Advances and Applications in Mathematical Sciences
Volume 22, Issue 2, December 2022, Pages 581-595

AN APPROXIMATE APPROACH FOR THE SYSTEM OF FREDHOLM INTEGRAL EQUATIONS

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Abstract

The aim of this article is to solve the system of linear Fredholm integral equations (FIE) of the second kind by Adomian Decomposition Method (ADM). This method can be easily used to solve linear equations effectively and approximately. The numerical results are comparing with the exact solutions. The solutions are convergent with the exact solutions and the computational work can be done easily with the software Mathematica. Some numerical examples are demonstrated based on this method.

1. Introduction

Calculus is also known as infinitesimal calculus or the calculus of infinitesimals. In 17th century, "Infinitesimal calculus" was developed by Isaac Newton and Gottfried Wilhelm Leibniz. Calculus has also used in science, engineering, and economics. It describes courses of elementary mathematical analysis, which is a lat in word. It has two types namely Differential calculus and Integral calculus.

Differential calculus is a subfield of calculus concerned with the study of the rates at which quantity change. Differential calculus is the applications of derivative of functions. Differentiations is said to be the process for finding the derivative of functions. If the derivative is a linear, it takes functions as input and gives second functions as output.

2020 Mathematics Subject Classification: 33E30, 45B05, 45F05, 65F10.

Keywords: Integral equations, Fredholm integral equations, System of Linear integral equations, Adomian Decomposition Method.

Received April 15, 2022; Accepted July 1, 2022

Generally, an integral equation is an unknown function under one or more functions appears under an integral sign [6]. A function y is a real value of s and an interval is $[c, d]$, the definite integral is,

$$\int_c^d y(s)ds \tag{1.1}$$

In 1823, the integral equation was first introduced by Abel. He derived generalization of a problem whose solution involved with the solution of an integral equation. Integral Equations classified numerous types [16]. If the limits of integration are both fixed, then it is called Fredholm equations [18]. Fredholm equations is said to be the linear integral equations, due to the linear behavior of the unknown function under the integral sign [7].

In recent years, system of Fredholm integral equations (FIE) of the second kind has been developed by many methods. The numerical solvability of such system has been derived by various approximate numerical methods. Such as Taylor's-series expansion Method [3] [8], Galerkin Method [12], Homotophy Analysis Method (HAM) [10], Reproducing kernel Hilbert space method (RKHS) [2] and others also. Integral equations are encountered in a variety of applications in potential theory, geophysics, electricity and magnetism, radiation and control systems.

This article express the Adomian Decomposition Method (ADM) [17] for the system of Fred-holm Integral equations (FIE) in the form:

$$\begin{cases} y_1(u) = g_1(u) + \int_c^d h(u, s, y_1(s))ds \\ y_2(u) = g_2(u) + \int_c^d h(u, s, y_2(s))ds \\ . \\ . \\ . \\ y_n(u) = g_n(u) + \int_c^d h(u, s, y_1(s), \dots, y_n)ds \end{cases} \tag{1.2}$$

Where $u \in [c, d]$ and $h(u, s)$ is kernel of Fredholm Integral Equations [14]. The ADM [1] can be handled easily. An approximate solutions can be obtained easily in this method.

2. Preliminary

2.1 Definition. An integral equation of the form,

$$y(u) = g(u) + \lambda \int_{c}^{d} h(u, s) y(s) ds \tag{2.1}$$

where, λ is a real or complex parameter, $h(u, s)$ is the kernel of the integral equation, $y(u)$ is the unknown function.

The integral equations,

$$f(u) y(u) = g(u) + \lambda \int_{c}^{d} h(u, s) y(s) ds \tag{2.2}$$

There are three kinds of integral equations to the value of $f(u)$.

2.2 Definition. The integral equations,

$$g(u) + \lambda \int_{c}^{d} h(u, s) y(s) ds = 0 \tag{2.3}$$

is called first kind of integral equation, where the value of $f(u)$ is equal to zero.

2.3. Definition. The integral equations,

$$y(u) = g(u) + \lambda \int_{c}^{d} h(u, s) y(s) ds \tag{2.4}$$

is called second kind of integral equation, where the value of $f(u)$ is equal to one.

2.4 Definition. The integral equations,

$$f(u) y(u) = g(u) + \lambda \int_{c}^{d} h(u, s) y(s) ds \tag{2.5}$$

is called third kind of integral equation, where the value of $f(u)$ is neither one nor zero.

