Advances and Applications in Mathematical Sciences
Volume 20, Issue 9, July 2021, Pages 1985-1994

CAUCHY INTEGRAL FORMULA FOR BI-POLYANALYTIC FUNCTIONS ON THE QUARTER PLANE

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Abstract

Explicit representation of Cauchy's Integral formula for bi-polyanalytic functions is obtained on the quarter plane, which is a very important tool in finding out the solutions of different forms of boundary conditions arising from well known problems of complex analysis like Neumann, Dirichlet and Schwarz.

1. Introduction

The complex form of the bi-analytic function can be reduced to the system

$$\frac{p+1}{2} f_{\bar{z}} - \frac{p-1}{2} f_z = \frac{\gamma - p}{4\gamma} \phi + \frac{\gamma + p}{4\gamma} \bar{\phi}, \ \phi_{\bar{z}} = 0 \tag{1.1}$$

with real constants p, $0 < p \leq 1$, and $\gamma \neq 0, 1, p^2$. Theory of bi-polyanalytic functions was introduced in [2]. In general, we can define a polyanalytic function as a solution to the equation

$$\partial_{\bar{z}}^{n} \phi = 0, \tag{1.2}$$

where n is a natural number [1]. The general solution to (2) can be written as

2010 Mathematics Subject Classification: 30G20, 31A30, 35J55, 30E25.

Keywords: Bi-polyanalytic functions, Cauchy-Pompeiu representation, Gauss theorem, boundary value problems for analytic functions.

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Received July 2, 2021; Accepted July 16, 2021

$$\phi(z) = \sum_{p=0}^{n-1} a_p(z)\bar{z}^p \tag{1.3}$$

with analytic coefficients a_p. In this paper, the system

$$f_{\bar{z}} = \frac{\gamma - 1}{4\gamma}\phi + \frac{\gamma + 1}{4\gamma}\bar{\phi},\ \partial_{\bar{z}}^n \phi = 0 \tag{1.4}$$

will be considered on the quarter plane $\mathbb{Q}$ and f will be called a bi-polyanalytic function. In case of unit disc and upper half plane, the system (4) have been studied extensively in [3, 8].

2. Cauchy Integral Formula for Bi-polyanalytic Functions

Let $f : \mathbb{Q}_1 \to \mathbb{C}$ satisfy $|f(x)| \le c|x|^{-\epsilon}$ for $|x| > K, \epsilon > 1$, and for $0 \le \mu \le n-1$, ϕ satisfies the regularity condition $|\partial_{\bar{z}}^{\mu}\phi(x)| \le c\frac{|x|^{-\mu-\epsilon}}{\log|x|}$ for $|x| > K$ where $Q_1 = \{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$.

The Cauchy formula given here is more explicit in comparison with that in the unit Disc [6]. The following formula can also be obtained for the unit disc $\mathbb{D}$.

Theorem 2.1. *Any solution to* (4) *i.e. any bi-polyanalytic function of order n can be written as*

$$f(z) = \frac{1}{2\pi i}\int_0^{+\infty}\frac{f(t)}{t-z}dt - \frac{1}{2\pi i}\int_0^{+\infty}\frac{f(it)}{t+iz}dt$$

$$-\frac{\gamma-1}{4\gamma}\sum_{\mu=1}^{n-1}\frac{(-1)^{\mu}}{(\mu)!}\frac{1}{2\pi i}\int_0^{\infty}\frac{(t-\bar{z})^{\mu}}{t-z}\partial_{\bar{\varsigma}}^{\mu}\phi(t)dt$$

$$+\frac{\gamma-1}{4\gamma}\sum_{\mu=1}^{n}\frac{1}{(\mu)!}\frac{1}{2\pi i}\int_0^{+\infty}\frac{(it-\bar{z})^{\mu}}{t+iz}\partial_{\bar{\varsigma}}^{\mu-1}\phi(it)dt$$

$$-\frac{\gamma-1}{4\gamma}\left[\sum_{\mu=1}^{n-1}(-1)^{\mu-1}b_{\mu}\frac{1}{2\pi i}\int_0^{+\infty}[(t-z)^{\mu}\overline{\partial_{\bar{\varsigma}}^{\mu-1}\phi(t)}\right.$$

