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ON FUZZY IMPLICATIVE IDEALS IN Z-ALGEBRAS

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Abstract

This paper, introduces the notions of fuzzy implicative ideals and fuzzy sub-implicative ideals in Z-algebras. Further, the relationship between fuzzy Z-ideal, fuzzy implicative ideal and fuzzy sub-implicative ideal are obtained.

1. Introduction

In 1966, Imai and Iseki [2, 3] introduced two new classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 2017, Chandramouleeswaran et al. [1] introduced the concept of Z-algebras as a new structure of algebra based on propositional calculus. In order to deal with the problem of uncertainty in the real physical world, in 1965 Zadeh [13] introduced the notion of fuzzy sets. In 1991, Xi [12] applied the concept of fuzzy sets to BCK-algebras. In 1992, Meng and Xin [5] introduced the concept of implicative ideals in BCI-algebras. In 1999, Jun et al. [4] studied the

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concept of fuzzy implicative ideals and constructed a fuzzy characteristic implicative ideal in BCK-algebras. In our earlier papers [6-11], we introduced the notions of fuzzy Z-Subalgebras and fuzzy Z-ideals in Z-algebras. In this paper, we introduce the concept of fuzzy implicative ideals in Z-algebras and prove some interesting results.

2. Preliminaries

Definition 2.1 [1]**.** A Z-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

(Z1) $x * 0 = 0$

(Z2) $0 * x = x$

(Z3) $x * x = x$

(Z4) $x * y = y * x$ when $x \neq 0$ and $y \neq 0 \, \forall \, x, y \in X$.

Definition 2.2 [1]**.** Let $(X, *, 0)$ be a Z-algebra and I be a subset of X. Then, I is called an Z-ideal of X, if it satisfies the following conditions: For all x, y in X,

(i) $0 \in I$

(ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$.

Definition 2.3 [1]**.** Let $(X, *, 0)$ and $(Y, *', 0')$ be two Z-algebras. A mapping $h : (X, *, 0) \to (Y, *', 0')$ is said to be a *Z-homomorphism* of Z-algebras if $h(x * y) = h(x) *' h(y)$ for all $x, y \in X$.

Definition 2.4 [13]**.** A fuzzy set A in X is characterized by a membership function: $\mu_A : X \to [0, 1]$.

Definition 2.5 [7]**.** Let $(X, *, 0)$ be a Z-algebra. A fuzzy set A in X with membership function μ_A is said to be a fuzzy Z-ideal of a Z-algebra X if it satisfies the following conditions: For all x, y in X,

(i) $\mu_A(0) \geq \mu_A(x)$

(ii) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$.

Definition 2.6 [4]**.** A fuzzy set μ in a BCK-algebra X is called a fuzzy implicative ideal of X if:

(i) $\mu(0) \geq \mu(x)$ and

(ii) $\mu_A(x) \geq \min\{\mu((x * (y * x) * z), \mu(z)\}$ for all $x, y, z \in X$.

3. Fuzzy Implicative Ideals in Z-Algebras

In this section, we introduce the notions of fuzzy implicative ideals and fuzzy sub-implicative ideals in Z-algebras. Also, the relationship between fuzzy implicative ideal, fuzzy sub-implicative ideal and fuzzy Z-ideal of a Z-algebra are discussed.

Definition 3.1. A Z-algebra $(X, *, 0)$ is said to be an implicative if it satisfies the condition $(x * (x * y)) * (y * x) = y * (y * x)$, for all $x, y \in X$.

Example 3.2. Consider the Z-algebra $X = \{0, 1, 2, 3\}$ with the binary operation * defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	0	1	2	2
2	0	2	2	1
3	0	2	1	3

Then, $(X, *, 0)$ is an implicative Z-algebra.

Example 3.3. Consider the Z-algebra $X = \{0, 1, 2, 3\}$ with the operation $*$ defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	0	1	3	3
2	0	3	2	1
3	0	3	1	3

Then, $(X, *, 0)$ is a Z-algebra. But, it is not an implicative Z-algebra since $(1 * (1 * 2)) * (2 * 1) = 3 \neq 1 = 2 * (2 * 1)$.

