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RETAILER'S OPTIMAL ORDERING POLICY FOR DETERIORATING ITEM'S WITH INFLATION ORDER LINKED TRADE CREDIT AND PARTIAL BACKLOGGING

S. R. SINGH and RINKI CHAUDHARY

Department of Mathematics
C.C.S. University, Meerut, India
E-Mail: shivrajsingh@gmail.com
bcrinkichaudhary67@gmail.com

Abstract

In this paper, we present an economic order quantity model for decaying goods under inflation. Shortages are allowed which is partial backlogged and partial permissible delays in payment also allowable which is based on the order quantity. The major purpose of this paper is to establish optimal order and backlog policies to minimize optimum replenishment time and optimum cycle length and total inventory cost for retailers with these time values. Obtaining for this purpose various numerical and theoretical results are given which shows the model is validated numerically. Sensitivity analysis of the most favorable solution has been given with respect to the various parameters of the inventory organization.

1. Introduction

The economic order size inventory model assumes that permissible delay period allows the buyer to buy the supplies without immediate complete payment, while a supplier is required to pay the buyer upon receipt of the goods. Whenever demand increases then the supplier offers retailers credit period. Furthermore, if the ordered quantity is large then the provider desire to offer improved terms of trade provided. In an emerging economy, it is common and acceptable for small and micro-retailers to offer a business loan because these retailers need the economic income to pay when goods are received. In fact, the larger quantity ordered, improved the trading conditions

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a trader can usually proposal to extend the delay period to reduce the retailer's economic position. Shortly, if a vendor orders extra items from the trader, the vendor can gain worse credit expressions, take benefit of amount discounts and maybe recognize superior income. Firstly, Haley and Higgins [7] presented an inventory model with trade credit financing. After some time Goyal [5] described an EOQ model with the conditions of the trade credit period. Huang [8] designed an inventory model, assuming that the supplier offers retailers a trade credit period, although the ordered q is less than the predetermined quantity d_q. But to enjoy the trade credit period the retailers must pay a part of the total purchasing cost immediately. After that, he/she would pay the remaining amount at the end of the permissible delay period.

Table 1. Summary of works related to inflation, trade credit and shortage.

Source	Inflation	Trade credit	Order depended trade credit	Shortages	EOQ/EPQ
Chang et al. (2003)	✓	Full	✓		EOQ
Chung and Liao (2004)		Full	✓		EOQ
Ouyang et al. (2009)			✓		EOQ
Chen et al.(2014)		✓			EOQ
Guchhait et al. (2014)		Full	✓		EOQ
Shastri et al. (2015)	✓	Full	✓	Partial	EOQ
Vandana and Sharma (2016)			✓		EOQ
Sunil Tiwari (2019)		Partial	✓	✓	EOQ
This paper	✓	Partial	✓	Partial	EOQ

The literature is complete with papers on permissible delay in payment such as Singh et al. [11], Singh and Singh [12], Das et al. [4]. This paper is an extension of the previous study by including the following realistic situation: (1) a supplier offers trade credit period based on the ordered quantity to the retailer (2) effect of inflation. The best refill cycle time and the inventory level time to reach zero are obtained.

2. Notation and Assumptions

A Replenishment cost per order.

C purchasing cost ($/unit).

P selling price of the product ($/unit), $P > C$.

H holding cost ($/unit/year)

S backordering cost ($/unit/year).

L goodwill loss when loss of unsatisfied demand ($/unit) interest earned ($/unit/ year) interest charged ($/unit/ year).

D the demand rate. The deterioration rate, $0 \leq \theta < 1$.

M the length of the permissible delay in years offered by the supplier the minimum order quantity at which the delay in payments is permitted order quantity per order.

S maximum shortage level the fraction of the total purchase cost which payment allowed to be postponed when a vendor order quantity q is fewer than the minimum order quantity. Time interval that units are decline to zero due to both demand and deterioration quantity of inventory (unit/year) at time t.

r Inflation rate.

These are the following assumption.

1. Allows retailers a delay in supplier's offer partial trade credit even though they order less than a prearranged quantity. For this condition the retailer will have to pay a portion of the total procure cost immediately, where β is the portion of the late payment per order by suppliers, $0 < \beta < 1$.

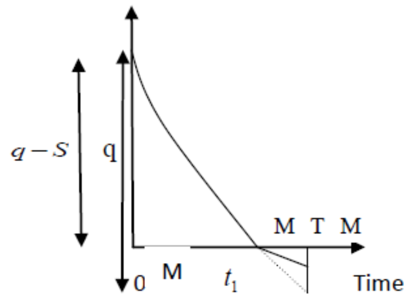
2. Replenishments are expeditiously and lead time is zero and time horizon is infinite.

3. The stocking structure consists of single items with a constant decline rate and demand is constant.

4. In this paper shortages are allowed which is partial backlogging and the backlogging rate exp $(-\delta t)$, the backlogging parameter δ is positive constant.

5. No replacement or repair of decaying goods is made during a given cycle.

3. Inventory model formulation



Inventory level at time t

Inventory level is $I(t)$ at time $t = 0$, due to demand and deterioration inventory level becomes zero at time t_1, at this time shortage occur up to time T. The rate change of inventory level is governed by the following differential equation.

$$I_1'(t) + \theta I_1(t) = -D, \quad 0 \leq t \leq t_1 \tag{1}$$

$$I_1'(t) = -De^{-\delta t}, \quad t_1 \leq t \leq T, (t_1) = 0 \tag{2}$$

Solution of equation (1) with the help of boundary condition

$$I_1(t) = \frac{D}{\theta}(e^{\theta(t_1 - t)} - 1), \qquad 0 \leq t \leq t_1. \tag{3}$$

Solution of equation (2) with the help of boundary condition

$$I_2(t) = \frac{D}{\delta}(e^{-\delta t} - e^{\delta t_1}), \quad t_1 \leq t \leq T. \tag{4}$$

The maximum backorder is given by

$$S = -I_2(T) = -\frac{D}{\delta}(e^{-\delta T} - e^{-\delta t_1}),\ I(0) = q - S \tag{5}$$

$$q = I_{\max} + S = I(0) + S$$

$$q = \frac{D}{\theta}(e^{\theta t_1} - 1) + \frac{D}{\delta}(e^{-\delta t_1} - e^{-\delta T}) \tag{6}$$

The relevant cost parameter for the retailer consists of the following elements.

1. Ordering cost $O_c = A$

2. Holding cost $H_c \dfrac{ChD}{\theta}\left(\dfrac{(e^{\theta t_1} - e^{-rt_1})}{\theta + r} + \dfrac{(e^{rt_1} - 1)}{r}\right)$

3. Shortage cost $S_c \dfrac{sD}{\delta}\left(e^{-(\delta+r)t}\left(\dfrac{1}{r} - \dfrac{1}{\delta + r}\right) + e^{-rT}\left(\dfrac{e^{-\delta T}}{\delta + r} - \dfrac{e^{-\delta t_1}}{r}\right)\right)$

4. Lost sale cost $L_C = LD\left(e^{-rT}\left(\dfrac{e^{-\delta T}}{r + \delta}\dfrac{1}{r}\right) + e^{-rt_1}\left(\dfrac{1}{r} - \dfrac{e^{-\delta t_1}}{r + \delta}\right)\right)$

5. Deterioration cost $D_c \dfrac{CD\theta}{\theta}\left(\dfrac{(e^{\theta t_1} - e^{-rt_1})}{\theta + r} + \dfrac{(e^{rt_1} - 1)}{r}\right)$

(i) Two cases arise on interest earned and interest pay on the basis of the values of q and q_d, (i) $q < q_d$ (ii) $q \geq q_d$.

