes of graphs, we refer [14].

Proposition 6. *The MILD number of a complete bipartite graph $K_{p,q}$ is $p + q - 2$.*

Proof. Let U and W be the partites of $K_{p,q}$, Suppose a vertex u_1 of U devout dominates a vertex w_1 of W. Then being adjacent to w_1, all the remaining vertices in U cannot be devout dominators and, being adjacent to u_1, all the vertices of $W \setminus \{w_1\}$ cannot be non-codewords. Next, let a vertex w_2 devout dominate a vertex u_2. Then being adjacent to u_2 all the vertices (other than w_2) in W cannot be devout dominators and, being adjacent to w_2, all the vertices of U (other than u_2) cannot be non-codewords. Thus all the remaining vertices will have to be non-devout dominating codewords, and the result follows.

Definition 7. The p − crown graph (where $p \geq 3$) is defined as a graph with vertex set $\{x_0, x_1, \ldots, x_{p-1}, y_0, y_1, \ldots, y_{p-1}\}$ and the edge set $\{(x_i, y_j) : 0 \leq i, j \leq p-1, i \neq j\}$. In other words, it is a $K_{p,p}$ complete bipartite graph with horizontal edges removed. Figure 3a shows the 4 − crown graph.

Proposition 8. The MILD number of a p − crown graph, $p \geq 4$ is $2p - 4$.

Proof. Let U and W be the partites of the p − crown. Suppose a vertex u_1 of U devout dominates a vertex w_1 of W, then only one vertex, say w_1', is not adjacent to u_1 and hence can afford to become another non-codeword from W. Also, only one vertex u_1' can afford to become another devout dominator from U. Let u_1' devout dominate w_1'.

Let a vertex $u_2 \neq u_1$ adjacent to w_1 be devout dominated by a vertex w_2. Then by similar argument done in the previous paragraph, only one more code pair, say u_2' and w_2', is possible. Apart from the eight vertices of the four code pairs discussed until now, all other vertices (if any) must become non-devout dominating codewords and the result follows.

Definition 9. The cartesian product of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ with an edge (x, y) adjacent to (x', y') if and only if (1) $x = x'$ and $yy' \in E(H)$, or (2) $y = y'$ and $xx' \in E(G)$. It is written as $G \square H$.

Definition 10. The $k-$ book graph is defined as the cartesian product $B_k = S_{k+1} \square P_2$ where P_3 is the path with two vertices and S_{k+1} is a star graph with $k+1$ vertices. Figure 3b shows the book graphs B_3.

Proposition 11. The MILD number of the book graph B_k is $k+1$.

Proof. Consider a $(k+1)-$ star printed on a page. Suppose we pull all the vertices above the page in such a way that the original vertices are intact, but there are new vertices where we pull and leave, and also, the path in which the vertices were pulled become edges. Then we have a $k-$ book graph, and there are two sets of vertices, say A on the page and B above the page. We can see that every vertex in A is adjacent to a unique vertex in B and vice-versa. Making all the vertices of A codewords (or non-codewords) and all the vertices of B non-codewords (or codewords), we have $(k+1)$ code pairs, which proves the result.

Definition 12. A $k-$ crossed prism graph for a positive even k is a graph obtained by taking two copies of a cycle C_k, say, C_1 and C_2 with vertices $\{u_1, u_2, \ldots, u_k\}$ and $\{v_1, v_2, \ldots, v_k\}$ respectively and, adding edges (u_p, v_{p+1}) and (u_{p+1}, v_p) for $p = 1, 3, \ldots, k-1$. Figure 4a shows the 6-crossed prism graph.

Proposition 13. *The MILD number of $k-$ crossed prism graph is k.*

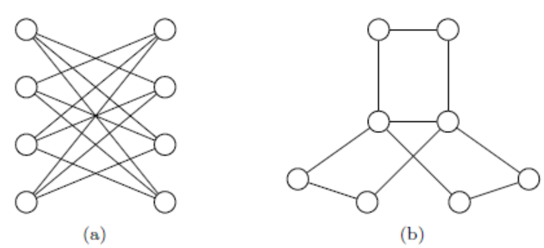

Figure 3. $4-$ Crown graph and the book graph B_3.