Definition 3.4. A Z-algebra $(X, *, 0)$ is called medial if $x * (x * y) = y$, for all $x, y \in X$.

Example 3.4. Consider the Z-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

Then, $(X, *, 0)$ is an implicative Z-algebra.

Definition 3.6. A Fuzzy set A of a Z-algebra $(X, *, 0)$ with membership function μ_A is called a fuzzy implicative ideal of X if it satisfies the following condition:

(i) $\mu_A(0) \geq \mu_A(x)$

(ii) $\mu_A(x) \geq \min\{\mu_A(x * (y * x) * z), \mu_A(z)\}$, for all $x, y, z \in X$.

Example 3.7. Consider the Z-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	0	1	1	2
2	0	3	2	3
3	0	2	3	3

Define a fuzzy set A of X with membership function μ_A is given by $\mu_A(x) = 0.5$ for all $x \in X$. Then A is a fuzzy implicative ideal of X.

Using Definition 3.1 and Definition 2.5, the following Proposition can be proved.

Proposition 3.8. *In a Z-algebra X, every fuzzy implicative ideal is a fuzzy Z-ideal.*

Theorem 3.9. *A be a fuzzy Z-ideal of a Z-algebra X. Then A is a fuzzy implicative ideal of X if and only if A satisfies the following inequality: for all* $x, y \in X$, $\mu_A(x) \geq \mu_A(x * (y * x))$.

Proposition 3.10. *Let* $\{A_i \mid i \in \Omega\}$ *be a family of fuzzy implicative ideals of a Z-algebra X. Then* $\bigcap_{i\in\Omega} A_i$ *is a fuzzy implicative ideal of X.*

Proof. Let $x \in X$. *Then* $\mu_{\underset{i\in\Omega}{\cup} A_i}(0) = \sup_{i\in\Omega}(\mu_{A_i}(0)) \geq \sup_{i\in\Omega}(\mu_{A_i}(x))$ $= \mu_{\underset{i\in\Omega}{\cup} A_i}(x)$

Let $x, y, z \in X$. Then we have

$$\mu_{\underset{i\in\Omega}{\cup} A_i}(x) = \sup_{i\in\Omega}(\mu_{A_i}(x)) \geq \sup_{i\in\Omega}\{\min\{\mu_{A_i}(x * (y * x) * z), \mu_{A_i}(z)\}\}$$

$$= \min\{\sup_{i\in\Omega} \mu_{A_i}(x * (y * x) * z), \sup_{i\in\Omega} \mu_{A_i}(z)\}$$

$$= \min\{\mu_{\underset{i\in\Omega}{\cup}} A_i(x * (y * x) * z), \mu_{\underset{i\in\Omega}{\cup}} A_i(z)\}$$

Hence $\bigcap_{i\in\Omega} A_i$ is a fuzzy implicative ideal of a *Z*-algebra *X*.

Theorem 3.11. *A fuzzy set A of a Z-algebra* $(X, *, 0)$ *is a fuzzy implicative ideal if and only if for any* $t \in [0, 1]$, $U(\mu_A; t) = \{x \in X \mid \mu_A(x) \geq t\}$ *is an implicative ideal of X where* $U(\mu_A; t) \neq \phi$.

Note. Hereafter, $U(\mu_A; t)$ is called level implicative ideal of a *Z*-algebra *X*.

Theorem 3.12. *Let A be a fuzzy implicative ideal of a Z-algebra X then two level implicative ideals* $U(\mu_A; t_1)$ *and* $U(\mu_A; t_2)$ *(with* $t_1 < t_2$*) of A are equal if and only if there is no* $x \in X$ *such that* $t_1 \leq \mu_A(x) < t_2$.

Proof. Let A be a fuzzy implicative ideal of a Z-algebra X. Assume that $U(\mu_A; t_1) = U(\mu_A; t_2)$ for some $t_1 < t_2$ and there exists $x \in X$ such that $t_1 < \mu_A(x) < t_2$. Then $U(\mu_A; t_2)$ is a proper subset of $U(\mu_A; t_1)$ which is a contradiction. Hence there is no $x \in X$ such that $t_1 < \mu_A(x) < t_2$.

