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GRAPH THEORETIC PROPERTIES FOR $\Gamma_{pm}(R)$ AND RESISTANCE DISTANCE BASED INDICES

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Abstract

Let R be a finite commutative ring with unity and $M_1, M_2, \ldots, M_r$ be the maximal ideals of R. The product maximal graph whose vertices are all the elements of R and two different vertices x and y are adjacent if and only if the product $xy \in M_i$, $i = 1, 2, \ldots, r$. In this paper, the concept of rank, girth, domination, coloring, connectivity, planarity, Hamiltonian of the product maximal graph are interpreted. Moreover, some topological indices using resistance distance matrix for $\Gamma_{pm}(R)$ are explored.

1. Introduction

Associating a graph with algebraic properties is an active research focus in algebraic graph theory and has concerned considerable attention. This is an area of mathematics in which methods of abstract algebra are employed in studying various graph invariants and tools in graph theory. For further notations we follow Dummit and Foote [3] for algebra and Frank Haray [5] for graph theory.

In 1988, being the first to associate a graph to a ring. Beck [1] introduced and studied the zero divisor graph of a commutative ring R. This graph turns out to best exhibit the properties of the set of zero-divisors and other related properties of a commutative ring. The zero-divisor graph translate the

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algebraic properties of a ring to graph theoretical tools, thus help in exploring interesting results in both graph theory and abstract algebra. Since then the zero divisor graph as well as many other graphs associated with rings have been extensively studied.

The spectral graph theory is the study of the eigenvalues of different matrices associated with graphs. The spectrum of the adjacency matrix can tell us the number of vertices, the number of edges and the number of closed walks of any length. It can also tell us whether or not a graph is bipartite, regular and, if it is regular, whether or not it is connected and what its girth is. The spectrum of the Laplacian matrix of a graph determines the number of vertices, the number of edges, the number of connected components and spanning trees as well as whether the graph is regular, and if it is regular, its girth. Graph properties such as diameter, chromatic number, independence number, clique number and connectivity are all related to a graph's spectrum too. For an arbitrary graph G, a subset $S \subset V(G)$ is said to be a cut-set if there exist distinct vertices u and $v \in V(G) - S$ such that every path in G from u and v involves at least one element of S. A cut-set of minimum cardinality is called a minimum cut-set of G and the cardinality of minimum cut-set is called the vertex connectivity of G, denoted by $\kappa(G)$.

Domination is most prominent area of graph theory. The course of domination was inducted by Ore and Claud Berge. A broad introduction to "Domination in graph" by Haynes, Hedetniemi and Slate. The generalized k-connectivity $\kappa_k(G)$ of a graph G was introduced by Chartrand et al. in 1984. The connectivity is one of the most basic concepts of graph-theoretic subjects, both in a combinatorial sense and an algorithmic sense. Graph connectivity theory are essential in network applications, routing transportation network, network tolerance etc., Graph coloring is a special case of graph labeling [10] it is assignment of labels traditionally called 'colors' to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring vertices of a graph such that no two adjacent vertices are of the same color; this is called vertex coloring. Similarly, an edge coloring assigns a color to each edge so that no two adjacent edges are of the same color.

A graph G is said to be planar if it can be drawn in the plane so that no two edges of G intersect at a point other than a vertex. Such a drawing of a planar graph is called a planar embedding of the graph. A Hamiltonian graph may be defined as: if there exists a closed walk in the connected graph that visits every vertex of the graph exactly once without repeating edges, then such a graph is called Hamiltonian graph.

The distance matrix is one of the matrix representations of graphs in algebraic graph theory. It is defined in a similar way as the adjacency matrix. The distance matrix has several applications not only in telecommunication, but also in chemistry. Several topological indices [6, 8, 9], which characterise the molecular graph of chemical compounds, have been derived from the study of the distance matrix. The concept of resistance distance originated from electrical circuit theory. If we view G as an electrical network N by replacing each edge of G with unit resistors. Then the resistance distance between v_i and v_j denoted by Ω_{ij}, is defined as the net effective resistance between the corresponding nodes in the electrical networks N.

The resistance distance matrix was introduced in 1993 by Klein and Randic [4]. They used from concept from the theory of resistive electrical networks (Kirchhoff's law) and the theory of graphs. A merging of concepts from these two theories was achieved by viewing an electrical network as a connected graph, such that the vertices of the graph corresponds to the junctions in the electrical network and the edges of the graph to unit resistors. Then the effective resistance between pairs of vertices is a graphical distance.

This paper concerned the product maximal graph. In 2021, the product maximal graph was established by D. Kalamani and G. Ramya [7] for the finite commutative ring with unity. In [11], they describe the domination number for the product maximal graph.

In this present report we describe the Weiner index [2], Hyper Weiner index, Resistance distance matrix [12], Kirchhoff's index, Kirchhoff's sum index for the product maximal graph. Also we discussed properties like rank, nullity, girth, connectivity, planarity, Hamiltonian and edge domination number for the product maximal graph.

2. Preliminaries

In this section, the essential definitions of the domination, coloring, connectivity, rank, nullity, girth, resistance distance and some topological indices are specified.