Proof. Of the two cycles that the crown prism is made of, every vertex of a cycle is adjacent to a unique vertex of another cycle. By the argument in the previous proposition, there are $2k/2$ code pairs, i.e., there are k codewords.

Definition 14. A wheel graph W_k of order $k \geq 4$ is made up of cycle

C_{k-1} and a vertex (called the hub) adjacent to every vertex of the cycle. The gear graph G_k is a graph resulting from the subdivision of the edges of the cycle in the wheel graph W_{k+1}.

It has $2k+1$ vertices and $3k$ edges. Because of subdivision, the outer cycle is always even and, the hub is a part of k 4-cycles. With every cycle being even, a gear graph is bipartite and hence, is also called a bipartite wheel graph. We now find the MILD number of a gear graph using the following proposition.

Proposition 15. [12] *For a path* P_n, $\gamma_{ml}(P_n) = \lceil n/2 \rceil$

Proposition 16. [12] *For a cycle* C_n, $\gamma_{ml}(C_n) = \gamma_{ml}(C_n) + c$ *where*

$$c = \begin{cases} 1, & n = 4k+2,\ k \in Z^{+} \\ 0, & otherwise \end{cases}$$

Proposition 17. *For the gear graph*

$$G_k,\ \gamma_{ml}(G_k) = \begin{cases} k+1, & where\ 2k \equiv 0 (\mathrm{mod}\ 4) \\ k+2, & otherewise \end{cases}$$

Proof. Consider the outer cycle of G_k which has $2k$ vertices. By Proposition 16, if $2k \equiv 0(\mathrm{mod}\ 4)$ then k vertices must be codewords, otherwise vertices must be codewords and all this will happen only when the hub is a codeword.

Thus, the MILD number is $k+1$ in the former case and $k+2$ in the latter.

Definition 18. A k-regular graph with $2k$ vertices is called a k-hypercube, denoted by Q_k. Figure 5 shows the hypercube Q_4.

Proposition 19. *For the hypercube* Q_k, $\gamma_{ml}(Q_k) = 2^{k-1}$.

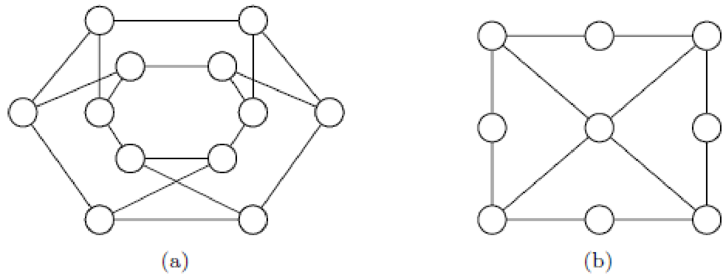

Figure 4. 6 − crossed prism graph and the gear graph G_4.

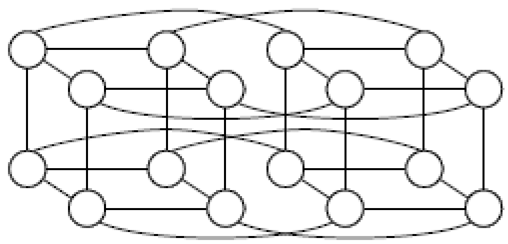

Figure 5. Hypercube graph Q_4.

Proof. It is a well known fact that a hypercube Q_k can be constructed by taking two copies of Q_{k-1} and adding an edge from a vertex of a copy of Q_{k-1} to the corresponding vertex of another copy of Q_{k-1}. The joining edges form a perfect matching in such a way that every vertex of a copy of Q_{k-1} is adjacent to only one vertex of another copy of Q_{k-1}. By making all the vertices of one copy of Q_{k-1} codewords, there will be 2_{k-1} code pairs, which proves the result.

4. Conclusion

The problems of finding the MILD numbers of chordal (and hence, split) graphs and bipartite graphs are proved to be NP-complete. The results extend for the connected MILD numbers as well. However, when restricted to some classes of bipartite graphs, we could find the MILD numbers.

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