Conversely, suppose that there is no $x \in X$ such that $t_1 < \mu_A(x) < t_2$. Since $t_1 < t_2$, we get $U(\mu_A; t_2) \subseteq U(\mu_A; t_1)$. (1)

If $x \in U(\mu_A; t_1)$ then $\mu_A(x) \geq t_1$ and so $\mu_A(x) \geq t_2$, because $\mu_A(x)$ does not lie between t and s. Hence $x \in U(\mu_A; t_2)$. Hence $U(\mu_A; t_1) \subseteq U(\mu_A; t_2)$. (2)

From (1) and (2) we get $U(\mu_A; t_1) = U(\mu_A; t_2)$.

Remark 3.13. As a consequence of Theorem 3.9, the level implicative ideals of a fuzzy implicative ideal A of a finite Z-algebra X form a chain $U(\mu_A; t_0) \subset U(\mu_A; t_1) \subset \ldots \subset U(\mu_A; t_t) = X$, where $t_0 > t_1 > t_2 > \ldots > t_r$.

Theorem 3.14. *Let X be a finite Z-algebra and A be a fuzzy implicative ideal of X. If* $\text{Im}(A) = \{t_1, \ldots t_n\}$, *then the family of implicative ideals* $U(\mu_A; t_i)$, $i = 1, 2, \ldots, n$, *constitutes all the level implicative ideals of A.*

Proof. Let $t \in [0, 1]$ and $t \notin \text{Im}(A)$. Suppose $t_1 < t_2 < \ldots < t_n$ without loss of generality. If $t \leq t_1$, then $U(\mu_A; t_1) \subseteq U(\mu_A; t)$. Since $U(\mu_A; t_1) = X$, $U(\mu_A; t) = X$ and $U(\mu_A; t_1) = U(\mu_A; t)$.

If $t > t_n$, then $U(\mu_A; t) = \phi$ obviously.

If $t_1 < t < t_{i+1} (1 \leq i \leq n-1)$, then there is no $x \in X$ such that $1 \leq \mu_A(x) < t_{i+1}$. It follows from Theorem 3.12 that $U(\mu_A; t) \subseteq U(\mu_A; t_{i+1})$. This shows that for any $t \in [0, 1]$, the level implicative Z-ideal $U(\mu_A; t)$ is in $\{U(\mu_A; t) | i = 1, 2, \ldots, n\}$.

Lemma 3.15. *Let X be a Z-algebra and A be a fuzzy implicative ideal of X. If* $\text{Im}(A)$ *is finite, say* $\{t_1, t_2, \ldots, t_n\}$ *then for any* $t_i, t_j \in \text{Im}(A)$, $U(\mu_A; t_i) = U(\mu_A; t_j)$ implies $t_i = t_j$.

Theorem 3.16. *Let A and B be two fuzzy implicative ideals of a Z-algebra*

X with identical family of level implicative ideals. If $\mathrm{Im}(A) = \{t_1, t_2, \ldots, t_m\}$ *and* $\mathrm{Im}(B) = \{q_1, q_2, \ldots, q_m\}$, *where* $t_1 > t_2 > \ldots > t_m$ *and* $q_1 > q_2 > \ldots > q_,$ *then*

(i) $m = n.$

(ii) $U(\mu_A; t_i) = U(\mu_B; q_i)$ $i = 1, \ldots, m.$

(iii) *If* $x \in X$ *such that* $\mu_A(t) = t_i$ *then* $\mu_B(x) = q_i$, $i = 1, \ldots, m.$

Corollary 3.17. *Let A and B be two fuzzy implicative ideals of a Z-algebra X with identical family of level implicative ideals. Then* $\mathrm{Im}(A) = \mathrm{Im}(B)$ *if and only if* $A = B.$

Theorem 3.18. *Let A be a fuzzy set in a Z-algebra X with* $\mathrm{Im}(A) = \{t_0, t_1, \ldots, t_k\}$ *where* $t_0 > t_1 > t_2 > \ldots > t_k$. *If there exists a chain of implicative ideals of* $X : I_0 \subset I_1 \subset \ldots \subset I_k = X$ *such that* $\mu_A(I_n^*) = t_n$ *where* $I_n^* = I_n - I_{n-1}$, $I_{-1} = \phi$, $n = 0, 1, \ldots, k$, *then A is a fuzzy implicative ideal of X.*

