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DOMINATION UNIFORM SUBDIVISION NUMBER OF G^{---}

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Abstract

Let $G = (V, E)$ be a simple undirected graph. A subset D of $V(G)$ is said to be dominating set if every vertex of $V(G) - D$ is adjacent to at least one vertex in D. The minimum cardinality taken over all minimal dominating sets of G is the domination number of G and is denoted by $\gamma(G)$. The domination uniform subdivision number $usd_{\gamma}(G)$ is the least positive integer k such that the subdivision of any k edges from G results in a graph having domination number greater than that of G. In this paper, we characterize sd_{γ}-critical graphs on G^{---}. Also we determine bounds of $usd_{\gamma}(G^{---})$ according to the $diam(G)$.

1. Introduction

Let $G = (V, E)$ be a simple undirected graph of order n and size m. If $v \in V(G)$, then the neighborhood of v is the set $N_G(v)$ (or $N(v)$) consisting of all vertices u which are adjacent to v. The closed neighborhood is

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$NN_G[v] = N_G(v) \cup \{v\}$. The degree of v in G is $|N(v)|$ and is denoted by $\deg(v)$. The minimum degree of G is $\min\{\deg_G(v) : v \in V(G)\}$ and is denoted by $\delta(G)$. A vertex v is said to be pendant vertex if $\deg(v) = 1$. A path, a cycle and a complete graph on n vertices are denoted by P_n, C_n and K_n respectively. A complete bipartite graph is denoted by $K_{m,n}$. A graph is said to be connected if there exists a path between any pair of vertices. Otherwise it is said to be disconnected. The distance $d(u, v)$ between two vertices u and v of a connected graph G is defined to be the length of any shortest path joining u and v. A shortest $u - v$ path is often called as geodesic. The diameter of a connected graph G is the length of any longest geodesic and is denoted by $diam(G)$.

A subset D of $V(G)$ is said to be dominating set if every vertex of $V(G) - D$ is adjacent to at least one vertex in D. The minimum cardinality taken over all minimal dominating sets of G is the domination number of G and is denoted by $\gamma(G)$.

The domination subdivision number introduced by Arumugam, Velammal in [13]. Its bound was obtained in [1] and several authors characterized trees according to their domination subdivision number. Also many results have also been obtained on the parameters sd_{dd}, $sd_{\gamma c}$ and $sd_{\gamma t}$. An edge $e = uv$ is said to be subdivided if it is deleted and replaced by a $u - v$ path of length two with a new internal vertex w (subdividing vertex). $G \wedge \{e\}$ is the graph obtained by subdividing the edge e. The domination subdivision number of a graph G is the minimum number of edges whose subdivision increases the domination number. It can also be defined as $sd_\gamma(G) = \min\{|E'| : \gamma(G \wedge E') > \gamma(G)\}$.

A domination uniform subdivision number of G is the least positive integer k such that the sub division of any k edges from G results in a graph having domination number greater than that of G and is denoted by $usd_\gamma(G)$. If it does not exist, then $usd_\gamma(G) = 0$. This number was introduced and studied in [3].

A subset $S \subseteq E(G)$ is said to be stable subdivision set if

$\gamma(G \wedge S) = \gamma(G)$. A stable subdivision set S is said to be maximum stable subdivision set if there is no stable subdivision set S' such that $|S'| > |S|$. $usd_{\gamma}(G) = |S| + 1$, where S is a maximum stable subdivision set of G. In [4] we have studied domination uniform subdivision number of $G \circ K_1$ for some standard graphs.

Wu and Meng [5] generalized the concept of total graphs to a total transformation graph G^{xyz} with $x, y, z \in \{+, -\}$ where G^{+++} is precisely the total graph of G, and G^{---} is the complement of G^{+++}. Each of these eight kinds of transformation graph G^{xyz} appears to have some nice properties; for instance, their diameters are small in most cases [5], and their edge A connectives are equal to their minimum degree etc. [8, 14]. Several authors discussed various concepts on transformation graphs [2, 9, 10, 11, 14].

The transformation graph G^{---} of G is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows: (a) two elements in $V(G)$ are adjacent if and only if they are non-adjacent in $G(b)$ two elements in $E(G)$ are adjacent if and only if they are non-adjacent in G and (c) an element of $V(G)$ and an element of $E(G)$ are adjacent if and only if they are non-incident in G. The domination subdivision number of the transformation graph G^{-+-} was studied in [2]. In [11], the domination subdivision number of G^{---} has been investigated. In [14], Wiener Index of transformation graph G^{---} has been determined. In this paper we study the domination uniform subdivision number of G^{---}.

Terms not defined here are used in the sense of [6].

2. Main Results

In this section, we characterize sd_{γ}-critical graphs on G^{---}. We determine the exact value of usd_{γ} for a graph with diameter one and 2. Also we obtain the upper bound of usd_{γ} for diameter greater than or equal to 2.