$$+ i(it - z)^{\mu} \overline{\partial_{\varsigma}^{\mu-1} \phi(it)}] dt$$

$$+ \sum_{\mu=1}^{n} \frac{(-1)^{\mu}}{(\mu-1)!} \frac{1}{2\pi i} \int_{0}^{+\infty} [(t-z)^{\mu-1} \log|t-z|^2 \overline{\partial_{\varsigma}^{\mu-1} \phi(t)}$$

$$+ i(it - z)^{\mu-1} \log|it - z|^2 \overline{\partial_{\varsigma}^{\mu-1} \phi(it)}]] dt, \tag{2.5}$$

where $b_{\mu} = (((1 + 1/2!)1/2 + 1/3!)1/3 + \ldots + 1/\mu!)1/\mu$.

Proof. Making use of Cauchy-Pompeiu formula [5] on the quarter disc

$$\mathbb{Q}_{1R} = \{z : |z| < R, \operatorname{Re} z > 0, \operatorname{Im} z > 0\},$$

we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} \frac{f(\varsigma)}{\varsigma - z} d\varsigma - \frac{\gamma - 1}{4\gamma} \frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{\phi(\varsigma)}{\varsigma - z} d\xi d\eta$$

$$- \frac{\gamma + 1}{4\gamma} \frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{\overline{\phi(\varsigma)}}{\varsigma - z} d\xi d\eta. \tag{2.6}$$

Let $z \in \mathbb{Q}_{1R}$ be fixed and consider $W(\varsigma) = \frac{1}{\varsigma - z}$ and

$$w(\varsigma) = T_{0,n} W(\varsigma) = \frac{(-1)^n}{(n-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{(\overline{\tilde{\varsigma} - \varsigma})^{n-1}}{\tilde{\varsigma} - \varsigma} \frac{d\tilde{\xi} d\tilde{\eta}}{\tilde{\varsigma} - z}$$

$$\tilde{w}(\varsigma) = T_{n,0} W(\varsigma) = \frac{(-1)^n}{(n-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{(\tilde{\varsigma} - \varsigma)^{n-1}}{\overline{\tilde{\varsigma} - \varsigma}} \frac{d\tilde{\xi} d\tilde{\eta}}{\tilde{\varsigma} - z},$$

where $T_{0,n}$, $T_{n,0}$ are Cauchy Pompeiu operators of higher order on $\mathbb{Q}_{1R}$ which are defined on Disc and half disc $\mathbb{H}_R [1, 4, 7, 8]$. Note that $\partial_{\bar{\varsigma}}^{n} w(\varsigma) = W(\varsigma)$, therefore applying Gauss theorem, we have

$$\sum_{\mu=1}^{n} \frac{(-1)^{\mu}}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} \partial_{\bar{\varsigma}}^{n-\mu} w(\varsigma) \partial_{\varsigma}^{\mu-1} \phi(\varsigma) d\varsigma$$

$$= \sum_{\mu=1}^{n} \frac{(-1)^{\mu}}{\pi} \int_{\mathbb{Q}_{1R}} \{\partial_{\bar{\varsigma}}^{n-\mu+1} w(\varsigma) \partial_{\bar{\varsigma}}^{\mu-1} \varphi(\varsigma) + \partial_{\bar{\varsigma}}^{\mu-\mu} w(\varsigma) \partial_{\bar{\varsigma}}^{\mu} \varphi(\varsigma)\} d\xi d\eta$$

$$= -\frac{1}{\pi}\int_{\mathbb{Q}_{1R}} \partial_{\bar{\varsigma}}^{n} w(\varsigma)\varphi(\varsigma) d\xi d\eta + \frac{(-1)^n}{\pi}\int_{\mathbb{Q}_{1R}} \partial_{\bar{\varsigma}}^{n} w(\varsigma)\varphi(\varsigma) d\xi d\eta$$

$$= -\frac{1}{\pi}\int_{\mathbb{Q}_{1R}} \frac{\phi(\varsigma)}{\varsigma - z} d\xi d\eta$$

(ϕ being a polyanalytic function and $\partial_{\bar{\varsigma}}^{n} w(\varsigma) = \frac{1}{\varsigma - z}$) i.e.