Case (1) $q < q_d$. At time $t = 0$ the vendor pay to trader $C(1 - \beta)q$ amount and the remaining amount $C\beta q$ pays the at time M. In this case, there arise three subcases depend on the values of t_1, T and M.

(1.1) $0 \leq M \leq t_1 \leq T$, (1.2) $0 \leq t_1 \leq M \leq T$, (1.3) $0 \leq t_1 \leq T \leq M$

Sub-case (1.1) $0 \leq M \leq t_1 \leq T$.

Since $q < q_d$ the vendor has to pays $C(1 - \beta)q$ to the supplier inventory when the goods are received. Further, as $M \leq t_1$ the vendor still has some inventory on hand, when paying the suppliers remains of the procure cost at time M. Interest paid

$$I_c = DCi_P\left[(1 - \beta)M\left(\frac{1}{\theta}(e^{\theta t_1} - 1) + \frac{1}{\delta}(e^{-\delta t_1} - e^{-\delta T})\right) + \frac{1}{\theta}\left(e^{rM}\left(\frac{e^{\theta(t_1 - M)}}{\theta + r} - \frac{1}{r}\right)\right.\right.$$

$$\left.\left.+ e^{-rt_1}\left(\frac{1}{r} - \frac{1}{\theta + r}\right)\right)\right]$$

Interest earn continues to accrue at the rate of time commencing between

0 to M.

Interest earned per cycle $I_E = Pi_e\left[D\left(\frac{-Me^{-rM}}{r} - \frac{e^{-rM}}{r^2} + \frac{1}{r^2}\right)\right]$

Subcase (1.2) $0 \le t_1 \le M \le T$

Interest paid $I_c = Ci_p(1-\beta)M\left(\frac{1}{\theta}(e^{\theta t_1} - 1) + \frac{1}{\delta}(e^{-\delta t_1} - e^{-\delta T})\right)$

Interest earned $I_E = DPi_e\left[\frac{-t_1}{r} e^{-rt_1} + \frac{(1 - e^{-rt_1})}{r^2} - \frac{t_1}{r}(e^{-rM} - e^{-rt_1})\right].$

Subcase (1.3) $0 \le t_1 \le T \le M$.

In this sub-case, the interest payable and the interest earned are identical to subcase 1.2.

Case (2) $q \ge q_d$.

Subcase (2.1). $0 \le M \le t_1 \le T$.

In this case, the retailer has some accumulation even when the seller pays the total purchase cost. Therefore interest is paid per cycle

$$I_c = \frac{Ci_pD}{\theta}\left(e^{-rM}\left(\frac{e^{\theta(t_1-M)}}{\theta + r} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{1}{\theta + r}\right)\right).$$

Interest earned $I_E = Pi_e\left[D\left(\frac{-Me^{-rM}}{r} - \frac{e^{-rM}}{r^2} + \frac{1}{r^2}\right)\right]$

Subcase (2.2) $0 \le t_1 \le M \le T$.

Interest paid $I_c = 0$

Interest earned $I_E = DPi_e\left[\frac{-t_1}{r} e^{-rt_1} + \frac{(1 - e^{-rt_1})}{r^2} - \frac{t_1}{r}(e^{-rM} - e^{-rt_1})\right].$

Subcase (2.3) $0 \le t_1 \le T \le M$.

In this sub-case, the interest payable and the interest earned are identical to subcase 2.2.

$$TC_{1.1} = \frac{1}{T}\left[\begin{array}{l} A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta t_1} - e^{-rt_1})}{\theta + r} + \frac{(e^{-rt_1} - 1)}{r}\right) + \frac{sD}{\delta}\left(e^{-(\delta + r)t_1}\left(\frac{1}{r} - \frac{1}{\delta + r}\right)\right. \\ \left. + e^{-rT}\left(\frac{e^{\delta T}}{\delta + r} - \frac{e^{-\delta t_1}}{r}\right)\right) + LD\left(e^{-rT}\left(\frac{e^{-\delta T}}{r+\delta} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{e^{-\delta t_1}}{r+\delta}\right)\right) \\ + Ci_P D\left((1-\beta)M\left(\frac{1}{\theta}(e^{\theta t_1} - 1) + \frac{1}{\delta}(e^{\delta t_1} - e^{-\delta T})\right) + \frac{1}{\theta}\left(e^{-rM}\left(\frac{e^{\theta(t_1 - M)}}{\theta + r} - \frac{1}{r}\right)\right.\right. \\ \left.\left.\left. + e^{-rt_1}\left(\frac{1}{r} - \frac{1}{\theta + r}\right)\right)\right)\right) - Pi_e\left[D\left(\frac{-Me^{rM}}{r} - \frac{e^{-rM}}{r^2} + \frac{1}{r^2}\right)\right] \end{array}\right]$$

$$TC_{1.2}(t_1, T) = TC_{1.3}(t_1, T)$$

$$= \frac{1}{T}\left\{A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta t_1} - e^{-rt_1})}{\theta + r} + \frac{(e^{-rt_1} - 1)}{r}\right) + \frac{sD}{\delta}\left(e^{-(\delta + r)t_1}\left(\frac{1}{r} - \frac{1}{\delta + r}\right)\right.\right.$$

$$\left. + e^{-rT}\left(\frac{e^{-\delta T}}{\delta + r} - \frac{e^{-\delta t_1}}{r}\right)\right)$$

$$+ LD\left(e^{-rT}\left(\frac{e^{-\delta T}}{r + \delta} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{e^{-\delta t_1}}{r + \delta}\right)\right) + Ci_P D(1 - \beta)M\left(\frac{1}{\theta}(e^{\theta t_1} - 1)\right.$$

$$\left. + \frac{1}{\delta}(e^{-\delta t_1} - e^{-\delta T})\right)$$

$$\left. - DPi_e\left[\frac{-t_1}{r}e^{-rt_1} + \frac{(1 - e^{-rt_1})}{r^2} - \frac{t_1}{r}(e^{-rM} - e^{-rt_1})\right]\right\}$$

$$TC_2(t_1, T) = \frac{1}{T}(O_C + H_C + D_C + S_C + L_C + I_C - I_E)$$

$$TC_2(t_1, T) = \begin{cases} TC_{2.1}(t_1, T), & \text{if } 0 \le M \le t_1 \le T \\ TC_{2.2}(t_1, T), & \text{if } 0 \le t_1 \le M \le T \\ TC_{2.3}(t_1, T), & \text{if } 0 \le t_1 \le T \le M \end{cases}$$

