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APPLICATION OF ANDUALEM AND KHAN TRANSFORM (AKT) TO RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE, RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL AND MITTAG-LEFFLER FUNCTION

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Abstract

The application of fractional calculus increases from time to time. That is why; researchers are always searching for new techniques (analytic, approximate or numerical) to solve such fractional calculus. Based on such a motivation, in this work, we have developed a new method known as Andualem and Khan Transform (AKT) to solve Riemann-Liouville fractional derivative, Riemann-Liouville fractional integral and Mittag-Leffler function. More exactly, here, first we have introduced a new technique (AK) with counter examples of some basic functions and then we have applied it to fractional derivatives and fractional integration and the solutions are successfully obtained. AK can also be applied to similar nonlinear problems in future.

Introduction

In recent years, fractional calculus (FC) has gained considerable achievements in various fields of science and engineering. Fractional calculus is the field of mathematical analysis which deals with the investigation and

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applications of integrals and derivatives of arbitrary order. Fractional Calculus is used in many problems, for example in engineering, physics, economics, biological processes, etc. Many physical problems [1-7] are modelled by using fractional differential equations (FDE) more accurately than classical differential equations [8-11]. Over the years, many mathematicians, using their own notation and approach, have found various definitions that fit the idea of a non-integer order integral or derivative. One version that has been popularized in the world of fractional calculus is the Riemann-Liouville definition.

One of the first to use Fractional Calculus for a problem was the Norwegian mathematician Niels Henrik Abel. In 1823 he applied it in the formulation of his solution for the Tautochrone Problem. The idea of this problem is to find the curve of a frictionless wire which lies in the plane such that the time required for a particle to slide down to the lowest point of the curve independent is of where the particle is placed. In this paper, we discussed important concepts of AK transform, and we present the application of AK transform for Riemann-Liouville integral, Riemann Liouville fractional derivative, and Mittag-Leffler function.

AK Transform

A new transform called the AK Transform of the function $y(t)$ belonging to a class A, where:

$$A = \left\{ y(t) : \exists N,\ \eta_1,\ \eta_2 > 0,\ |\, y(t)\,| < ne^{\frac{|t|}{\eta_i}},\ \text{if}\ t \in (-1)^i \times [0, \infty) \right\}$$

The AK transform of $y(t)$ denoted by $M_i[y(t)] = \bar{y}(s, \beta)$ and is given by:

$$\bar{y}(s, \beta) = M_i\{y(t)\} = s\int_0^\infty y(t)\, e^{-\frac{s}{\beta}t}\, dt \qquad (1)$$

Table 1. AK Transform of Some Basic functions.

$f(t)$	$M_i(f(t))$
C(constant)	$C\beta$
t^n	$\Gamma(n+1)\dfrac{\beta^{n+1}}{s^n}$
$e^{\lambda t}$	$\dfrac{s\beta}{s-\lambda\beta}$
$\sin(t)$	$\dfrac{s\beta^2}{\beta^2+s^2}$
$f^{(n)}(t)$	$\left(\dfrac{\beta}{s}\right)^{-n}\bar{f}(s,\beta)-\sum_{k=0}^{n-1}\dfrac{\beta^{k-n}}{s^{k-n-1}}f^{(k)}(0),\ n\geq 1$

The sufficient condition for the existence of AK transform.

If the function $y(t)$ is piecewise continues in every finite interval $0\leq t\leq\alpha$ and of exponential order β for $t>\beta$. Then its AK transform $\bar{y}(s,\beta)$ exists.

Proof. For any positive α, we have

$$s\int_0^\infty y(t)e^{-\frac{st}{\beta}}dt = s\int_0^\alpha y(t)e^{-\frac{st}{\beta}}dt + s\int_\alpha^\infty y(t)e^{-\frac{st}{\beta}}dt$$

Since the function $y(t)$ is piecewise continues in every finite interval $0\leq t\leq\alpha$, then the first integral on the right hand side exists. Besides, the second integral on the right hand side exists, since the function $y(t)$ is of exponential order β for $t>\beta$.