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \frac{f(\varsigma)}{\varsigma - z} d\varsigma - \frac{\gamma - 1}{4\gamma}\frac{1}{\pi}\int_{\mathbb{Q}_{1R}} \frac{\phi(\varsigma)}{\varsigma - z} d\xi d\eta - \frac{\gamma + 1}{4\gamma}\frac{1}{\pi}\int_{\mathbb{Q}_{1R}} \frac{\overline{\phi(\varsigma)}}{\varsigma - z} d\xi d\eta.$$

$$\sum_{\mu=1}^{n} \frac{(-1)^{\mu}}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \partial_{\varsigma}^{n-\mu} w(\varsigma)\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma) d\varsigma = -\frac{1}{\pi}\int_{\mathbb{Q}_{1R}} \frac{\phi(\varsigma)}{\varsigma - z} d\xi d\eta \tag{2.7}$$

and similarly

$$\sum_{\mu=1}^{n} \frac{(-1)^{\mu}}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \partial_{\varsigma}^{n-\mu} \tilde{w}(\varsigma)\overline{\partial_{\bar{\varsigma}}^{\mu-1}\phi(\xi)} d\bar{\varsigma} = -\frac{1}{\pi}\int_{\mathbb{Q}_{1R}} \frac{\overline{\phi(\varsigma)}}{\varsigma - z} d\xi d\eta. \tag{2.8}$$

Now, we compute the first integral of left hand sides of (7) and left hand side of (8) explicitly.

$$\partial_{\bar{\varsigma}}^{n-\mu} w(\varsigma) = T_{0, n-(n-\mu)} W(\varsigma) = T_{0, \mu} W(\varsigma)$$

$$= \frac{(-1)^{\mu}}{(\mu - 1)!}\frac{1}{\mu}\int_{\partial\mathbb{Q}_{1R}} \frac{\left(\overline{\tilde{\varsigma} - \varsigma}\right)^{(\mu-1)}}{(\tilde{\varsigma} - \varsigma)}\frac{d\tilde{\eta} d\tilde{\xi}}{(\tilde{\varsigma} - \varsigma)}$$

$$= \frac{(-1)^{\mu}}{(\mu - 1)!}\frac{1}{\pi}\int_{\partial\mathbb{Q}_{1R}} \frac{1}{\mu}\frac{\partial}{\partial\bar{\tilde{\varsigma}}}\left(\frac{\left(\overline{\tilde{\varsigma} - \varsigma}\right)^{\mu}}{(\tilde{\varsigma} - \varsigma)}\right)\frac{d\tilde{\eta} d\tilde{\xi}}{(\tilde{\varsigma} - z)}$$

$$= \frac{(-1)^{\mu}}{\mu!}\left[\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \frac{\overline{(\tilde{\varsigma} - \varsigma)^{\mu}}}{(\tilde{\varsigma} - \varsigma)(\tilde{\varsigma} - z)} d\bar{\varsigma} + \frac{(z - \varsigma)^{\mu}}{(\varsigma - z)}\right]$$

$$= \frac{1}{\mu!}\left[\tilde{\psi}_{\mu}(\varsigma, z) + \frac{\overline{(\varsigma - z)}^{\mu}}{\varsigma - z}\right],$$

where

$$\widetilde{\psi}_\mu(\varsigma, z) = \frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \frac{\overline{(\varsigma - \widetilde{\varsigma})}^\mu}{(\widetilde{\varsigma} - \varsigma)(\widetilde{\varsigma} - z)} d\widetilde{\varsigma}.$$

Note that

$$\widetilde{\psi}_0(\varsigma, z) = \frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \frac{1}{(\widetilde{\varsigma} - \varsigma)(\widetilde{\varsigma} - z)} d\widetilde{\varsigma} = 0.$$