2.5 Definition. A Fredholm Integral Equations is of the form,

$$y(u) = g(u) + \lambda \int_{c}^{d} h(u, s) y(s) ds \tag{2.6}$$

In this integral, both limits are constants.

2.6 Definition. A volttera integral equation is of the form,

$$y(u) = g(u) + \lambda \int_{c}^{x} h(u, s) y(s) ds \tag{2.7}$$

Here, the upper limit is variable and the lower limit is constant.

2.7 Definition. If the integral equations,

$$y(u) = g(u) + \lambda \int_{c}^{d} h(u, s) y(s) ds \tag{2.8}$$

is Linear, then the linear operation is perform in unknown functions under integral sign.

2.8 Definition. In the second kind of integral equation,

$$y(u) = g(u) + \lambda \int_{c}^{d} h(u, s) y(s) ds \tag{2.9}$$

where, the function $g(u)$ is identically zero. This equation is called Homogeneous Integral Equations.

2.9 Definition. In the second kind of integral equation,

$$y(u) = g(u) + \lambda \int_{c}^{d} h(u, s) y(s) ds \tag{2.10}$$

where, the function $g(u)$ is not zero. This equation is called Non-Homogeneous Integral Equations.

3. The Adomian Decomposition Method

The Adomian decomposition method was introduced and extended by George Adomian. It is an effective and approximate method for finding analytical solutions of both linear and non-linear problems consider, the first term of Fredholm Integral Equations,

$$y_1(u) = g_1(u) + \int_c^d h(u, s) y_1(s) ds \tag{3.1}$$

The second term of Fredholm Integral Equations,

$$y_2(u) = g_2(u) + \int_c^d h(u, s) y_2(s) ds \tag{3.2}$$

This leads to m^{th} term of Fredholm Integral Equations,

$$y_m(u) = g_m(u) + \int_c^d h(u, s) y_m(s) ds \tag{3.3}$$

Where, $m = 0, 1, 2, \ldots$

Adomian decomposition method consists of the unknown functions $y(u)$. A sum of m-number of components defined by the series,

$$y(u) = \sum_{m=0}^{\infty} y_m(u) \tag{3.4}$$

Consider, i^{th} term of equation (2) and applied ADM is,

$$y_i(u) = g_i(u) + \int_c^d h(u, s, y_1(s), y_2(s), \ldots, y_n(s)) ds \tag{3.5}$$

The canonical form of the equation is given by,

$$y_i(u) = g_i(u) + L_i(u) \tag{3.6}$$

Where,

$$L_i(u) = \int_c^d h(u,\ s,\ y_1(s),\ y_2(s),\ \dots,\ y_n(s))ds \tag{3.7}$$

Using Adomian decomposition method, let,

$$y_i(u) = \sum_{m=0}^{\infty} y_{im}(u) \tag{3.8}$$

and

$$L_i(u) = \sum_{m=0}^{\infty} K_{im} \tag{3.9}$$

Where, Kim, $m = 0, 1, 2, \dots$ are polynomials. It depends on $y_{10}, \dots, y_{n0} \dots, y_{nm}$. These are called Adomian polynomials. Hence, equation (14) can be written as,

$$\begin{cases} y_{i0}(u) = g_i(u) \\ y_{i,m+1}(u) = K_{im}(y_{10}, \dots, y_{1m}, \dots, y_{n0}, \dots, y_{nm}), \\ i = 1, \dots, n; m = 0, 1, 2, \dots \end{cases} \tag{3.10}$$

The approximate solution of the truncated series,

$$y_{i,m+1}(u) = y_{i,m}(u) \tag{3.11}$$

Consider, α is a parameter for convenient. To obtain the Adomian polynomials for a linear function of equation (16) and equation (17), it can be rewritten as,

$$y_{i\alpha}(u) = \sum_{m=0}^{\infty} \alpha^m y_{im}(u) \tag{3.12}$$

and

$$L_{i\alpha}(u) = \sum_{m=0}^{\infty} \alpha^m K_{im} \tag{3.13}$$

From (21) we have,

$$K_{im} = \frac{1}{m!}\left[\frac{d}{d\alpha^m} L_{i\alpha}(y_1, y_2, y_n)\right]_{\alpha=0} \tag{3.15}$$