Lemma 2.1. *For any graph G,* $usd_{\gamma}(G) \geq 1$ *iff* $N_G(u) \cap N_G(v) \neq \emptyset$ *for some pair of vertices in any minimum dominating set of G.*

Theorem 2.2. *For* $n \geq 7$, $usd_{\gamma}(K_n^{---}) = (n-2)(n-3)+1$.

Proof of Theorem 2.2. We have $\gamma(K_n^{---}) = 3$. let u and v be vertices of K_n. Let e_1, e_2 and e_3 be mutually independent edges in K_n. Then any γ-set of K_n^{---} is of the form $\{e_1, e_2, e_3\}$, $\{u, v, uv\}$ or $\{e_1, e_2, x\}$ where x is incident with neither e_1 nor e_1. Since degree of subdividing vertex v of G^{---} is two none of the minimum dominating sets of a derived graph G^* obtained by subdividing one or more edges of G^{---} containing v. Then minimum dominating set S^* of G^* must contain any one of the minimum dominating set of G^{---}.

Now we consider the dominating set $\{e_1, e_2, e_3\}$. Let $e_1 = u_1u_2$, $e_2 = u_3u_4$ and $e_3 = u_5u_6$. Let S_1 be an edge set in G^{---} consists of edges joining e_1 to all the edges in $\langle V(G) - \{u_1, u_2, u_3, \ldots, u_6\}\rangle$ and edges joining e_2 to all the edges in $\langle V(G) - \{u_1, u_2, u_3, \ldots, u_6\}\rangle$. Then $|S_1| = (n-6)(n-7)$. Let $S_2 = \{e_1u_3, e_1u_4, e_2u_5, e_2u_6, e_3u_1, e_3u_2, e_1e_2, e_1e_3, e_2e_3\}$. Then $|S_2| = 9$.

Let $S_3 = \{e_1u_7, \ldots, e_1u_n, e_2u_7, \ldots, e_2u_n\}$. Then $|S_3| = 2(n-6)$.

Let S_4 be set of edges in G^{---} consists of edges joining e_1 to all the adjacent edges of e_2 which are incident with a vertex of $V(G) - \{u_1, u_2, u_3, \ldots, u_6\}$ in G. Then $|S_4| = 2(n-6)$. Let S_5 be set of edges in G^{---} consists of edges joining e_2 to all the adjacent edges of e_3 which are incident with a vertex of $V(G) - \{u_1, u_2, u_3, \ldots, u_6\}$ in G. Then $|S_5| = 2(n-6)$. Let S_6 be set of edges in G^{---} consists of edges joining e_3 to all the adjacent edges of e_1 which are incident with a vertex of $V(G) - \{u_1, u_2, u_3, \ldots, u_6\}$ in G. Then $|S_6| = 2(n-6)$. Take $S' = S_1 \cup S_2 \cup \ldots \cup S_6$. Then S is a maximal domination subdivision stable set of G^{---} and

$$|S'| = (n-6)(n-7) + 9 + 2(n-6) + 6(n-6)$$

$$= (n-6)[n-7+2+6] + 9$$

$$= (n-6)(n+1) + 9$$

Now we consider the dominating set $\{u, v, uv\}$. Let S_7 be edge set of G^{---} consists of edges joining u to all the edges in $\langle V(G) - \{u, v\}\rangle$ and edges joining v to all the edges in $\langle V(G) - \{u, v\}\rangle$ in G. Therefore $|S_7| = (n-2)(n-3)$. We can easily verify that S_7 is a maximal subdivision domination subdivision stable set.

Now we consider dominating set $\{e_1, e_2, x\}$ where x is incident with neither e_1 nor e_2. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$. Let S_8 be edge set of G^{---} consists of edges joining e_1 to all the edges in $\langle V(G) - \{u_1, v_1, u_2, v_2, x\}\rangle$ and edges joining e_2 to all the edges of $\langle V(G) - \{u_1, v_1, u_2, v_2, x\}\rangle$ in G. Therefore $|S_8| = (n-5)(n-6)$. Let S_8 be set of edges of G^{---} consists of edges joining e_1 to all the vertices in $\langle V(G) - \{u_1, v_1, u_2, v_2, x\}\rangle$ and edges joining e_2 to all the edges of $\langle V(G) - \{u_1, v_1, u_2, v_2, x\}\rangle$ in G. Therefore $|S_9| = (n-5)$. Let S_9 be set of edges of G^{---} consists of edges joining e_1 to all the vertices in $\langle V(G) - \{u_1, v_1, u_2, v_2, x\}\rangle$ and edges joining e_2 to all the edges of $\langle V(G) - \{u_1, v_1, u_2, v_2, x\}\rangle$ in G. Therefore $|S_9| = n-5$. Let S_{10} be set of edges of G^{---} consists of edges joining e_1 to all the adjacent edges of e_2 whose end vertices in $\langle V(G) - \{u_1, v_1, u_2, v_2, x\}\rangle$, edges joining e_2 to all the adjacent edges of e_1 whose end vertices in $\langle V(G) - \{u_1, v_1, u_2, v_2, x\}\rangle$ and edges joining e_1 to all the incident edges of x whose end vertices in $\langle V(G) - \{u_1, v_1, u_2, v_2, x\}\rangle$. Therefore $|S_{10}| = 5(n-5)$. Let S_{11} be set of edges of G^{---} consists of edges joining e_1 to e_2, e_1 to x and e_2 to x. Then $|S_{11}| = 3$. Let $S'' = S_8 \cup S_9 \cup S_{10} \cup S_{11}$. We can easily verify that S'' is a maximal domination subdivision stable set.