Where,

$$TC_{2.1}(t_1,\, T) = \frac{1}{T}\left[\begin{array}{l} A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta t_1} - e^{-rt_1})}{\theta + r} + \frac{(e^{-rt_1} - 1)}{r}\right) \\ + \frac{sD}{\delta}\left(e^{-(\delta+r)t_1}\left(\frac{1}{r} - \frac{1}{\delta + r}\right) + e^{-rT}\left(\frac{e^{\delta T}}{\delta + r} - \frac{e^{-\delta T}}{r}\right)\right) \\ + LD\left(e^{-rT}\left(\frac{e^{-\delta T}}{r+\delta} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{e^{-\delta t_1}}{r+\delta}\right)\right) + \frac{Ci_p D}{\theta} \\ \left(e^{-rM}\left(\frac{e^{\theta(t_1 - M)}}{\theta + r} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{1}{\theta + r}\right)\right) - \\ Pi_e\left[D\left(\frac{-Me^{-rM}}{r} - \frac{e^{-rM}}{r^2} + \frac{1}{r^2}\right)\right] \end{array}\right]$$

$$TC_{2.2}(t_1,\, T) = TC_{2.3}(t_1,\, T)$$

$$= \frac{1}{T}\left[\begin{array}{l} A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta t_1} - e^{-rt_1})}{\theta + r} + \frac{(e^{-rt_1} - 1)}{r}\right) + \frac{sD}{\delta}\left(e^{-(\delta+r)t_1}\left(\frac{1}{r} - \frac{1}{\delta + r}\right)\right. \\ \left. + e^{-rT}\left(\frac{e^{-\delta T}}{\delta + r} - \frac{e^{-\delta t_1}}{r}\right)\right) + LD\left(e^{-rT}\left(\frac{e^{-\delta T}}{r+\delta} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{e^{-\delta t_1}}{r+\delta}\right)\right) - \\ DPi_e\left[\frac{-t_1}{r}e^{-rt_1} + \frac{(1 - e^{-rt_1})}{r^2} - \frac{t_1}{r}(e^{-rM} - e^{-rt_1})\right] \end{array}\right]$$

4. Model Analysis and Solution Processer

Now we talk about the hypothetical aspects for each case of the proposed inventory model.

Case 1. In this case, there arise three sub-cases:

Subcase 1.1. $0 \le M \le t_1 \le T$.

The essential condition for minimizing the total cost $TC_{1.1}(t_1,\, T)$ are

$$\frac{\partial TC_{1.1}(t_1,\, T)}{\partial t_1} = 0 \text{ and } \frac{\partial TC_{1.1}(t_1,\, T)}{\partial T} = 0.$$

Thus, let $\eta_1 = \frac{\partial TC_{1.1}(t_1,\, T)}{\partial t_1} = 0$ and $\nu_1 = \frac{\partial TC_{1.1}(t_1,\, T)}{\partial T} = 0$ yields

$$\eta_1 = \frac{1}{T}\left\{\frac{D(h+\theta)C}{\theta}\left(\frac{(\theta e^{\theta t_1} + re^{-rt_1})}{\theta + r} - e^{-rt_1}\right) + \frac{sD}{r}(-e^{-(\delta+r)t_1} + e^{-rT}e^{-\delta t_1})\right.$$

$$\left. + LD(e^{-(r+\delta)t_1} - e^{-rt_1}) + ci_P D\left((1-\beta)M(e^{\theta t_1} - e^{-\delta t_1}) + \frac{e^{-rM+\theta(t_1-M)}}{\theta + r} - \frac{e^{-rt_1}}{\theta + r}\right)\right\} = 0$$

$$\nu_1 = \frac{\partial TC_{1.1}(t_1, T)}{\partial T} = \frac{-1}{T^2}\left[A + \frac{D(h+\theta)C}{Gq}\left(\frac{(e^{\theta t_1} - e^{rt_1})}{\theta + r} + \frac{(e^{-rt_1} - 1)}{r}\right)\right.$$

$$+ \frac{sD}{\delta}\left(e^{(\delta+r)t_1}\left(\frac{1}{r} - \frac{1}{\delta + r}\right) + e^{-rT}\left(\frac{e^{-\delta T}}{\delta + r} - \frac{e^{-\delta t_1}}{r}\right)\right) + LD\left(e^{-rT}\left(\frac{e^{-\delta T}}{r + \delta} - \frac{1}{r}\right)\right.$$

$$\left. e^{-rt_1}\left(\frac{1}{r} - \frac{e^{-\delta t_1}}{r + \delta}\right)\right) + Ci_P D\left((1-\beta)M\frac{1}{\theta}(e^{\theta t_1} - 1) + \frac{1}{\delta}(e^{-\delta t_1} - e^{-\delta T}) + \frac{1}{\theta}\right.$$

$$\left.\left.\left(e^{-rM}\left(\frac{e^{\theta(t_1-M)}}{\theta + r} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{1}{\theta + r}\right)\right)\right) - Pi_e\left[D\left(\frac{-Me^{-rM}}{r} - \frac{e^{-rM}}{r^2} + \frac{1}{r^2}\right)\right]\right]$$

$$+ \frac{1}{T}\left[\frac{sD}{\delta}(e^{-\delta t_1}e^{-rT} - e^{-(\delta+r)T}) + LD(-e^{-(\delta+r)T} + e^{-rT})\right.$$

$$\left. + Ci_p D(1-\beta)Me^{-\delta T}\right] = 0. \tag{8}$$

Solving equation (7),

$$T = \frac{e^{\delta t_1}}{SD}\left[\begin{array}{l}\frac{D(h+\theta)C}{(\theta+r)}(e^{\theta t_1} - e^{-rt_1}) + \frac{SD}{r}(-e^{-(\delta+r)t_1} + e^{-\delta t_1}) + LD(e^{(\delta+r)t_1} - e^{-rt_1}) \\ + Ci_p D\left((1-\beta)M(e^{\theta t_1} - e^{-\delta t_1}) + \frac{e^{rM+\theta(t_1-M)}}{\theta + r} - \frac{e^{-rt_1}}{\theta + r}\right)\end{array}\right].$$

Putting these values in equation (8) leads to

$$\nu_1(t_1) = \frac{-1}{T_1^2}\left[\begin{array}{l} A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta t_1} - e^{-rt_1})}{\theta + r} + \frac{(e^{-rt_1} - 1)}{r}\right) + \frac{sD}{\delta}\left(e^{-(\delta+r)t_1}\left(\frac{1}{r} - \frac{1}{\delta + r}\right)\right. \\ \left. + e^{-rT}\left(\frac{e^{\delta T}}{\delta + r} - \frac{e^{-\delta t_1}}{r}\right)\right) + LD\left(e^{-rT_1}\left(\frac{e^{-\delta T_1}}{r+\delta} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{e^{-\delta t_1}}{r+\delta}\right)\right) \\ + Ci_p D\left((1-\beta)M\left(\frac{1}{\theta}(e^{\theta t_1} - 1) + \frac{1}{\delta}(e^{\delta t_1} - e^{-\delta T_1})\right)\right) \\ + \frac{1}{\theta}\left(e^{-rM}\left(\frac{e^{\theta(t_1 - M)}}{\theta + r} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{1}{\theta + r}\right)\right) \\ - Pi_e\left[D\left(\frac{-Me^{-rM}}{r} - \frac{e^{-rM}}{r^2} + \frac{1}{r^2}\right)\right] \end{array}\right]$$

$$+ \frac{1}{T_1}\left[\frac{sD}{\delta}(e^{-\delta t_1}e^{-rT_1} - e^{-(\delta+r)T_1}) + LD(-e^{-(\delta+r)T_1} + e^{-rT_1}) + Ci_P D(1-\beta)Me^{-\delta T_1}\right] = 0.$$