We have,

$$K_{im}(y_{10}, \ldots, y_{1m}, \ldots, y_{n0}, \ldots, y_{nm}) = \int_c^d \sum_{m=0}^{\infty} h_{ij}(u, s) y_{jm}(s) ds \tag{3.16}$$

so, the system of linear Fredholm Integral Equations for equation (18) as follows:

$$\begin{cases} y_{i0}(u) = g_i(u) \\ y_{i,m+1}(u) = \int_c^d \sum_{m=0}^{\infty} h_{ij}(u, s) y_{jm}(s) ds, \; i = 1, \ldots, n; m = 0, 1, 2 \end{cases} \tag{3.17}$$

4. Theorem. *If* $H : [c, d] \times [c, d] \to \mathbb{R}$ *and* $f : [c, d] \to \mathbb{R}$ *are continuous and if*

$$\sup_{c \le u \le d} |\lambda| \int_c^d |H(u, s)| ds < 1.$$

There exists an unique continuous $g : [c, d] \to \mathbb{R}$ *and then it satisfies the Fredholm Integral Equations.*

Proof. Consider, the Fredholm Integral Equation,

$$y(u) = g(u) + \lambda \int_c^d (u, s) y(s) ds \tag{4.1}$$

Define, an mapping $M : C([c, d]) \to C([c, d])$ by,

$$My(u) = g(u) + \lambda \int_c^d (u, s) y(s) ds \tag{4.2}$$

Since, $C([c, d])$ is complete and continuous real valued functions with supremum norm,

$$\| y \| = \sup \{| y(u) | : u \in [c, d]\} \tag{4.3}$$

A fixed point of M is equivalent to a solution of Fredholm Integral Equation.

Let, $h(u, s)$ be a continuous function on $[c, d] \times [c, d]$.

Since, g be a contraction function on $[c, d]$ and λ be a real number such that sup

$$\sup_{c \leq u \leq d} |\lambda| \int_c^d |H(u, s)| ds < 1.$$

By contraction mapping theorem,

$$\| My_1 - My_2 \| = \sup c \leq u \leq d \| My_1(u) - My_2(u) \|$$

$$\| My_1 - My_2 \| = \sup_{c \leq u \leq d} |\lambda| \left| \int_c^d H(u, s)(y_1(s) - y_2(s)) ds \right|$$

$$\| My_1 - My_2 \| \leq \sup_{c \leq u \leq d} |\lambda| \left| \int_c^d H(u, s)(y_1(s) - y_2(s)) \right| ds$$

$$\| My_1 - My_2 \| \leq \| y_1 - y_2 \| \sup_{c \leq u \leq d} |\lambda| \left| \int_c^d |(u, s)| ds \right.$$

$$\| My_1 - My_2 \| \leq \| y_1 - y_2 \| \tag{4.4}$$

Therefore, M is a contraction mapping. Hence, $My = y$ and so the Fredholm Integral Equation has a unique solution in $C([c, d])$.

5. Numerical Analysis

Algorithms

Step 1. Start the Mathematica Software.

Step 2. Choose equation (35) and substitute the given values.

Step 3. The first iteration values can be calculated.

Step 4. If the equation (37) is an approximate solution, and go to step 5, otherwise the next iteration will be started.

Step 5. Print the approximate values.

Example 1. Consider, the system of Fredholm integral equations with the exact solutions $y_1(u) = u^2$ and $y_2(u) = -u + u^2 + u^3$.

$$\begin{cases} y_1(u) - \int_0^1 ((u-s)^3 y_1(s) + (u-s)^2 y_2(s))ds = g_1(u) \\ y_2(u) - \int_0^1 ((u-s)^4 y_1(s) + (u-s)^3 y_2(s))ds = g_2(u) \end{cases} \tag{5.1}$$

Where, $g_1(u) = \frac{1}{20} - \frac{11}{30}u + \frac{5}{3}u^2 - \frac{1}{3}u^3$ and $g_2(u) = \frac{-1}{30} - \frac{41}{30}u + \frac{3}{20}u^2 + \frac{23}{12}u^3 - \frac{1}{3}u^4, \in [0, 1]$ The FIE of the form,

$$\begin{cases} y_1(u) - f_1(u) + \int_0^1 ((u-s)^3 y_1(s) + (u-s)^2 y_2(s))ds \\ y_2(u) = f_2(u) + \int_0^1 ((u-s)^4 y_1(s) + (u-s)^3 y_2(s))ds \end{cases} \tag{5.2}$$