Now to this, we have following

$$\left|s\int_0^\infty y(t)e^{-\frac{st}{\beta}}dt\right| \leq s\int_0^\infty\left|y(t)e^{-\frac{st}{\beta}}dt\right|dt$$

$$\leq s\int_0^\infty e^{-\frac{st}{\beta}}dt|\, y(t)\,|\, dt \leq s\int_0^\infty e^{-\frac{st}{\beta}}Me^{\alpha t}dt$$

$$= Ms\int_0^\infty e^{-\frac{(s-\alpha\beta)t}{\beta}}dt$$

$$= -\frac{Ms\beta}{s-\alpha\beta}\lim_{\tau\to\infty} e^{-\frac{(1-s\alpha)t}{s}}\Bigg]_0^\tau$$

$$= \frac{Ms\beta}{s-\alpha\beta}$$

and this improper integral is convergent for all, $s > \alpha\beta$. Thus $M_i\{y(t)\} = \bar{y}(s, \beta)$.

Convolution property of AK transform

Let $\bar{f}(s, \beta)$ and $\bar{g}(s, \beta)$ denote the AK transform of $f(t)$ and $g(t)$ respectively. Then

$$M_i[f * g] = \frac{1}{s}\bar{f}(s, \beta)\,\bar{g}(s, \beta).$$

Proof. The convolution of two functions $f(t)$ and $g(t)$ is given by:

$$f * g = \int_0^\infty f(\tau)g(t-\tau)d\tau$$

$$M_i[f * g] = s\int_0^\infty\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right]e^{-\frac{s}{\beta}t}dt$$

$$= s\int_0^\infty f(\tau)\left[\int_0^\infty g(t-\tau)\,e^{-\frac{s}{\beta}t}dt\right]d\tau$$

Using substitution method $u = t - \tau \Rightarrow u + \tau = t$, we get

$$M_i[f * g] = s\int_0^\infty f(\tau)\left[\int_0^\infty g(u)\,e^{-\frac{(us+\tau s)}{\beta}}du\right]d\tau$$

$$M_i[f * g] = s\int_0^\infty f(\tau)\,e^{-\frac{s\tau}{\beta}}d\tau\int_0^\infty g(t)\,e^{-\frac{st}{\beta}}dt = \frac{1}{s}\bar{f}(s, \beta)\,\bar{g}(s, \beta).$$

Mittag-Leffler Function

The Mittag-Leffler function is a generalization of the exponential function and it is one of the most important functions that are related to fractional differential equations.

Definition. The one and two-parameter Mittag-Leffler functions are defined, respectively, by:

$$E_a(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(an+1)}, a > 0$$

$$E_{a,b}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(an+1)}, \; a > 0, \; b > 0$$

If $a = 1 = b$, then

$$E_{1,1}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+1)} = e^x$$

Taking AK transform both sides, we get

$$\Rightarrow M_i(E_{1,1}(x)) = M_i\left[\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+1)}\right] = \frac{s\beta}{s-\beta}.$$

AK transform of Mittag-Leffler function

The AK transform of the Mittag-Leffler function is given by the following theorem:

$$M_i^{-1}\left[\frac{\frac{s^{a-b+1}}{\beta^{a-b}}}{\left(\frac{s}{\beta}\right)^a - \lambda}\right] = t^{b-1}E_{a,\, b}(\lambda t^a).$$

Proof. By using the definition of AK transform, we have

$$M_i[x^{b-1}E_{a,\, b}(at^a)] = M_i\left[\sum_{n=0}^{\infty}\frac{x^{b-1}(\lambda x^a)^n}{\Gamma(an+b)}\right]$$

$$= s\int_0^{\infty}\sum_{n=0}^{\infty}\frac{x^{b-1}(\lambda x^a)^n}{\Gamma(an+b)}e^{-\frac{s}{\beta}t}dt$$

$$= s\int_0^{\infty}t^{b-1}e^{-\frac{s}{\beta}t}\sum_{n=0}^{\infty}\frac{\lambda^n x^{a^n}}{\Gamma(an+b)}dt$$

$$= \sum_{n=0}^{\infty}\frac{\lambda^n}{\Gamma(n+1)},\, s\int_0^{\infty}e^{-\frac{s}{\beta}t}t^{an+b-1}dt$$

$$= \sum_{n=0}^{\infty}\frac{\lambda^n}{\Gamma(n+1)}M_i[t^{an+b-1}].$$

Since $M_i[t^n] = \Gamma(n+1)\frac{\beta^{n+1}}{s^n}$. Therefore,

$$M_i[t^{an+b-1}] = \Gamma(an+b)\frac{\beta^{an+b}}{s^{an+b-1}}$$

Hence,

$$M_i[x^{b-1}E_{a,\, b}(at^a)] = \sum_{n=0}^{\infty}\frac{\lambda^n}{\Gamma(an+b)}\cdot\Gamma(an+b)\frac{\beta^{an+b}}{s^{an+b-1}}$$

$$= \sum_{n=0}^{\infty}\lambda^n\cdot\frac{\beta^{an+b}}{s^{an+b-1}}$$

$$= \frac{\beta^b}{s^{b-1}} \sum_{n=0}^{\infty} \left(\frac{\lambda \beta^a}{s^a}\right)^n$$