Thus the left hand side of (7) may be written as

$$\sum_{\mu=1}^{n} \frac{(-1)^\mu}{2\pi i}\int_{\mathbb{Q}_{1R}}\left(\frac{1}{\mu!}\left[\widetilde{\psi}_\mu(\varsigma, z) + \frac{\overline{(\varsigma - z)}^\mu}{\varsigma - z}\right]\right)\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\varsigma$$

$$= \sum_{\mu=1}^{n} \frac{(-1)^\mu}{\mu!}\left[\frac{1}{2\pi i}\int_{\mathbb{Q}_{1R}} \widetilde{\psi}_\mu(\varsigma, z)\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\varsigma + \frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \frac{\overline{(\varsigma - z)}^\mu}{\varsigma - z}\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\varsigma\right]$$

$$= \sum_{\mu=1}^{n} \frac{(-1)^\mu}{\mu!}\left[\frac{1}{\pi}\int_{\mathbb{Q}_{1R}} \partial_{\bar{\varsigma}}\widetilde{\psi}_\mu(\varsigma, z)\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\xi d\eta + \frac{1}{\pi}\int_{\mathbb{Q}_{1R}} \widetilde{\psi}_\mu(\varsigma, z)\partial_{\bar{\varsigma}}^{\mu}\phi(\varsigma)d\xi d\eta\right]$$

$$+ \sum_{\mu=1}^{n} \frac{(-1)^\mu}{\mu!}\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \frac{\overline{(\varsigma - z)}^\mu}{\varsigma - z}\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\varsigma$$

$$= \sum_{\mu=1}^{n} (-1)^\mu\Big[\frac{1}{2\pi i}\int_{\mathbb{Q}_{1R}} \frac{1}{(\mu-1)!}\widetilde{\psi}_{\mu-1}(\varsigma, z)\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\eta d\xi$$

$$+ \frac{1}{\pi}\int_{\mathbb{Q}_{1R}} \widetilde{\psi}_\mu(\varsigma, z)\partial_{\bar{\varsigma}}^{\mu}\phi(\varsigma)d\xi d\eta\Big] + \sum_{\mu=1}^{n} \frac{(-1)^\mu}{\mu!}\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \frac{\overline{(\varsigma - z)}^\mu}{\varsigma - z}\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\varsigma$$

$$= \frac{(-1)^n}{\pi}\int_{\mathbb{Q}_{1R}} \widetilde{\psi}_n(\varsigma, z)\partial_{\bar{\varsigma}}^{n}\phi(\varsigma)d\xi d\eta\Big] + \sum_{\mu=1}^{n} \frac{(-1)^\mu}{\mu!}\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \frac{\overline{(\varsigma - z)}^\mu}{\varsigma - z}\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\varsigma$$

$$= \sum_{\mu=1}^{n} \frac{(-1)^\mu}{\mu!}\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \frac{\overline{(\varsigma - z)}^\mu}{\varsigma - z}\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\varsigma, \tag{2.9}$$

since

$$\partial_{\bar{\varsigma}}\widetilde{\psi}_\mu(\varsigma, z) = \mu\widetilde{\psi}_{\mu-1}(\varsigma, z)$$

and ϕ is polyanalytic. Thus the left hand side of (8) can be written and is equal to

$$\overline{-\sum_{\mu-1}^{n} \frac{(-1)}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} T_{0,\mu} \overline{W}(\varsigma) \partial_{\overline{\varsigma}}^{\mu-1} \phi(\varsigma) d\varsigma.} \tag{2.10}$$

For $\mu - 1$, $T_{0,1}\overline{W}(\varsigma) = -\widetilde{\psi}(\varsigma, z)$ and for $\mu \geq 2$, we have

$$T_{0,\mu}\overline{W}(\varsigma) = \frac{(-1)^{\mu}}{(\mu-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{(\overline{\widetilde{\varsigma}-\varsigma})^{(\mu-1)}}{(\widetilde{\varsigma}-\varsigma)} \frac{1}{\overline{\widetilde{\varsigma}-z}} d\widetilde{\xi} d\widetilde{\eta}$$

$$= \frac{(-1)^{\mu}}{(\mu-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{(\overline{\widetilde{\varsigma}-z} + \overline{z-\varsigma})^{\mu-1}}{(\widetilde{\varsigma}-\varsigma)} \frac{1}{\overline{\widetilde{\varsigma}-z}} d\widetilde{\xi} d\widetilde{\eta}$$