Again putting $t_1 = M$ in equation (9)

$$\Delta_1 = \frac{-1}{T_1^2}\left[\begin{array}{l} A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta M} - e^{-rM})}{\theta + r} + \frac{(e^{-rM} - 1)}{r}\right) \\ + \frac{sD}{\delta}\left(e^{-(\delta+r)M}\left(\frac{1}{r} - \frac{1}{\delta + r}\right) + e^{-rT}\left(\frac{e^{\delta T}}{\delta + r} - \frac{e^{-\delta M}}{r}\right)\right) \\ + LD\left(e^{-rT_1}\left(\frac{e^{-\delta T_1}}{r+\delta} - \frac{1}{r}\right) + e^{-rM}\left(\frac{1}{r} - \frac{e^{-\delta M}}{r+\delta}\right)\right) \\ Ci_P D\left((1-\beta)M\left(\frac{1}{\theta}(e^{\theta M} - 1) + \frac{1}{\delta}(e^{-\delta M} - e^{-\delta T_1})\right)\right) \\ + \frac{1}{\theta}\left(e^{-rM}\left(\frac{1}{\theta + r} - \frac{1}{r}\right) + e^{-rM}\left(\frac{1}{r} - \frac{1}{\theta + r}\right)\right) \\ - Pi_e\left[D\left(\frac{-Me^{-rM}}{r} - \frac{e^{-rM}}{r^2} + \frac{1}{r^2}\right)\right] \end{array}\right]$$

$$+ \frac{1}{T_1}\left[\frac{sD}{\delta}(e^{-\delta M}e^{-rT_1} - e^{-(\delta+r)T_1})\right.$$

$$\left. + LD(-e^{-(\delta+r)T_1} + e^{-rT_1}) + Ci_p D(1-\beta)Me^{-\delta T_1}\right] = 0 \qquad (10)$$

Lemma 4.1.

(1) *If* $\Delta_1 < 0$, *then the total cost has its minimum value* $t_1 = M$.

(2) *If* $\Delta_1 \geq 0$, *then the total cost has its minimum value at* $(t_1, T) = (M, T_1)$.

Proof. It can be easily verified.

Subcase 1.2. $0 \leq t_1 \leq M \leq T$.

The essential conditions for minimizing the total cost $TC_{1.2}(t_1, T)$ are

$$\frac{\partial TC_{1.2}(t_1, T)}{\partial t_1} = 0 \text{ and } \frac{\partial TC_{1.2}(t_1, T)}{\partial T} = 0.$$

Thus, let $\eta_2 = \dfrac{\partial TC_{1.2}(t_1, T)}{\partial t_1} = 0$ and $\nu_2 = \dfrac{\partial TC_{1.2}(t_1, T)}{\partial t_1} = 0$

$$\eta_2 = \frac{D}{T}\Bigg[\frac{D(h+\theta)c}{\theta}\left(\frac{(\theta e^{\theta t_1} + re^{-rt_1})}{\theta + r} - e^{-rt_1}\right) + \frac{sD}{r}(-e^{-(\delta+r)t_1} + e^{-rT}e^{-\delta t_1})$$

$$+ LD(e^{-(r+\delta)t_1} - e^{-rt_1}) + Ci_p M(1-\beta)M(e^{\theta t_1} - e^{-\delta t_1}) - Pi_e(2t_1 e^{-rt_1}$$

$$+\frac{1}{r}(e^{rM} - e^{-rt_1}))\Bigg] = 0. \tag{11}$$

$$\nu_2 = -\frac{1}{T^2}\left[\begin{array}{l} A + \dfrac{D(h+\theta)C}{\theta}\left(\dfrac{(e^{\theta t_1} - e^{-rt_1})}{\theta + r} + \dfrac{(e^{-rt_1} - 1)}{r}\right) \\ + \dfrac{sD}{\delta}\left(e^{-(\delta+r)t_1}\left(\dfrac{1}{r} - \dfrac{1}{\delta + r}\right) + e^{-rT}\left(\dfrac{e^{\delta T}}{\delta + r} - \dfrac{e^{-\delta T}}{r}\right)\right) \\ + LD\left(e^{-rT}\left(\dfrac{e^{-\delta T}}{r+\delta} - \dfrac{1}{r}\right) + e^{-rt_1}\left(\dfrac{1}{r} - \dfrac{e^{-\delta t_1}}{r+\delta}\right)\right) \\ + Ci_p D(1-\beta)M\left(\dfrac{1}{\theta}(e^{\theta t_1} - 1) + \dfrac{1}{\delta}(e^{-\delta t_1} - e^{-\delta T})\right) \\ - DPi_e\left[\dfrac{-t_1}{r}e^{-rt_1} + \dfrac{(1 - e^{-rt_1})}{r^2} + \dfrac{t_1}{r}(e^{-rM} - e^{-rt_1})\right] \end{array}\right]$$

$$+ \frac{1}{T}\left[\frac{sD}{\delta}(e^{-\delta t_1}e^{-rT} - e^{-(\delta+r)T}) + LD(e^{-(\delta+r)T} - e^{-rT}) + Ci_p D(1-\beta)Me^{\delta T}\right].$$

Now solving equation (11) for T

$$T=\frac{e^{\delta t_1}}{SD}\left[\begin{array}{l} D(h+\theta)C\dfrac{(e^{\theta t_1}-e^{-rt_1})}{\theta+r}+\dfrac{sDe^{-\delta t_1}(1-e^{-rt_1})}{r}+LDe^{-\delta t_1}(1-e^{rt_1}) \\ +Ci_P D(1-\beta)M((e^{\theta t_1}-e^{-\delta t_1})DPi_e(2t_1e^{rt_1}+\dfrac{1}{r}(e^{-rM}-e^{-rt_1})) \end{array}\right].$$