Again from geometric series, we have

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\Rightarrow M_i[x^{b-1} E_{a,\, b}(at^a)] = \frac{\beta^b}{s^{b-1}} \sum_{n=0}^{\infty} \left(\frac{\lambda \beta^a}{s^a}\right)^n = \frac{\beta^b}{s^{b-1}} \left(\frac{1}{1 - \frac{\lambda \beta^a}{s^a}}\right)$$

$$= \frac{\beta^b}{s^{b-1}} \left(\frac{s^a}{s^a - \lambda \beta^a}\right)$$

$$= \frac{\frac{s^{a-b+1}}{\beta^{a-b}}}{\left(\frac{s}{\beta}\right)^a - \lambda}$$

$$M_i^{-1}\left[\frac{\frac{s^{a-b+1}}{\beta^{a-b}}}{\left(\frac{s}{\beta}\right)^a - \lambda}\right] = x^{b-1} E_{a,\, b}(ax^a).$$

AK Transform of Riemann-Liouville fractional integral

Riemann-Liouville fractional integral is defined as:

$$aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t - \tau)^{p-1} f(\tau)\, d\tau \tag{2}$$

where the symbol $aD_t^{-p} f(t)$ denotes fractional integration of order p. This definition was derived from Cauchy formula, when we replace the integer n by $p > 0$, $p \in \mathbb{R}$ Note: when $a = 0$, we will use the notation $D^{-p} f(t)$ to denote the fractional integration of order and defined as:

$$D^{-p}f(t) = \frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1} f(\tau)\,d\tau \tag{3}$$

Now, taking the AK transform both sides of equation (3), we get

$$M_i(D^{-p}f(t)) = M_i\left(\frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1} f(\tau)\,d\tau\right)$$

$$= M_i\left(\frac{1}{\Gamma(p)} t^{p-1} * f(t)\right)$$

$$= \frac{1}{\Gamma(p)} M_i(t^{p-1} * f(t))$$

$$= \frac{1}{\Gamma(p)}\frac{1}{s}\,\bar{g}(s,\beta)\bar{f}(s,\beta), \text{ where } g(t) = t^{p-1}$$

$$= \left(\frac{s}{\beta}\right)^{-p}\bar{f}(s,\beta).$$

AK Transform of Riemann-Liouville fractional derivative

Definition. The Riemann-Liouville fractional derivative of order $p \in \mathbb{R},\ p > 0$ is given by

$$aD_t^p f(t) = \frac{1}{\Gamma(m-p)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-p-1} f(\tau)\,d\tau \tag{4}$$

$$= \frac{d^m}{dt^m}(aD_t^{-(m-p)}f(t)),\ m-1 \le p \le m \tag{5}$$

Equation (4) can be rewritten as:

$$aD_t^p f(t) = \frac{1}{\Gamma(m-p)}\int_0^t (t-\tau)^{m-p-1} f(\tau)\,d\tau$$

$$= \frac{1}{\Gamma(m-p)}(t^{m-p-1} * f^{(m)}(\tau)) \tag{6}$$

Taking the AK transform both sides of equation (6)

$$M_i[aD_t^p f(t)] = M_i\left(\frac{1}{\Gamma(m-p)}(t^{m-p-1} * f^{(m)}(\tau))\right)$$

$$= \frac{1}{\Gamma(m-p)} M_i(t^{m-p-1} * f^{(m)}(\tau))$$

$$= \left(\frac{\beta}{s}\right)^{-m} \bar{f}(s, \beta) - \sum_{k=0}^{m-1} \frac{\beta^{k-m+1}}{s^{k-m}} f^{(k)}(0).$$

Example. Consider the following fractional differential equations

$${}_0^c D_t^{\alpha} y(t) + ay(t) = 0,\ t > 0,\ y(0) = A,\ 0 < \alpha < 1. \tag{7}$$

Solution. Since from equation (6), we have

$$0D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau$$

$$= \frac{1}{\Gamma(m-p)}(t^{m-\alpha-1} * f^{(m)}(\tau)) \quad m-1 < \alpha < m \tag{8}$$