Using Adomian decomposition method,

$$\begin{cases} y_{11}(u) \approx 0.05 - 0.3667u + 1.6667u^2 - 0.3333u^3 \\ y_{20}(u) \approx -0.3333 - 0.6833u + 0.15u^2 + 1.9167u^3 - 0.3333u^4 \end{cases} \tag{5.3}$$

For the first iteration, we have

$$\begin{cases} y_{11}(u) - \int_0^1 ((u-s)^3 y_{10}(t) + (u-s)^2 y_{20}(s))ds \\ y_{11}(u) \approx 0.03316 - 0.31761u + 0.85412u^2 - 0.92360u^3 - 0.3388u^4 \end{cases}$$

$$\begin{cases} y_{21}(u) - \int_0^1 ((u-s)^4 y_{10}(s) + (u-s)^3 y_{20}(s))ds \\ y_{21}(u) \approx 0.03316 - 0.31761u + 0.85412u^2 - 0.92360u^3 - 0.3388u^4 \end{cases}$$

The approximation solutions of above terms is,

$$\begin{cases} y_{12}(u) \approx 0.00057 - 0.00007u + 0.9959u^2 - 0.00559u^3 \\ y_{22}(u) \approx -0.00017 - 1.0009u + 1.00411u^2 + 0.99311u^3 - 0.00559u^4 \end{cases} \tag{5.4}$$

For the second iteration, we have

$$\begin{cases} y_{12}(u) = \int_0^1 ((u-s)^3 y_{11}(s) + (u-s)^2 y_{21}(s))ds \\ y_{12}(u) \approx 0.00063 + 0.00050u + 0.00323u^2 - 0.00499u^3 \end{cases}$$

$$\begin{cases} y_{22}(u) = \int_0^1 ((u-s)^3 y_{11}(s) + (u-s)^3 y_{21}(s))ds \\ y_{22}(u) \approx 0.00021 + 0.00054u - 0.00299u^2 + 0.00565u^3 - 0.00499u^4 \end{cases}$$

The approximation solution of above terms is,

$$\begin{cases} y_{13}(u) \approx -0.00005 + 0.00043u + 0.99912u^2 + 0.00059u^4 \\ y_{23}(u) \approx 0.0004 - 1.00037u + 1.00111u^2 + 0.99876u^3 + 0.00059u^4 \end{cases} \tag{5.5}$$

Similarly, the components can be calculated for $m = 3, 4, \ldots$

Eight terms of the approximation solution is given by,

$$\begin{cases} y_{18}(u) \approx -0.00002 + 0.00007u + 1.00005u^2 + 0.00004u^3 \\ y_{28}(u) \approx 8.64367 * 10^{-6} - 0.99992u + 0.99995u^2 + 1.00004u^3 + 0.00004u^4 \end{cases} \tag{5.6}$$

Numerical Results:

Table 1. $y_{1,11}(u)$.

u	Exact solution	Approximate solution	Error
0	0	0.00002	-2×10^{-05}
0.1	0.01	0.01003	-3×10^{-05}
0.2	0.04	0.04004	-4×10^{-05}
0.3	0.09	0.09005	-5×10^{-05}
0.4	0.16	0.16006	-6×10^{-05}
0.5	0.25	0.25008	-7×10^{-05}
0.6	0.36	0.36009	-8×10^{-05}

0.7	0.49	0.49011	-1.1×10^{-04}
0.8	0.64	0.64013	-1.3×10^{-04}
0.9	0.81	0.81015	-1.5×10^{-04}
1	1	1.00018	-1.8×10^{-04}

Table 2. $y_{2,11}(u)$.

u	Exact solution	Approximate solution	Error
0	0	-0.0000008	8×10^{-07}
0.1	-0.089	-0.089001	1×10^{-06}
0.2	-0.152	-0.151995	-5×10^{-06}
0.3	-0.183	-0.182989	-1×10^{-05}
0.4	-0.176	-0.175982	-1.8×10^{-05}
0.5	-0.125	-0.124975	-2.5×10^{-05}
0.6	-0.024	-0.023966	-3.36×10^{-05}
0.7	0.133	0.133045	-4.5×10^{-05}
0.8	0.352	0.352059	-5.9×10^{-05}
0.9	0.639	0.639077	-7.7×10^{-05}
1	1	1.0001	-1×10^{-04}

Algorithms

Step 1. Start the Mathematica Software.

Step 2. Choose equation (40) and substitute the given values.