In equation (12) yields

$$\nu_2=-\frac{1}{T_2^2}\left[\begin{array}{l} A+\dfrac{D(h+\theta)C}{\theta}\left(\dfrac{(e^{\theta t_1}-e^{-rt_1})}{\theta+r}+\dfrac{(e^{-rt_1}-1)}{r}\right) \\ +\dfrac{sD}{\delta}\left(e^{-(\delta+r)t_1}\left(\dfrac{1}{r}-\dfrac{1}{\delta+r}\right)+e^{-rT}\left(\dfrac{e^{-\delta T_2}}{\delta+r}-\dfrac{e^{-\delta t_1}}{r}\right)\right) \\ +LD\left(e^{-rT_2}\left(\dfrac{e^{-\delta T_2}}{r+\delta}-\dfrac{1}{r}\right)+e^{-rt_1}\left(\dfrac{1}{r}-\dfrac{e^{-\delta t_1}}{r+\delta}\right)\right) \\ +Ci_P D(1-\beta)M\left(\dfrac{1}{\theta}(e^{\theta t_1}-1)+\dfrac{1}{\delta}(e^{-\delta t_1}-e^{-\delta T_2})\right) \\ -DPi_e\left[\dfrac{-t_1}{r}e^{-rt_1}+\dfrac{(1-e^{-rt_1})}{r^2}+\dfrac{t_1}{r}(e^{-rM}-e^{-rt_1})\right] \end{array}\right]$$

$$+\frac{1}{T_2}\left[\frac{sD}{\delta}(e^{-\delta t_1}e^{-rT_2}-e^{-(\delta+r)T_2})+LD(e^{-(\delta+r)T_2}-e^{-rT_2})\right.$$

$$\left.+Ci_P D(1-\beta)Me^{-\delta T_2}\right]. \tag{13}$$

Again substituting $t_1=T_A$ in equation (13), we obtain

$$\Delta_2 = -\frac{1}{T_2^2}\begin{bmatrix} A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta T_A} - e^{-rT_A})}{\theta + r} + \frac{(e^{-rT_A} - 1)}{r}\right) \\ + \frac{sD}{\delta}\left(e^{-(\delta+r)T_A}\left(\frac{1}{r} - \frac{1}{\delta + r}\right) + e^{-rT_2}\left(\frac{e^{\delta T_2}}{\delta + r} - \frac{e^{-\delta T_A}}{r}\right)\right) \\ + LD\left(e^{-rT_2}\left(\frac{e^{-\delta T_2}}{r+\delta} - \frac{1}{r}\right) + e^{-rT_A}\left(\frac{1}{r} - \frac{e^{-\delta T_A}}{r+\delta}\right)\right) \\ + Ci_P D(1-\beta)M\left(\frac{1}{\theta}(e^{\theta T_A} - 1) + \frac{1}{\delta}(e^{-\delta T_A} - e^{-\delta T_2})\right) \\ - DPi_e\left[\frac{-T_A}{r}e^{-rT_A} + \frac{(1 - e^{-rT_A})}{r^2} + \frac{T_A}{r}(e^{-rM} - e^{-rT_A})\right. \\ \left. + \frac{1}{r}(e^{-rM} - e^{-rT_A})T_A\right] \end{bmatrix}$$

$$+ \frac{1}{T_2}\left[\frac{sD}{\delta}(e^{-\delta T_A}e^{-rT_2} - e^{-(\delta+r)T_2})\right.$$

$$\left. + LD(e^{-(\delta+r)T_2} - e^{-rT_2}) + Ci_P D(1-\beta)Me^{-\delta T_2}\right] \tag{14}$$

consequently, the following lemma is planned.

Lemma 4.2.

(1) *If* $\Delta_2 \leq 0 \leq \Delta_1$, *then the total cost has its minimum value* $t_1 \in (T_A, M]$.

(2) *If* $\Delta_2 \leq 0$, *then the total cost has its minimum value* $(t_1, T) = (T_A, T_2]$.

3) *If* $\Delta_1 \leq 0$, *then the total cost has its minimum value at* $(t_1, T) = (M, T_2]$.

Proof. It can be easily verified.

Sub-case $0 \leq t_1 \leq T \leq M$.

In this sub-case, the total cost is identical to the total cost of sub-case 1.2

Consequently, the following lemma is obtained.

Lemma 4.3.

(1) *If* $\Delta_3 \geq 0$, *then the total cost has its minimum value* $t_1 < M$

(2) *If* $\Delta_3 < 0$, *then the value of* $T \in (0, T_A)$.

Proof. It can be easily verified.

Case 2. $q \geq q_d$, there arise three sub-cases:

Subcase 2.1. $0 \leq M \leq t_1 \leq T$. The essential situation for minimizing the total cost $TC_{2.1}(t_1, T)$ are

$$\frac{\partial TC_{2.1}(t_1, T)}{\partial t_1} = 0 \text{ and } \frac{\partial TC_{2.1}(t_1, T)}{\partial T} = 0.$$

Thus, let $\eta_4 \frac{\partial TC_{2.1}(t_1, T)}{\partial t_1} = 0$ and $\nu_4 \frac{\partial TC_{2.1}(t_1, T)}{\partial T} = 0$ yields

$$\eta_4 = \frac{1}{T}\left[\begin{array}{l} \frac{D(h+\theta)C}{\theta}\left(\frac{(\theta e^{\theta t_1} + re^{-rt_1})}{\theta + r} - e^{-rt_1}\right) \\ + \frac{sD}{r}(-e^{(\delta+r)t_1} + e^{-rT}e^{-\delta t_1}) + LD(e^{-(r+\delta)t_1} - e^{-rt_1}) \\ + Ci_P D\left(\frac{e^{-rM+\theta(t_1-M)}}{\theta+r} - \frac{e^{-rt_1}}{\theta+r}\right) \end{array}\right] = 0 \tag{15}$$

$$\nu_4 = \frac{-1}{T^2}\left\{A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta t_1} - e^{-rt_1})}{\theta+r} + \frac{(e^{-rt_1} - 1)}{r}\right) + \frac{sD}{\delta}\left(e^{-(\delta+r)t_1}\left(\frac{1}{r} - \frac{1}{\delta+r}\right)\right.\right.$$

$$\left. + e^{-rT}\left(\frac{e^{-\delta T}}{\delta + r} - \frac{e^{-\delta t_1}}{r}\right)\right)$$

$$+ LD\left(e^{-rT}\left(\frac{e^{-\delta T}}{r+\delta} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{e^{-\delta t_1}}{r+\delta}\right)\right) + Ci_P D\left(\frac{1}{\theta}\left(e^{-rM}\left(\frac{e^{\theta(t_1-M)}}{\theta+r} - \frac{1}{r}\right)\right.\right.$$

$$\left.\left. + e^{-rt_1}\left(\frac{1}{r} - \frac{1}{\theta+r}\right)\right)\right)$$

$$\left. - Pi_e\left[D\left(\frac{-Me^{-rM}}{r} - \frac{e^{-rM}}{r^2} + \frac{1}{r^2}\right)\right]\right\} + \frac{1}{T}\left[\frac{sD}{\delta}(e^{\delta t_1}e^{-rT} - e^{-(\delta+r)T} + e^{-rT})\right.$$

$$+ LD(- e^{-(\delta+r)T} + e^{-rT})\Big] = 0 \tag{16}$$

From (15)

$$T = \frac{e^{-\delta t_1}}{SD}\Bigg[D(h+\theta)C\frac{(e^{\theta t_1} - e^{-rt_1})}{\theta + r} + \frac{sDe^{-\delta t_1}(1 - e^{-rt_1})}{r} + LDe^{-\delta t_1}(1 - e^{-rt_1})$$

$$+ Ci_P DM\left(\frac{e^{-rM+\theta(t_1-M)}}{\theta+r} - \frac{e^{-rt_1}}{\theta+r}\right)\Bigg].$$