Definition 2.1. The rank of a graph G, denoted by $\rho(G)$, is the number of non-zero eigenvalues of the adjacency matrix A of the graph G. The *nullity* of a graph is defined as the multiplicity of the eigenvalue zero in the spectrum of the adjacency matrix of the graph. It is denoted as $\eta(G)$. Consider A be a real and symmetric matrix then, $\dim A = \rho(A) + \eta(A)$.

Definition 2.2. The *girth* of a graph G, denoted by $Girth(G)$ is the length of a shortest cycle in G.

Definition 2.3. A set of edges D of $G(V, E)$ is called an *edge dominating set* if every edge of $E - D$ is adjacent to an element of D. The minimum cardinality of an edge dominating set is called an *edge domination number* and it is denoted by $\gamma'(G)$.

Definition 2.4. For $k \geq 1$ an integer, a *k-fair dominating set*, abbreviated kFD-set is a dominating set D such that $|N(v) \cap D| = k$ for every vertex v in $V - D$ where $N(v)$ is the open neighbourhood of v. The minimum cardinality of the k-fair dominating set is called *k-fair dominating number* and it is denoted by $\gamma_{kFD}(\Gamma_{pm}(R))$.

Definition 2.5. A dominating set D of a graph is said to be an *annihilator dominating set*, if its induced subgraph $V - D$ is a graph containing only isolated vertices. The minimum cardinality of an annihilator dominating set is called an annihilator domination number.

Definition 2.6. The *chromatic index* of a graph is the minimum number of colors needed to color the edges of graphs so that no two adjacent edges share the same color. It is denoted by $\chi'(G)$.

Definition 2.7. The (*vertex*) *connectivity* $\kappa(G)$ of an undirected graph or digraph is the smallest number of vertices whose deletion separates or trivializes the graph. The minimum number of edges whose removal makes G disconnected is called *edge connectivity* $\kappa'(G)$ of G.

Definition 2.8. A connected graph G is called *Hamiltonian* graph if there is a cycle which includes every vertex of G and the cycle is called Hamiltonian cycle.

Definition 2.9. Suppose G is a connected graph with the set of vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$ and d_{ij} represent the length of the shortest path between v_i and v_j. Then the *distance matrix* of G, denoted by $D(G)$, as an $n \times n$ matrix whose $(i, j)^{\text{th}}$ entry is d_{ij}. It is real symmetric and also has trace equal to zero.

Definition 2.10. The *Wiener index* of a graph $G = (V, E)$, denoted by $W(G)$, was introduced in 1947 by chemist Harold Wiener as the sum of distances between all vertices of G.

$$W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}$$

whereas the *Hyper Weiner index* of a graph G is

$$WW(G) = \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} + d_{ij}^2\right]$$

Definition 2.11. The *resistance distance* between vertices i and j, denoted by Ω_{ij}, is defined to be the effective electrical resistance between them if each edge of G is replaced by a unit resistor. A famous distance-based topological index as the *Kirchhoff index*,

$$Kf = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\Omega_{ij})$$

is defined as the sum of resistance distances between all pairs of vertices in G.

Definition 2.12. The *Kirchhoff sum index* of a graph G is defined as

$$KSf = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Omega_{ij}/d_{ij}$$

where Ω_{ij} is an elements in resistance distance matrix and d_{ij} is an elements of distance matrix.

3. Product Maximal Graph

In this section, we attempt an enormous result for the product maximal graph. Throughout this paper, n is the number of vertices, m is the number of elements in M. The product maximal graph is a connected graph with n vertices and $mC_2 + m(n-m)$ edges. We begin with our definition of the product maximal graph.

Definition 3.1 [7]**.** The Product maximal graph of a commutative ring R is abbreviated as $\Gamma_{pm}(R)$ where R be the finite commutative ring with 1(identity). Let $M_1, M_2, \ldots, M_r$ be the maximal-ideals of R. The product maximal graph whose vertices are all the elements of R and two different vertices x and y are adjacent if and only if the product $xy \in M_i$, $i = 1, 2, \ldots, r$.

3.1. Rank, Nullity and Girth of the product maximal graph

In this section, we have discussed rank using characteristics polynomial, nullity and also girth of the product maximal graph.

Theorem 3.1. *Let R be the finite commutative ring with unity. Then the rank of $\Gamma_{pm}(R)$ is $\rho(\Gamma_{pm}(R)) = m+1$ where m is the number of elements in $M = \bigcup_{i=1}^{r} M_i$ and M_i's are the maximal ideal.*

Proof. Let $\Gamma_{pm}(R)$ be the product maximal graph with n vertices. Then the adjacency matrix $A = [a_{ij}]$ of $\Gamma_{pm}(R)$ is the real symmetric matrix where

$$a_{ij} = \begin{cases} 1 \text{ if } v_i \in M \text{ or } v_j \in M \text{ for } i \neq j \\ \quad 0 \text{ otherwise} \end{cases}, 1 \leq i, j \leq n.$$

One of the eigenvalue of A is zero, since A is singular.