Taking the AK transform both sides of equation (7)

$$M_i({}_0^c D_t^{\alpha} y(t) + ay(t) = 0)$$

$$M_i({}_0^c D_t^{\alpha} y(t) + aM_i[y(t)] = 0)$$

Now using the definition of AK transform for fractional derivative, we have

$$\left(\frac{\beta}{s}\right)^{-\alpha} \bar{y}(s, \beta) - \sum_{k=0}^{m-1} \frac{\beta^{k-\alpha+1}}{s^{k-\alpha}} y^{(k)}(0) + a\bar{y}(s, \beta) = 0 \tag{9}$$

Since $0 < \alpha < 1$ and from equation (8) $m-1 < \alpha < m$, implies $m = 1$.

Therefore, equation (9) can be rewritten as:

$$\left(\frac{\beta}{s}\right)^{-\alpha} \bar{y}(s, \beta) - \sum_{k=0}^{0} \frac{\beta^{k-\alpha+1}}{s^{k-\alpha}} y^{(k)}(0) + a\bar{y}(s, \beta) = 0$$

$$\Rightarrow \left(\frac{\beta}{s}\right)^{-\alpha} \bar{y}(s, \beta) - \sum_{k=0}^{0} \frac{\beta^{-\alpha+1}}{s^{-\alpha}} y^{(k)}(0) + a\bar{y}(s, \beta) = 0$$

Using the given initial condition, we get

$$\left[\left(\frac{\beta}{s}\right)^{-\alpha} + a\right] \bar{y}(s, \beta) = A \frac{\beta^{-\alpha+1}}{s^{-\alpha}}$$

$$\Rightarrow \bar{y}(s, \beta) = A \cdot \frac{\frac{s^{\alpha}}{\beta^{\alpha-1}}}{\left(\frac{s}{\beta}\right)^{a} + a}$$

Applying inverse AK transform

$$y(t) = AM_t^{-1}\left(\frac{\frac{\beta^{-\alpha+1}}{s^{-\alpha}}}{\left(\frac{\beta}{s}\right)^{-\alpha} + a}\right)$$

$$= AM_i^{-1}\left[\frac{\frac{s^{\alpha}}{\beta^{\alpha-1}}}{\left(\frac{s}{\beta}\right)^{a} + a}\right]$$

Since

$$M_i^{-1}(t^{b-1}E_{a,\, b}(\lambda t^{a})) = \frac{\frac{s^{a-b+1}}{\beta^{a-b}}}{\left(\frac{s}{\beta}\right)^{a} - \lambda}$$

Now comparing the exponents of the two terms, we have

$$a = a - b + 1 \Rightarrow b = 1, \text{ and } \lambda = -a$$

$$\Rightarrow y(t) = A[E_{a,\, 1}(-at^{a})]$$

Let $a = 1$, then

$$y(t) = A[E_{a,1}(-at^{a})]$$

$$= A\sum_{n=0}^{\infty}\frac{(-at)^n}{\Gamma(n+1)} = A\cdot e^{-at}.$$

Example. Consider the following non-homogeneous initial value problem

$$0D_x^a y(t) + 0D_x^b y(x) = h(x),\ x > 0,\ 0 < a < b < 1$$

$$y(0) = C.$$

Solution. Taking the AK transform of both side of the above IVP, we have

$$M_i[0D_x^a y(x) + 0D_x^b y(x) = h(x)]$$

$$\Rightarrow \left(\frac{\beta}{s}\right)^{-a}\bar{y}(s,\beta) - \sum_{k=0}^{m-1}\frac{\beta^{k-a+1}}{s^{k-a}}y^{(k)}(0) + \left(\frac{\beta}{s}\right)^{-b}\bar{y}(s,\beta) - \sum_{k=0}^{m-1}\frac{\beta^{k-b+1}}{s^{k-b}}y^{(k)} = \bar{h}(s,\beta)$$

$$0 < a < b < 1,\ m-1 < a < m \Rightarrow m = 1$$

$$\left(\frac{\beta}{s}\right)^{-a}\bar{y}(s,\beta) - \frac{\beta^{-a+1}}{s^{-a}}y(0) + \left(\frac{\beta}{s}\right)^{-b}\bar{y}(s,\beta) - \frac{\beta^{-b+1}}{s^{-b}}y(0) = \bar{h}(s,\beta).$$