Substituting these value in 16 and putting $t_1 = T_A$

$$\Delta_4 = \frac{-1}{T_4^2}\Bigg\{A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta T_A} - e^{-rT_A})}{\theta + r} + \frac{(e^{-rT_A} - 1)}{r}\right) + \frac{sD}{\delta}\Bigg(e^{-(\delta+r)T_A}\left(\frac{1}{r} - \frac{1}{\delta + r}\right)$$

$$+ e^{-rT_4}\left(\frac{e^{-\delta T_4}}{\delta + r} - \frac{e^{-\delta T_A}}{r}\right)\Bigg)$$

$$+ LD\left(e^{-rT_4}\left(\frac{e^{-\delta T_4}}{r+\delta} - \frac{1}{r}\right) + e^{-rT_A}\left(\frac{1}{r} - \frac{e^{-\delta T_A}}{r+\delta}\right)\right) + Ci_P D\Bigg(\frac{1}{\theta}\Bigg(e^{-rM}\left(\frac{e^{\theta(T_A-M)}}{\theta + r} - \frac{1}{r}\right)$$

$$+ e^{-rT_A}\left(\frac{1}{r} - \frac{1}{\theta + r}\right)\Bigg)\Bigg)$$

$$- Pi_e\left[D\left(\frac{-Me^{-rM}}{r} - \frac{e^{-rM}}{r^2} + \frac{1}{r^2}\right)\right]\Bigg\} + \frac{1}{T_4}\Bigg[\frac{sD}{\delta}(e^{-\delta T_A}e^{-rT_4} - e^{-(\delta+r)T_4})$$

$$+ LD(-e^{-(\delta+r)T_4} + e^{-rT_4})\Bigg] = 0. \tag{17}$$

Lemma 4.4.

(1) *If* $\Delta_4 \le 0$, *then the total cost has its minimum value* $t_1 > M$.

(2) $\Delta_4 \le 0$, *then the total cost has its minimum value at* $(t_1, T) = (T_A, T_1]$

Proof. It can be easily verified.

Subcase 2.2. $0 \le t_1 \le M \le T$. The essential conditions for minimizing the total cost $TC_{2.2}(t_1, T)$ are $\frac{\partial TC_{2.1}(t_1, T)}{\partial t_1} = 0$ and $\frac{\partial TC_{2.1}(t_1, T)}{\partial T} = 0$.

Thus, let $\eta_5 \dfrac{\partial TC_{2.2}(t_1, T)}{\partial t_1} = 0$ and $\nu_5 \dfrac{\partial TC_{2.2}(t_1, T)}{\partial T} = 0$ yields

$$\eta_5 = \frac{D}{T}\Bigg[\frac{(e^{\theta t_1} - e^{-rt_1})(h+\theta)C}{\theta + r} + \frac{S}{r}(e^{-rT}e^{-\delta t_1} - e^{-(\delta+r)t_1}) + L(e^{-(\delta+r)t_1} - e^{-rt_1})$$

$$- Pi_e(2t_1 e^{-rt_1} + \frac{1}{r}(e^{-rM} - e^{-rt_1}))\Bigg] = 0 \quad (18)$$

$$\nu_5 = -\frac{1}{T^2}\begin{bmatrix} A + \dfrac{D(h+\theta)C}{\theta}\left(\dfrac{(e^{\theta t_1} + re^{-rt_1})}{\theta + r} - \dfrac{(e^{-rt_1} - 1)}{r}\right) \\ + \dfrac{sD}{\delta}\left(e^{-(\delta+r)t_1}\left(\dfrac{1}{r} - \dfrac{1}{\delta + r}\right) + e^{-rT}\left(\dfrac{e^{-\delta T}}{\delta + r} - \dfrac{e^{-\delta t_1}}{r}\right)\right) \\ + LD\left(e^{-rT}\left(\dfrac{e^{-\delta T}}{r+\delta} - \dfrac{1}{r}\right) + e^{-rt_1}\left(\dfrac{1}{r} - \dfrac{e^{-\delta t_1}}{r+\delta}\right)\right) \\ - DPi_e\left[\dfrac{-t_1}{r}e^{-rt_1} + \dfrac{(1 - e^{-rt_1})}{r^2} + \dfrac{t_1}{r}(e^{-rM} - e^{-rt_1})\right] \end{bmatrix}$$

$$+ \frac{1}{T}\left[\frac{sD}{\delta}(e^{-\delta t_1}e^{-rT} - e^{-(\delta+r)T}) + LD(e^{-(\delta+r)T} + e^{-rT})\right] \quad (19)$$

$$T = \frac{e^{-\delta t_1}}{SD}\Bigg[D(h+\theta)C\frac{(e^{\theta t_1} - e^{-rt_1})}{\theta + r} + \frac{sDe^{-\delta t_1}(1 - e^{-rt_1})}{r} + LDe^{-\delta t_1}$$

$$(1 - e^{-rt_1}) - DPi_e(2t_1 e^{-rt_1} + \frac{1}{r}(e^{-rM} - e^{-rt_1}))\Bigg] = 0.$$

This value substituting in (19)

$$\nu_5 = -\frac{1}{T_5^2}\left[\begin{array}{l} A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta t_1} + e^{-rt_1})}{\theta + r} + \frac{(e^{-rt_1} - 1)}{r}\right) \\ + \frac{sD}{\delta}\left(e^{-(\delta+r)t_1}\left(\frac{1}{r} - \frac{1}{\delta + r}\right) + e^{-rT_5}\left(\frac{e^{-\delta T_5}}{\delta + r} - \frac{e^{-\delta t_1}}{r}\right)\right) \\ + LD\left(e^{-rT_5}\left(\frac{e^{-\delta T_5}}{r+\delta} - \frac{1}{r}\right) + e^{-rt_1}\left(\frac{1}{r} - \frac{e^{-\delta t_1}}{r+\delta}\right)\right) \\ - DPi_e\left[\frac{-t_1}{r}e^{-t_1} + \frac{(1 - e^{-rt_1})}{r^2} + \frac{t_1}{r}(e^{-rM} - e^{-rt_1})\right] \end{array}\right]$$

$$+ \frac{1}{T_5}\left[\frac{sD}{\delta}(e^{-\delta t_1}e^{-rT_5} - e^{-(\delta+r)T_5}) + LD(-e^{-(\delta+r)T_5} + e^{-rT_2}\right]. \quad (20)$$