Then $| S'' | = (n-5)(n-6)+(n-5)+5(n-5)+3$

$= (n-5)[n-6+1+5]+3$

$= n(n-5)+3$

Now S_7 is a maximum domination subdivision stable set of G^{---}. Hence $usd_\gamma(G^{---}) = (n-2)(n-3)+1$.

Observations 2.3.

(1) $usd_\gamma(P_n^{---}) = 2n-5$ for all $n \geq 4$

(2) $usd_\gamma(C_n^{---}) = 2n-7$ for all $n \geq 5$

(3) $usd_\gamma(K_{r,s}^{---}) = 2$ for all $r, s \geq 3$

(4) $usd_\gamma(K_{1,r}^{---}) = r+1$ for all $n \geq 3$

Theorem 2.4. *For any graph G, G^{---} is sd_γ-critical iff G has an isolated vertex.*

Proof of Theorem 2.4. Assume that G has an isolated vertex v. Then $\{v\}$ is a dominating set of G^{---} and so $\gamma(G^{---}) = 1$. Therefore subdivision of any edge of G^{---} increases domination number. Hence $usd_\gamma(G^{---}) = 1$. Thus G is sd_γ-critical.

Assume that G^{---} is sd_γ-critical. Then $usd_\gamma(G^{---}) = 1$. Suppose G has no isolated vertex.

Case (i). G is disconnected

Then G has at least two components G_1 and G_2. Then $\{u_1, u_2\}$, where $u_1 \in V(G_1) \cup E(G_1)$ and $u_2 \in V(G_2) \cup E(G_2)$ is minimum dominating set of G^{---} and so $\gamma(G^{---}) = 2$. Also there is an edge between u_1 and u_2 in G^{---}. Further, $\gamma(G^{---} \wedge u_1u_2) = 2$. Hence $usd_\gamma(G^{---}) > 1$.

Case (ii). G is connected

Subcase (i). $diam(G) = 1$.

Then $G = K_n$, $n \geq 3$. For $n < 7$, we can easily verify that $usd_{\gamma}(G^{---}) > 1$. By theorem 2.2, for $n \geq 7$, $usd_{\gamma}(G^{---}) = (n-2)(n-3) +1 > 1$.

Subcase (ii). $diam(G) = 2$.

Then $\gamma(G^{---}) = 3$. Also $N_{G^{---}}(u) \cap N_{G^{---}}(v) \neq \emptyset$ for any pair of vertices (u, v) of γ-set of G^{---}. Therefore $usd_{\gamma}(G^{---}) \geq 1$.

Subcase (iii). $diam(G) \geq 3$.

Then $\gamma(G^{---}) \geq 2$. Let $S = \{u, v\}$ be a minimum dominating set of G^{---}. Then u and v must be adjacent in G^{---}. Therefore $N_{G^{---}}(u) \cap N_{G^{---}}(v) \neq \emptyset$. Hence $usd_{\gamma}(G^{---}) \geq 1$.

In both the cases we get a contradiction. Hence G has an isolated vertex.

Theorem 2.5. *Let G be a connected graph. If* $diam(G) \geq 3$, *then* $usd_{\gamma}(G^{---}) \leq n + m - 4\delta(G) + 2$.

Proof of Theorem 2.5. Since $diam(G) \geq 3$, $\gamma(G^{---}) \geq 2$. Then there exists $x, v \in V(G)$ such that $d(u, v) \geq 3$. Therefore $S = \{u, v\}$ is minimum dominating set in G^{---}. In G^{---}, all the vertices in $N_G(u) \cup N_G(v)$ are adjacent to only one element of S. Similarly, all the edges incident with u or v in G are adjacent to only one element of S. The remaining vertices and edges of G are adjacent to both u and v in G^{---}. Therefore subdivision of $n + m - (2\deg(u) + 2\deg(v)) - 1$ edges in G^{---} does not increase the domination number. Hence maximum subdivision domination stable set of G^{---} contains at least $n + m - (2\deg(u) + 2\deg(v)) - 1$ edges.