$$= \frac{1}{(\mu-1)!} [\psi_{\mu-2}(\varsigma, z) + (\overline{\varsigma - z}) \binom{v-1}{1} \psi_{\mu-3}(\varsigma, z)$$

$$+ (\overline{\varsigma - z})^{2} \binom{v-1}{2} \psi_{\mu-2}(\varsigma, z) + \ldots$$

$$+ (\overline{\varsigma - z})^{\mu-2} \binom{\mu-1}{\mu-2} \psi_{0}(\varsigma, z) - (\overline{\varsigma - z})^{\mu-1} \widetilde{\psi}(\varsigma, z)]$$

$$= \frac{(-1)^{\mu}}{\mu!} [\frac{1}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} \frac{\overline{\widetilde{\varsigma}-\varsigma}}{(\widetilde{\varsigma}-\varsigma)(\widetilde{\varsigma}-z)} d\widetilde{\varsigma} + \frac{(z-\varsigma)^{\mu}}{(\varsigma-z)}].$$

Thus $T_{0,\mu}\overline{W}(\varsigma)$ can be expressed as

$$\frac{(1)}{(\mu-1)!} \left[\sum_{r=0}^{\mu-2} \binom{v-1}{r} (\overline{\varsigma - z})^{r} \psi_{\mu-2-r}(\varsigma, z) - (\overline{\varsigma - z})^{\mu-1} \widetilde{\psi}(\varsigma, z) \right],$$

where

$$\psi_{r}(\varsigma, z) = \frac{(-1)^{r}}{\pi} \int_{\mathbb{Q}_{1R}} \frac{(\overline{\widetilde{\varsigma}-z})^{r}}{(\widetilde{\varsigma}-\varsigma)} d\widetilde{\xi} d\widetilde{\eta}$$

and

$$\widetilde{\psi}(\varsigma, z) = \frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{1}{(\widetilde{\varsigma}-z)\overline{(\widetilde{\varsigma}-z)}} d\widetilde{\xi} d\widetilde{\eta}.$$

Note that $\partial_{\bar{\varsigma}}\psi_r(\varsigma, z) = (-1)^{r+1}(\overline{\varsigma - z})^r$ and $\tilde{\psi}(\varsigma, z) = h(\varsigma, z) - \log|\varsigma - z|^2$, where

$$h(\varsigma, z) = \frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}} \log|\tilde{\varsigma} - z|^2 \frac{d\tilde{\varsigma}}{\tilde{\varsigma} - \varsigma} \text{ and } \partial_{\bar{\varsigma}}h(\varsigma, z) = 0. \tag{2.11}$$

Thus, the conjugate of the expression in (10) is equal to

$$-\sum_{\mu=1}^{n}\frac{(-1)^{\mu}}{(\mu-1)!}\frac{1}{2\pi i}[\int_{\partial\mathbb{Q}_{1R}}[\sum_{r=0}^{\mu-2}\binom{\mu-1}{r}(\overline{\varsigma - z})^r\psi_{\mu-2-r}(\varsigma, z)$$

$$-(\overline{\varsigma - z})^{\mu-1}h(\varsigma, z)]\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\varsigma$$

$$+\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}(\overline{\varsigma - z})^{\mu-1}\log|\varsigma - z|^2\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\varsigma]. \tag{2.12}$$

It is understood that the first boundary integral is zero of $\mu = 1$.

Now, applying Gauss theorem and using equation (11), the first boundary integral in equation (12) can be written as

$$-\sum_{\mu=1}^{n}(-1)^{\mu}\frac{1}{\pi}\frac{1}{(\mu-1)!}\int_{\mathbb{Q}_R}\left[\sum_{r=0}^{\mu-2}\left[r(\overline{\varsigma - z})^{r-1}\binom{\mu-1}{r}\psi_{\mu-2-r}(\varsigma, z)\right.\right.$$

$$+(\overline{\varsigma - z})^r\binom{\mu-1}{r}\partial_{\bar{\varsigma}}\psi_{\mu-2-r}(\varsigma, z)$$

$$-(\overline{\varsigma - z})^{\mu-1}\partial_{\bar{\varsigma}}h(\varsigma, z) - (\mu-1)(\overline{\varsigma - z})^{\mu-2}h(\varsigma, z)]\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)$$