Applying the given initial conditions, we get

$$\Rightarrow \left(\frac{s}{\beta}\right)^{a}\bar{y}(s,\beta) - C\frac{s^a}{\beta^{a-1}} + \left(\frac{s}{\beta}\right)^{b}\bar{y}(s,\beta) - C\frac{s^b}{\beta^{b-1}} = \bar{h}(s,\beta)$$

$$\Rightarrow \left[\left(\frac{s}{\beta}\right)^{a} + \left(\frac{s}{\beta}\right)^{b}\right]\bar{y}(s,\beta) - C\left[\frac{s^a}{\beta^{a-1}} + \frac{s^b}{\beta^{b-1}}\right] = \bar{h}(s,\beta)$$

$$\Rightarrow \left[\left(\frac{s}{\beta}\right)^{a} + \left(\frac{s}{\beta}\right)^{b}\right]\bar{y}(s,\beta) = \bar{h}(s,\beta) + C\left[\frac{s^a}{\beta^{a-1}} + \frac{s^b}{\beta^{b-1}}\right]$$

$$\bar{y}(s,\beta) = \frac{\bar{h}(s,\beta)}{\left(\frac{s}{\beta}\right)^{a} + \left(\frac{s}{\beta}\right)^{b}} + C\frac{\frac{s^a}{\beta^{a-1}} + \frac{s^b}{\beta^{b-1}}}{\left(\frac{s}{\beta}\right)^{a} + \left(\frac{s}{\beta}\right)^{b}}$$

$$= \bar{h}(s, \beta)\frac{1}{\left(\frac{s}{\beta}\right)^a + \left(\frac{s}{\beta}\right)^b} + C\frac{\frac{s^a}{\beta^a}\cdot\beta + \frac{s^b}{\beta^b}\cdot\beta}{\left(\frac{s}{\beta}\right)^a + \left(\frac{s}{\beta}\right)^b}$$

$$= \bar{h}(s, \beta)\frac{1}{\left(\frac{s}{\beta}\right)^a + \left(\frac{s}{\beta}\right)^b} + C\beta$$

$$y(x) = M_i^{-1}\left[\bar{h}(s, \beta)\frac{1}{\left(\frac{s}{\beta}\right)^a + \left(\frac{s}{\beta}\right)^b}\right] + M_i^{-1}(C\beta)$$

$$= M_i^{-1}[\bar{h}(s, \beta)\bar{f}(s, \beta)] + C, \text{ where } \bar{f}(s, \beta) = \frac{1}{\left(\frac{s}{\beta}\right)^a + \left(\frac{s}{\beta}\right)^b}$$

$$\text{where } \bar{f}(s, \beta) = \frac{1}{\left(\frac{s}{\beta}\right)^a + \left(\frac{s}{\beta}\right)^b} = \frac{\left(\frac{s}{\beta}\right)^{-a}}{\left(\frac{s}{\beta}\right)^{b-a} + 1}$$

$$\Rightarrow f(x) = M_i^{-1}\left[\frac{\left(\frac{s}{\beta}\right)^{-a}}{\left(\frac{s}{\beta}\right)^{b-a} + 1}\right].$$

Since

$$M_i^{-1}(t^{b-1}E_{a,\, b}(\lambda t^a)) = \frac{\frac{s^{a-b+1}}{\beta^{a-b}}}{\left(\frac{s}{\beta}\right)^a - \lambda}$$

$$\Rightarrow \lambda = -1,\ a = b - a$$

$$f(x) = (t^{b-1}E_{b-a,\, b}(-t^a)).$$

Since the solution of the unknown function given by

$$y(x) = M_i^{-1}[\bar{h}(s, \beta)\bar{f}(s, \beta)] + C.$$

Now from convolution theorem, we have

$$f * g = \int_0^\infty f(\tau)g(t-\tau)d\tau$$

$$M_i(f * g) = \frac{1}{s}\bar{f}(s, \beta)\bar{g}(s, \beta)$$

$$y(x) = f(x) * g(x) + C$$

$$= \int_0^x f(x-\tau)\,h(\tau)\,d\tau + C$$

$$\Rightarrow y(x) = \int_0^x (x-\tau)^{b-1} E_{b-a,\, b}(-1(x-\tau))^{b-a} h(\tau)d\tau + C.$$

Conclusion

In this paper, we have developed a new integral method known as AK transform with counter examples of some basic functions, and also some fundamental theorems of this new method are provided. We use AK transform to solve linear homogeneous and non-homogenous fractional differential equations and we obtained an exact solution of in all examples. In future studies, AK transform can be applied for other problems in fluid mechanics, biomathematics, etc.

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