Again putting $t_1 = M$ and $t_1 = T_a$ in equation (20), we get

$$\Delta_5 = -\frac{1}{T_5^2}\left[\begin{array}{l} A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta M} - re^{-rM})}{\theta + r} + \frac{(e^{-rt_1} - 1)}{r}\right) \\ + \frac{sD}{\delta}\left(e^{-(\delta+r)M}\left(\frac{1}{r} - \frac{1}{\delta + r}\right) + e^{-rT_5}\left(\frac{e^{-\delta T_5}}{\delta + r} - \frac{e^{-\delta M}}{r}\right)\right) \\ + LD\left(e^{-rT_5}\left(\frac{e^{-\delta T_5}}{r+\delta} - \frac{1}{r}\right) + e^{-rM}\left(\frac{1}{r} - \frac{e^{-\delta M}}{r+\delta}\right)\right) \\ - DPi_e\left[\frac{-M}{r}e^{-rM} + \frac{(1 - e^{-rM})}{r^2}\right] \end{array}\right]$$

$$+ \frac{1}{T}\left[\frac{sD}{\delta}(e^{-\delta M}e^{-rT_5} - e^{-(\delta+r)T_5} + LD(-e^{-(\delta+r)T_5} + e^{-rT_5})\right]. \quad (21)$$

$$\Delta_6 = -\frac{1}{T_5^2}\left[\begin{array}{l} A + \frac{D(h+\theta)C}{\theta}\left(\frac{(e^{\theta T_A} - e^{-rT_A})}{\theta + r} - \frac{(e^{-rT_A} - 1)}{r}\right) \\ + \frac{sD}{\delta}\left(e^{-(\delta+r)T_A}\left(\frac{1}{r} - \frac{1}{\delta + r}\right) + e^{-rT_5}\left(\frac{e^{-\delta T_5}}{\delta + r} - \frac{e^{-\delta T_A}}{r}\right)\right) \\ + LD\left(e^{-rT_5}\left(\frac{e^{-\delta T_5}}{r+\delta} - \frac{1}{r}\right) + e^{-rT_A}\left(\frac{1}{r} - \frac{e^{-\delta T_A}}{r+\delta}\right)\right) \\ - DPi_e\left[\frac{-T_A}{r}e^{-rt_1} + \frac{(1 - e^{-rT_A})}{r^2} + \frac{T_A}{r}(e^{-rM} - e^{-rT_A})\right] \end{array}\right]$$

$$+\frac{1}{T}\left[\frac{sD}{\delta}(e^{-\delta T_A}e^{-rT_5}-e^{-(\delta+r)T_5})+LD(-e^{-(\delta+r)T_5}+e^{-rT_5})\right] \tag{22}$$

consequently, the following lemma is obtained.

Lemma 4.5.

(1) *If* $\Delta_5 \leq 0 \leq \Delta_6$, *then the total cost has its minimum value* $t_1 < M < T$.

(2) *If* $\Delta_5 > 0$, *then the total cost has its minimum value* $(t_1, T) = (M, T_5]$.

(3) *If* $\Delta_6 < 0$, *then the value of* $t \in (M, T_A)$ *minimizes* $TC_{2.2}(t_1, T)$.

Proof. It can be easily varified.

Subcase 2.3. $0 \leq t_1 \leq T \leq M$.

In this sub-case, the total cost is identical to the total cost of subcase-2.2 and the necessary condition for this subcase is the same as that equations (18) and (19). The following theorem is derived, based on lemmas.

Lemma 4.6.

(1.) *If* ≥ 0, *then the total cost has its minimum value at* $< M$.

(2.) *If* < 0, *then the total cost has its minimum value at*.

Theorem. *The following condition holds for* $q \geq q_d$.

Condition	$TC^*(t_1, T)$	t_1^*, T^*
$\Delta_6 < 0$	$TC^*(t_1, T) = \min$ $\{TC_{2.1}(t_1, T_1), TC_{2.3}(M, T_3)\}$	$t_1^*, T^* = T_1$ or T_3
$\Delta_4 > 0, \Delta_5 < 0$ and $\Delta_6 \geq 0$	$TC^*(t_1, T) = \min$ $\{TC_{2.1}(t_1, T_2), TC_{2.2}(t_1, T_4)\}$	$t_1^*, T^* = T_1$ or T_4
$\Delta_4 < 0$ and $\Delta_7 \geq 0$	$TC^*(t_1, T) = \min$ $\{TC_{2.1}(t_1, T_1), TC_{2.3}(t_1, T_3)\}$	$t_1^*, T^* = T_1$ or T_3

$\Delta_4 \geq 0, \Delta_5 < 0$ and $\Delta_6 < 0$	$TC^*(t_1, T) = \min \{TC_{2.1}(T_A, T_1), TC_{2.2}(t_1, T_4)\}$	$T^* = T_1$ or
$\Delta_4 \geq 0$ and $\Delta_7 \geq 0$	$TC^*(t_1, T) = \min \{TC_{2.1}(T_A, T_1), TC_{2.3}(t_1, T_3)\}$	$T^* = T_1$ or T_3

Step 1. put the values of all parameters.

Step 2. Compare between q and q_d if $q < q_d$, then go to step 3 otherwise go to step 4.

Step 3. Find Δ_1, Δ_2 and Δ_3 from equations (10), (14) and (18) respectively.

(I) If $\Delta_1 \leq 0, \Delta_3 < 0$, then $TC^*(t_1, T) = TC_{1.1}(t_1, T)$ and $T^* = T_1$ (by theorem 1) Otherwise attend step 5.

(II) If $\Delta_1 \leq 0$ but $\Delta_3 \geq 0$, then $TC^*(t_1, T) = \min\{TC_{1.1}(t_1, T_1), TC_{1.3}(t_1, T_3)\}$ and $T^* = T_1$ or T_3 (by theorem 1). Otherwise, attend step 5.

(III) If $\Delta_1 > 0, \Delta_3 \geq 0$ but $\Delta_2 < 0$ then $TC^*(t_1, T) = \min\{TC_{1.2}(t_1, T_2), TC_{1.3}(t_1, T_3)\}$ and $T^* = T_1$ or T_3 (by theorem 1). Otherwise, attend step 5.

(IV) If $\Delta_2 > 0, \Delta_3 \geq 0$, then $TC^*(t_1, T) = \min\{TC_{1.2}(T_A, T_2), TC_{1.3}(t_1, T_3)\}$ and $T^* = T_1$ or T_3 (by theorem 1). Otherwise, attend step 5.

(V) If $\Delta_2 > 0, \Delta_1$ but $\Delta_3 < 0$, then $TC^*(t_1, T) = TC_{1.2}(t_1, T_2)$ and $T^* = T_2$ (by theorem 1). Otherwise, attend step 5.

Step 4. Calculated Δ_4, Δ_5, and Δ_6 from equations (19), (23), (24) respectively.

(1) If $\Delta_6 < 0$, then $TC^*(t_1, T) = \min\{TC_{2.1}(t_1, T_1), TC_{2.3}(M, T_3)\}$ and $T^* = T_1$ or T_3 (by theorem 2). Otherwise, go to step 5.

(2) If $\Delta_4 > 0, \Delta_5 < 0$ and $\Delta_6 \geq 0$, then $TC^*(t_1, T) = \min\{TC_{2.1}(t_1, T_2),$ $TC_{2.2}(t_1, T_4)\}$ and $T^* = T_1$ or T_4 (by theorem 2). Otherwise, attend step 5.

(3) If $\Delta_4 < 0$ but $\Delta_6 \geq 0$, then $TC^*(t_1, T) = \min\{TC_{2.1}(t_1, T_1),$ $TC_{2.3}(t_1, T_3)\}$ and $T^* = T_1$ or T_3 (by theorem 1). If not attend step 5.

(4) If $\Delta_4 \geq 0, \Delta_5, \Delta_6 < 0$, then $TC^*(t_1, T) = \min\{TC_{2.1}(T_A, T_1), TC_{2.2}(t_1, T_4)\}$ and $T^* = T_1$ or T_4 (by theorem 2) or else attend step 5.