$$\left.+\left[\sum_{r=0}^{\mu-2}(\overline{\varsigma - z})^r\binom{\mu-1}{r}\psi_{\mu-2-r}(\varsigma, z) - (\overline{\varsigma - z})^{\mu-1}h(\varsigma, z)\right]\partial_{\bar{\varsigma}}^{\mu}\phi(\varsigma)\right]d\xi d\eta$$

$$= -\sum_{\mu=1}^{n}(-1)^{\mu}\frac{1}{\pi}\int_{\mathbb{Q}_R}[\frac{1}{(\mu-1)!}[\sum_{r=0}^{\mu-2}r\binom{\mu-1}{r}(\overline{\varsigma - z})^{r-1}\psi_{\mu-2-r}(\varsigma, z)$$

$$-(\mu-1)(\overline{\varsigma - z})^{\mu-2}h(\varsigma, z)]\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)$$

$$+\frac{1}{(\mu-1)!}\left[\sum_{r=0}^{\mu-2}(\overline{\varsigma-z})^{r}\binom{\mu-1}{r}\psi_{\mu-2-\mu}(\varsigma, z)-(\overline{\varsigma-z})^{\mu-1}h(\varsigma, z)\right]\partial_{\bar{\varsigma}}^{\mu}\phi(\varsigma)$$

$$+\sum_{r=0}^{\mu-2}\frac{(\overline{\varsigma-z})}{(\mu-1)!}\binom{\mu-1}{r}(-1)^{\mu-1-r}(\overline{\varsigma-z})^{\mu-2-r}\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)]d\xi d\eta$$

$$=-\sum_{\mu=1}^{n}\frac{(-1)^{\mu}}{\pi(\mu-1)!}\int_{\mathbb{Q}_R}\sum_{r=0}^{\mu-2}\binom{\mu-1}{r}(-1)^{\mu-1-r}(\overline{\varsigma-z})^{\mu-2}\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\xi d\eta$$

$$=-\sum_{\mu=1}^{n}\frac{(-1)^{\mu-1}}{\pi(\mu-1)!}\int_{\mathbb{Q}_R}(\overline{\varsigma-z})^{\mu-2}\partial_{\bar{\varsigma}}^{\mu-1}\phi(\varsigma)d\xi d\eta. \tag{2.13}$$

Making use of Gauss theorem repeatedly, the expression in (13) can be written equal to

$$-\sum_{\mu=1}^{n-1}\frac{(-1)^{\mu-1}}{\mu}(((1+1/2!)1/2+1/3!)1/3+\ldots+1/\mu!)\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}(\overline{\varsigma-z})^{\mu}\partial_{\bar{\varsigma}}^{\mu}\phi(\varsigma)d\varsigma$$

$$=-\sum_{\mu=1}^{n-1}(-1)^{\mu-1}b\mu\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}(\overline{\varsigma-z})^{\mu}\partial_{\bar{\varsigma}}^{\mu}\phi(\varsigma)d\varsigma,$$

where $b_{\mu} = (((1+1/2!)1/2+1/3!)1/3+\ldots+1/\mu!)1/\mu$.

Therefore, the left hand side of (8) is equal to

$$\sum_{\mu=1}^{n-1}(-1)^{\mu-1}b_{\mu}\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}(\varsigma-z)^{\mu}\overline{\partial_{\bar{\varsigma}}^{\mu}\phi(\varsigma)d\varsigma}$$

$$+\sum_{\mu=1}^{n-1}\frac{(-1)^{\mu+1}}{2\pi i}\frac{1}{\mu!}\int_{\partial\mathbb{Q}_{1R}}(\varsigma-z)^{\mu}\log|\varsigma-z|^{2}\overline{\partial_{\bar{\varsigma}}^{\mu}\phi(\varsigma)}d\bar{\varsigma}. \tag{2.14}$$

Since $f:\mathbb{Q}_{1R}\to\mathbb{C}$ satisfies $|f(x)|\le c|x|^{-\epsilon}$ for $|x|>K$, $\epsilon>1$, so