Proposition 3.19. *Let h be a Z-homomorphism from a Z-algebra* $(X, *, 0)$ *onto a Z-algebra* $(Y, *', 0')$ *and A be a fuzzy implicative ideal of X with the supremum property. Then the image of A denoted by* $h(A)$ *is a fuzzy implicative ideal of Y.*

Proposition 3.20. *Let* $h : (X, *, 0) \to (Y, *', 0')$ *be a Z-homomorphism of Z-algebras. If B is a fuzzy implicative ideal of Y, then* $h^{-1}(B)$ *defined by* $\mu_{h^{-1}(B)}(x) = \mu_B(h(x))$ *is a fuzzy implicative ideal of X.*

Proof. For any $x \in X$, we have

$$\mu_{h^{-1}(B)}(x) = \mu_B(h(x)) \leq \mu_B(0') = \mu_B(h(0)) = \mu_{h^{-1}(B)}(0).$$

Let $x, y \in X$. Then

$$\min\{\mu_{h^{-1}(B)}((x * (y * x)) * z), \mu_{h^{-1}(B)}(z)\}$$

$$= \min\{\mu_B(h((x * (y * x)) * z)), \mu_B(h(z))\}$$

$$= \min\{\mu_B((h(x)) *' (h(y) *' h(x))) *' h(z), \mu_B(h(z))\}$$

$$\leq \mu_B(h(x)) = \mu_{h^{-1}(B)}(x)$$

Hence $h^{-1}(B)$ is a fuzzy implicative Z-ideal of a Z-algebra X.

Corollary 3.21. *Let X be a Z-algebra. Then A is a fuzzy implicative ideal of X if and only if the set X_A is an implicative ideal of X, where* $X_A = \{x \in X \mid \mu_A(x) = \mu_A(0)\}$.

Definition 3.22. A fuzzy set A of a Z-algebra $(X, *, 0)$ with membership function μ_A is said to be a fuzzy sub-implicative ideal of X if it satisfies the following conditions:

(i) $\mu_A(0) \geq \mu_A(x)$

(ii) $\mu_A(y * (y * x)) \geq \min\{\mu_A(((x * (x * y)) * (y * x)) * z), \mu_A(z)\}$, *for all* $x, y, z \in X$.

Example 3.23. Consider the Z-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	0	1	1	2
2	0	1	2	2
3	0	2	2	3

Define a fuzzy set A of X with membership function μ_A is given by $\mu_A(x) = 0.6$ for all $x \in X$. Then A is a fuzzy sub-implicative ideal of X.

Proposition 3.24. *Let X be a Z-algebra. Then every fuzzy sub-implicative ideal of X is a fuzzy Z-ideal of X.*

Proof. Put $x = y$ in Definition 3.22 and using Definition 2.5.

Theorem 3.25. *Let X be an implicative Z-algebra. Then every fuzzy Z-ideal of X is a fuzzy sub-implicative ideal of X.*

Proof. Let A be a fuzzy Z-ideal of X. Let $x, y, z \in X$. Then, $\mu_A(0) \geq \mu_A(x)$ (1)

and $\mu_A(y * (y * x)) \geq \min\{\mu_A((y * (y * x)) * z), \mu_A(z)\}$. Since X is an implicative, $\mu_A(y * (y * x)) \geq \min\{\mu_A((x * (x * y) * (y * x)) * z), \mu_A(z)\}$. (2)

From (1) and (2), A is a fuzzy sub-implicative ideal of X.

Corollary 3.26. *In a medial Z-algebra X, every fuzzy Z-ideal of X is a fuzzy sub-implicative ideal of X.*

Proof. Let A be a fuzzy Z-ideal of a medial Z-algebra X. Let $x, y, z \in X$.