(5) If $\Delta_4, \Delta_6 \geq 0$, then $TC^*(t_1, T) = \min\{TC_{2.1}(T_A, T_1), TC_{2.3}(t_1, T_3)\}$ and $T^* = T_1$ or T_3 [by theorem 2] if not go to step 5.

Step 5. If the condition is not of the form of the above four cases then stop the process and produce the best solution. Similarly, we obtained the condition for $q \leq q_d$.

5. Numerical Analysis

To describe the process for obtaining the optimal solution, some representative examples are solved.

Example 1. Let $D = 1500$ units / year, $A = \$120$/ order, $P = \$15$/ item, C = \$10/item, M = .15 years, r = .065, s = \$/unit/year, L = \$10/unit/year, h = 0.0035, $\theta = 0.5$, $\delta = 0.5$ $i_P = 12\%$/ year, $i_e = 6\%$/ year, $q_d = 300$units, $\beta = 0.8$.

Sol. We find the optimal solution with the help of mathematica $q^* = 297.201 < q_{d,}$ $S^* = 13.7195$, $t_1^* = 0.180584$, $T^* = 0.19062 = T_1$ and $TC^*(t_1, T) = TC_{2.1}(t_1, T) = 1348.73$.

Example 2. Let $D = 1500$ units / year, $A = \$120$/ order, $P = \$15$/ item, C = \$10/item, M = .15 years, r = .065, s = \$15/unit/year, L = \$10/unit/year, h = 0.0035, $\theta = 0.5$, $\delta = 0.5 i_P = 12\%$/ year, $i_e = 6\%$/ year, $q_d = 300$units, $\beta = 0.8$.

Sol. We find the optimal solution with the help of mathematica. $q^* = 290.973 < q_{d,}$ $S^* 57.3205$, $t_1^* = 0.15$, $T^* = 0.19162 = T_1$ and $TC^*(t_1, T) = TC_{2.1}(t_1, T) = 1411.71$.

6. Sensitivity Analysis

Sensitivity analysis is given w.r.t. some parameters.

		t_1	T	TC	Q	St
	90	0.15	0.19162	1255.15	290.973	57.3205
A	120	0.15	0.19162	1411.71	290.973	57.3205
	150	0.15	0.19162	1568.27	290.973	57.3205
	180	0.15	0.19162	1724.83	290.973	57.3205
M	10/365	0.15	0.19162	1557.6	290.973	57.3205
	20/365	0.15	0.19162	1538.71	290.973	57.3205
	30/365	0.15	0.19162	1519.87	290.973	57.3205
	40/365	0.15	0.19162	1501.08	290.973	57.3205
	0.4	0.15	0.19162	1319.53	289.208	57.3205
θ	0.5	0.15	0.19162	1411.71	290.973	57.3205
	0.6	0.15	0.19162	1504.82	292.756	57.3205
	0.7	0.15	0.19162	1598.88	294.558	57.3205
	0.055	0.15	0.19162	1412.47	290.973	57.3205
r	0.065	0.15	0.19162	1413.71	290.973	57.3205
	0.075	0.15	0.19162	1414.95	290.973	57.3205
	0.085	0.15	0.19162	1415.18	290.973	57.3205
	10	0.15	0.19162	1380.83	290.973	57.3205
s	15	0.15	0.19162	1411.71	290.973	57.3205

	20	0.15	0.19162	1442.58	290.973	57.3205
	25	0.15	0.19162	1473.46	290.973	57.3205
	10	0.15	0.19162	1411.71	290.973	57.325
C	15	0.15	0.19162	1665.16	290.973	57.325
	20	0.13668	0.19162	1904.82	288.107	75.9187
	25	0.121848	0.19162	2089.93	285.229	96.7743
	10	0.15	0.19162	1499.71	290.973	57.3205
P	15	0.15	0.19162	1473.46	290.973	57.3205
	20	0.15	0.19162	1447.21	290.973	57.3205
	25	0.15	0.19162	1420.96	290.973	57.3205
	0.4	0.15	0.19162	1362.48	291.96	58.3077
	0.5	0.15	0.19162	1411.71	290.973	57.3205
δ	0.6	0.15	0.19162	1460.1	290.003	56.3502
	0.7	0.15	0.19162	1507.6	289.049	55.3964

From Table 3, It is noted that worth discussing conditions that will bear practical effect. Some implications are given below:

1. An increase in the length of the credit period the total cost $TC(t_1, T)$ decrease but the most favorable backorder level S^*, the most favorable order quantity q^*, the optimal refill cycle time T^* and the inventory level time to reach zero (t_1^*) remain unchanged.

2. An increase in the selling price P results in slight changes of the total optimal cost $TC(t_1, T)$ decrease but the most favorable backorder level S^*, the most favorable order quantity q^*, cycle time T^* and the inventory level time to reach zero (t_1^*) remain unchanged.

3. As the unit cost (C) increases than most favorable order quantity q^* and the inventory level time to reach zero (t_1^*) are decrease but the optimal replenishment cycle time T^* remains unchanged and total optimal cost $TC(t_1^*, T^*)$ and the most favorable backorder level S^*, are increases.

4. As the deterioration rate (θ) increases, shortage quantity S^*, cycle T^* and Inventory level time to reach zero (t_1^*) remain unchanged but the total optimal cost $TC(t_1^*, T^*)$ and the most favorable order quantity q^* are increases.

5. The replenishment cost (A) increases, then the inventory level time to reach zero (t_1^*), cycle time T^*, and the most favorable backorder level S^*, the most favorable order quantity q^* remains unchanged but the total optimal cost $TC(t_1^*, T^*)$ increase.

6. The partial backlogging parameter (δ) increases, then total optimal cost $TC(t_1, T)$ increase but the most favorable backorder level S^*, the most favorable order quantity q^* are decrease but cycle time T^* and the inventory level time to reach zero (t_1^*) remain unchanged.

7. An increase in the shortage cost (S), then total cost $TC(t_1, T)$ increase but the most favorable backorder level S^*, cycle time T^*, the most favorable order quantity q^* and the inventory level time to reach zero (t_1^*) remain unchanged.

8. The inflation rate r increases, the total cost $TC(t_1, T)$ increase but the most favorable backorder level S^*, the most favorable order quantity q^*, the inventory level time to reach zero (t_1^*) and cycle time T^* remain unchanged.

7. Conclusion

Here, in this paper, we have tried to study a real problem (deterioration, backlogging, and trade credits) in the context of retail sector, where sales data is abundant, product challenges exist, and business credit policy optimist is not there. In particular, we have built an instructive inventory model, in which the characteristics of retail characteristics are described as to how better the supply chain can be realized through better information about the importance of parameters and the effective effectiveness of the inventory coordination. The meaning of this paper is to determine the optimum sequence and backlog policies for retailers who want to reduce the total cost per unit time. The results show that if the retailer's order quantity is more than the minimum amount, then partial payment delays are allowed. If minimum amount ( ) is large, then the retailer will prefer a partial permissible delay in payment. The retailer will pay a partial amount on receipt of the goods and there will be a grace period for the payment of the remaining payments. Apart from this, it was shown that (1) the total cost of the retailer increases with the ordering cost; (2) a high reduction cost increases the total cost but reduces the level of the back; and (3) the expansion of an extended delay leads to a lower total cost.

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