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\frac{f(\varsigma)}{\varsigma-z}d\varsigma=\frac{1}{2\pi i}\int_{0}^{+\infty}\frac{f(t)}{t-z}dt-\frac{1}{2\pi i}\int_{0}^{+\infty}\frac{f(it)}{t+iz}dt. \tag{2.15}$$

Similarly using (9), we obtain

$$\lim_{R\to\infty}\frac{1}{\pi}\int_{\mathbb{Q}_{1R}}\frac{\phi(\varsigma)}{\varsigma - z}\,d\xi d\eta = \sum_{\mu=1}^{n}\frac{(-1)^{\mu}}{\mu!}\frac{1}{2\pi i}\int_{0}^{+\infty}\frac{(t-\bar{z})^{\mu}}{t-z}\,\partial_{\bar{\varsigma}}^{\mu-1}\phi(t)dt$$

$$-\sum_{\mu=1}^{n}\frac{(-1)^{\mu}}{\mu!}\frac{1}{2\pi i}\int_{0}^{+\infty}\frac{(-it-\bar{z})}{t+iz}\,\partial_{\bar{\varsigma}}^{\mu-1}\phi(it)dt \tag{2.16}$$

and using (12), we have

$$\lim_{R\to\infty}\frac{1}{\pi}\int_{\mathbb{Q}_{1R}}\frac{\overline{\phi(\varsigma)}}{\varsigma - z}\,d\xi d\eta = \sum_{\mu=1}^{n-1}(-1)^{\mu-1}b_{\mu}\frac{1}{2\pi i}\int_{0}^{+\infty}(t-z)^{\mu}\,\overline{\partial_{\bar{\varsigma}}^{\mu-1}\phi(t)}dt$$

$$+\sum_{\mu=1}^{n}\frac{(-1)^{\mu}}{2\pi i}\frac{1}{(\mu-1)!}\int_{0}^{+\infty}(t-z)^{\mu-1}\log|t-z|^{2}\,\overline{\partial_{\bar{\varsigma}}^{\mu-1}\phi(t)}dt$$

$$-\sum_{\mu=1}^{n-1}(-1)^{\mu-1}b_{\mu}\frac{1}{2\pi i}\int_{0}^{+\infty}(it-z)^{\mu}\,\overline{\partial_{\bar{\varsigma}}^{\mu-1}\phi(it)}(-idt)$$

$$-\sum_{\mu=1}^{n}\frac{(-1)^{\mu}}{2\pi i}\frac{1}{(\mu-1)!}\int_{0}^{+\infty}(it-z)^{\mu-1}\log|it-z|^{2}\,\overline{\partial_{\bar{\varsigma}}^{\mu-1}\phi(it)}(-idt)$$

$$=\sum_{\mu=1}^{n-1}(-1)^{\mu-1}b_{\mu}\frac{1}{2\pi i}\int_{0}^{+\infty}\left[(t-z)^{\mu}\,\overline{\partial_{\bar{\varsigma}}^{\mu-1}\phi(t)}+i(it-z)^{\mu}\,\overline{\partial_{\bar{\varsigma}}^{\mu-1}\phi(it)}\right]dt$$

$$+\sum_{\mu=1}^{n}\frac{(-1)^{\mu}}{2\pi i}\frac{1}{(\mu-1)!}\int_{0}^{+\infty}\left[(t-z)^{\mu-1}\log|t-z|^{2}\,\overline{\partial_{\bar{\varsigma}}^{\mu-1}\phi(t)}\right.$$

$$\left.+\,i(it-z)^{\mu-1}\log|it-z|^{2}\,\overline{\partial_{\bar{\varsigma}}^{\mu-1}\phi(it)}\right]dt. \tag{2.17}$$

Substituting (15), (16), (17) in (6), we obtain the representation (5). □

3. Conclusions

Using this representation of Cauchy Integral formula, we can find the solutions of Schwarz, Dirichlet, Riemann and Neumann boundary value problems for bi-polyanalytic functions on the Quarter plane $\mathbb{Q}$ as done in case of regular domains [5, 9, 10, 14]. Boundary value problems with Schwarz, Dirichlet, Riemann and Neumann boundary conditions have waste applications in physical world and are extensively used and solved in different planes [11, 12, 13, 15].

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