Step 3. The first iteration values can be calculated.

Step 4. If the equation (42) is an approximate solution, and go to step 5, otherwise the next iteration will be started.

Step 5. Print the approximate values.

Example 2. Consider, the system of Fredholm integral equations with the exact solutions $y_1(u) = u + 1$ and $y_2(u) = u^2 + 1$

$$\begin{cases} y_1(u) = \frac{u}{18} + \frac{17}{36} + \int_0^1 \frac{s+u}{3}(y_1(s) + y_2(s))ds \\ y_2(u) = u^2 - \frac{19}{12}u + 1 + \int_0^1 us(y_1(s) + y_2(s))ds \end{cases} \tag{5.7}$$

Using Adomian decomposition method,

$$\begin{cases} y_{10}(u) = \frac{u}{18} + \frac{17}{36} \approx 0.0566u + 0.4722 \\ y_{20}(u) = u^2 - \frac{19}{12}u + 1 \end{cases} \tag{5.8}$$

For the first iteration, we have

$$\begin{cases} y_{11}(u) = \int_0^1 \frac{usu}{3}(y_{10}(s) + y_{20}(s))ds \\ y_{21}(u) = \int_0^1 us(y_{10}(s) + y_{20}(s))ds \end{cases}$$

The approximation solution of above terms is,

$$\begin{cases} y_{12}(u) \approx 0.63127 + 0.40399u \\ y_{22}(u) \approx u^2 - 1.1061u + 1 \end{cases} \tag{5.9}$$

For the second iteration, we have

$$\begin{cases} y_{12}(u) = \int_0^1 \frac{u+s}{3}(y_{11}(s)y_{21}(s))ds \\ y_{22}(u) = \int_0^1 us(y_{11}(s)y_{21}(s))ds \end{cases}$$

The approximation solution of above terms is,

$$\begin{cases} y_{13}(u) \approx 0.7494 + 0.59449u \\ y_{23}(u) \approx u^2 - 1.751702u + 1 \end{cases}$$

$$\begin{cases} y_{13}(u) \approx 0.7494 + 0.59449u \\ y_{23}(u) \approx u^2 - 1.751702u + 1 \end{cases} \tag{5.10}$$

Similarly, the components can be calculated, for $n = 3, 4, \ldots$ Twelve terms of the approximation solution is given by,

$$\begin{cases} y_{1,12}(u) \approx 0.994924 + 0.992283u \\ y_{1,12}(u) \approx u^2 + 0.0251842u + 1 \end{cases} \tag{5.11}$$

Numerical Results:

Table 3. $y_{1,12}(u)$.

u	Exact solution	Approximate solution	Error
0	1	0.994924	5.0760×10^{-03}
0.1	1.1	1.0941523	5.8477×10^{-03}
0.2	1.2	1.1933806	6.6194×10^{-03}
0.3	1.3	1.2926089	7.3911×10^{-03}
0.4	1.4	1.3918372	8.1628×10^{-03}
0.5	1.5	1.4910655	8.9350×10^{-03}
0.6	1.6	1.5902938	9.7062×10^{-03}
0.7	1.7	1.6895221	1.0478×10^{-02}
0.8	1.8	1.7887504	1.1249×10^{-02}
0.9	1.9	1.8879787	1.2021×10^{-02}
1	2	1.987207	1.2793×10^{-02}

Table 4. $y_{2,12}(u)$.

u	Exact solution	Approximate solution	Error
0	1	1	0
0.1	1.01	1.01251842	-2.51842×10^{-03}
0.2	1.04	1.04503684	-5.03684×10^{-03}
0.3	1.09	1.09755526	-7.55526×10^{-03}
0.4	1.16	1.17007368	-1.00737×10^{-02}
0.5	1.25	1.2625921	-1.25921×10^{-02}
0.6	1.36	1.37511052	-1.51105×10^{-02}
0.7	1.49	1.50762894	-1.76289×10^{-02}
0.8	1.64	1.66014736	-2.01474×10^{-02}
0.9	1.81	1.83266578	-1.26658×10^{-02}
1	2	2.0251842	-1.51842×10^{-02}

6. Conclusion

This article, for the systems of linear Fredholm integral equations of the second kind, provides the use of Adomian decomposition method. A large number of iteration must be done to obtain a good approximation. Whereas, this method with less number of iteration provide better approximation. Further, fuzzy concepts can be used to solve the system of Fredholm integral equations of the second kind.

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