Then, $\mu_A(0) \geq \mu_A(x)$ (1)

and
$$\mu_A(y * (y * x)) = \mu_A(x) \geq \min\{\mu_A(x * z), \mu_A(z)\}$$
$$= \min\{\mu_A((y * (y * x)) * z), \mu_A(z)\}$$
$$= \min\{\mu_A((x * (x * y) * (y * x)) * z), \mu_A(z)\}.$$

Hence A is a fuzzy sub-implicative ideal of X.

Theorem 3.27. *If X is a Z-algebra satisfies the condition: for all $x, y, z \in X$; $\mu_A(y * z) \geq \mu_A((x * (x * y)) * z)$ then every fuzzy Z-ideal of X is a fuzzy subimplicative ideal of X.*

Proof. Let X be a Z-algebra satisfies the condition:

$\mu_A(y * z) \geq \mu_A((x * (x * y)) * z)$ for all $x, y, z \in X$. (1)

Let A be a fuzzy Z-ideal of X. For $x, y, z \in X$,

$$\mu_A((x * (x * y) * (y * x))) \geq \min\{\mu_A((x * (x * y) * (y * x)) * z), \mu_A(z)\}$$

Put $z = y * x$ in (1), $\mu_A(y * (y * x)) \geq \mu_A((x * (x * y) * (y * x)))$

$$\Rightarrow \mu_A(y * (y * x)) \geq \min\{\mu_A((x * (x * y) * (y * x)) * z), \mu_A(z)\}.$$

Therefore, A is a fuzzy sub-implicative ideal of X.

Theorem 3.28. *Let X be a medial Z-algebra satisfies the condition: for all $x, y \in X$, $\mu_A((x * (x * y)) * (y * x)) \geq \mu_A(x * (y * x))$.*

Then every fuzzy sub-implicative ideal of X is a fuzzy implicative ideal of X.

Proof. Let A be a fuzzy sub-implicative ideal of X. Then A is a fuzzy Z-ideal of X.

$\mu_A(0) \geq \mu_A(x)$, for all $x \in X$ (1)

For $x, y, z \in X$,

$$\mu_A(x) = \mu_A(y * (y * x)) \geq \mu_A((x * (x * y) * (y * x))) \geq \mu_A(x * (y * x))$$

$$\geq \min\{\mu_A((x * (y * x)) * z), \mu_A(z)\}$$

Put $z = x * (y * x)$,

$$\mu_A(x) \geq \min\{\mu_A((x * (y * x)) * (x * (y * x))), \mu_A(x * (y * x))\}$$

$$\geq \min\{\mu_A((x * (y * x)), \mu_A(x * (y * x)))\} = \mu_A(x * (y * x)).$$

Therefore, by Theorem 3.9, A is a fuzzy implicative ideal of X.

Theorem 3.29. *Let X be an implicative Z-algebra. Then every fuzzy implicative ideal of X is a fuzzy sub-implicative ideal of X.*

Proof. A is a fuzzy implicative ideal of X implies A is a fuzzy Z-ideal of X.

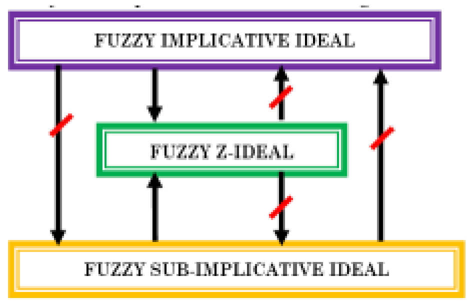
Then, $\mu_A(0) \geq \mu_A(x)$, for all $x \in X$ (1)

For $x, y, z \in X$, $\mu_A((x * (x * y) * (y * x)))$

$$\geq \min\{\mu_A((x * (x * y) * (y * x)) * z), \mu_A(z)\}$$

Since X is implicative, $\mu_A((x * (x * y) * (y * x))) = \mu_A(y * (x * y))$.

The following diagram gives the relations between fuzzy Z-ideal, fuzzy implicative ideal and fuzzy sub-implicative ideal of a Z-algebra



4. Conclusion

In this article, we have introduced fuzzy implicative ideals in Z-algebras and discussed their properties. We extend this concept in our research work.

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