Thus $usd_{\gamma}(G^{---}) \leq n + m - (2\deg(u) + 2\deg(v)) - 1 + 1$

$$\leq n + m - 4\delta(G).$$

Corollary 2.6. *If* $diam(G) \geq 3,$ *and G has at least two pendent vertices, then* $usd_{\gamma}(G^{---}) \leq n + m - 4.$

Theorem 2.7. *If* $diam(G) \geq 2,$ *then* $usd_{\gamma}(G^{---}) \leq (n-2)^2 + 1$ *for all n.*

Proof of Theorem 2.7. Since $diam(G) = 2,$ there exists $u, w \in V(G)$ such that $d(u, w) = 2.$

Let us take u, x, v, y, w be path of length two where $x = uv$ and $y = vw$ are edges of G. Then $S = \{u, x, v\}$ is a minimum dominating set of G^{---}. Since $diam(G) = 2, V(G) - \{u, v, w\}$ are adjacent to v or and adjacent to both v and w.

Case (i). There exists a vertex of degree 2.

Without loss of generality we assume that $\deg(v) = 2.$ All the vertices in $V(G) - \{u, v\}$ are adjacent to two elements of S in G^{---}. All the incident edges of w except y are adjacent to all the three elements of S in G^{---}. All the edges in $\langle V(G) - \{u, v, w\}\rangle$ are adjacent to all the three elements of S in G^{---} and $\langle V(G) - \{u, v, w\}\rangle$ has at most $\frac{(n-3)(n-4)}{2}$ edges. Therefore subdivision of at most $(n-2) + 2(n-3) + (n-3)(n-4) = (n-2)^2$ edges does not increase the domination number. Also maximum subdivision domination stable set has greater than or equal to $(n-2)^2$ edges of G^{---}. Hence $usd_{\gamma}(G^{---}) \leq (n-2)^2 + 1.$

Case (ii). There is no vertex of degree 2.

Then we just remove all the edges joining v to $\langle V(G) - \{u, v, w\}\rangle$ from the domination subdivision stable set. Therefore maximum domination subdivision stable set has greater than or equal to $(n-2)^2$ edges of G^{---}. Hence $usd_{\gamma}(G^{---}) \leq (n-2)^2 + 1.$

Theorem 2.8. *For any disconnected graph G with two components* G_1 *and* G_2, $usd_\gamma(G^{---}) = n + m - 2[\delta(G_1) + \delta(G_2)] + 1.$

Proof of Theorem 2.8. Since G is disconnected, $\gamma(G^{---}) = 2$ and minimum dominating set of G^{---} is $\{u, v\}$ where $u \in V(G_1) \cup E(G_1)$ and $v \in V(G_2) \cup E(G_2)$. Let $x \in V(G_1)$ with $\deg_G(x) = \delta(G_1)$. Then $\deg_{G^{---}}(x)$ is greater than or equal to $\deg_{G^{---}}(y)$, $y \in V(G_1) \cup E(G_1)$. Let $z \in V(G_2)$ with $\deg_G(z) = \delta(G_1)$. Then $\deg_{G^{---}}(z)$ is greater than or equal to $\deg_{G^{---}}(y)$, where $y \in V(G_2) \cup E(G_2)$. Then $\{x, z\}$ is a minimum dominating set of G^{---}. Let X be a set of incident edges x in G. Since $S_1 = (V(G_1) \cup E(G_1)) \setminus (N_{G^{---}}(x) \cup X)$ is adjacent to both x and z, subdivision of the set of edges joining x to all the elements in S_1 does not increase the domination number. Let Z be set of incident edges of z in G. Since $S_2 = (V(G_2) \cup E(G_2)) \setminus (N_{G^{---}}(z) \cup Z)$ is adjacent to both x and z, subdivision of the set of edges joining z to all the elements in S_2 does not increase the domination number.

Since for any $y \in V(G_1) \cup E(G_1)$, $\deg_{G^{---}}(y) \leq \deg(x)$ and for any $w \in V(G_2) \cup E(G_2)$, $\deg_{G^{---}}(w) \leq \deg(z)$. $S_1 \cup S_2$ is a maximum domination subdivision stable set of G^{---}. Hence $usd_\gamma(G^{---})$ $= | S_1 | + | S_1 | + 1$

$$= n_1 + m_1 - 2\delta(G_1) + n_2 + m_2 - 2\delta(G_2) + 1$$

$$= n + m - 2[\delta(G_1) + \delta(G_2)] + 1.$$

Corollary 2.9. *Let G be a graph with* $G_1, G_2, \ldots, G_n$ *components. Then*

$$usd_\gamma(G^{---}) = n + m - 2[\delta(G_1) + \delta(G_2) + \ldots + \delta(G_n)] + 1.$$

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