So, $\rho(\Gamma_{pm}(R)) < n$.

Next we find the remaining $(n-1)$ eigenvalues of the adjacency matrix A of $\Gamma_{pm}(R)$.

The characteristic equation of the product maximal graph is $|A - \lambda I| = 0$ where λ is the eigenvalues of A and I is the nth order unit matrix.

On expanding the above determinant, we get

$$\lambda^n - k_1\lambda^{n-1} + k_2\lambda^{n-2} - \ldots + (-1)^n k_n = 0.$$

where k_i is the sum of the minors of $(n - i)$ diagonal elements, $1 \le i \le n - 1$ and $k_n = |A|$.

For the adjacency matrix A of the product maximal graph,

$$k_1 = 0,\ k_n = |A| = 0$$

$$k_i = 0,\ \forall i > m + 1$$

$$\lambda^n - k_2\lambda^{n-2} - k_3\lambda^{n-3} - \ldots - k(m + 1)\lambda^{n-(m+1)} = 0.$$

Solving the characteristic equation, it is clear that, there are $m + 1$ number of non-zero eigenvalues of $\Gamma_{pm}(R)$.

Hence the rank of the matrix A is $m + 1$. This means that the rank of the product maximal graph $\rho(\Gamma_{pm}(R)) = m + 1$. □

The result established in theorem 3.1 can be generalized the nullity of the product maximal graph.

Corollary 3.2. *The nullity of the product maximal graph* $\Gamma_{pm}(R)$ *is* $\eta(\Gamma_{pm}(R)) = n - m - 1$.

Proof. From theorem 3.1, the number of nonzero eigenvalue of $\Gamma_{pm}(R)$ is $m + 1$. Therefore, the number of zero eigenvalues of $\Gamma_{pm}(R)$ is $n - m - 1$. □

For instance, the rank of the product maximal graph is $\rho(\Gamma_{pm}(Z_6)) = 4$ and nullity is $\eta(\Gamma_{pm}(Z_9)) = 5$ from figure 1.

Theorem 3.3. *For the product maximal graph,* $Girth(\Gamma_{pm}(R)) = 3$.

Proof. The girth of the complete graph K_n is 3 for all $n \ge 3$.

In the product maximal graph, the vertices which are in the maximal ideals form the complete subgraph.

Therefore $Girth(\Gamma_{pm}(R)) = 3.$ □

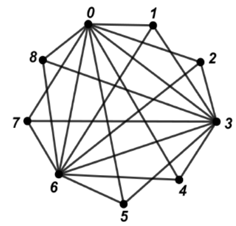

Figure 1. Product maximal graph $\Gamma_{pm}(Z_9)$.

3.2 Dominations for the product maximal graph

In this section, we explored some properties of dominations of the product maximal graph. In [11], they investigated the domination number and some parameters of the product maximal graph. Initially, we show that the edge domination number of $\Gamma_{pm}(R)$.

Theorem 3.4. *Let R be a finite commutative ring with unity and $M_1, M_2, \ldots, M_r$ be the maximal ideals of R. Then the edge domination for the product maximal graph $\Gamma_{pm}(R)$ is*

$$\gamma'(\Gamma_{pm}(R)) = \begin{cases} \dfrac{m}{2}, & m \text{ is even} \\ \left\lceil \dfrac{m}{2} \right\rceil, & m \text{ is odd} \end{cases}$$

where m is the number of elements in $M = \bigcup_{i=1}^{r} M_i$ and $\left\lceil \frac{m}{2} \right\rceil$ is the ceil which round sup the nearest integer of $\frac{m}{2}$.

Proof. Let $V(\Gamma_{pm}(R))$ and $E(\Gamma_{pm}(R))$ be the vertex set and edge set of the product maximal graph respectively.

Let E_1 be the set of edges whose both end vertices are in maximal ideals. Therefore $|E_1| = mC_2$.

Let E_2 be the set of edges whose one end vertex in maximal ideal and other end is not in maximal ideals. Therefore $| E_2 | = m(n - m)$.

Let E_3 be the set of edges whose both end vertices which are not in maximal ideals. Obviously the element which are not in maximal ideals form the co-clique. Therefore $| E_3 | = 0$.

Let D be the non-empty subset of the edge set of the product maximal graph.

Claim: D is the edge dominating set.

Case 1. Let $D \subseteq E_1$

Then the subgraph $E(\Gamma_{pm}(R)) - D$ whose edges are all the neighbour of D. Thus D is the edge dominating set. Therefore the number of dominating set in E_1 is

$$\begin{cases} \left\lceil \frac{m}{2} \right\rceil \leq | D | \leq mC_2, & \text{if } m \text{ is odd} \\ \frac{m}{2} \leq | D | \leq mC_2, & \text{if } m \text{ is even} \end{cases} \tag{1}$$

Case 2. Let $D \subseteq E_2$

Then the subgraph $E(\Gamma_{pm}(R)) - D$ whose edges are all the neighbour of D. Thus D is the edge dominating set. Therefore the number of dominating set in E_2 is

$$\begin{cases} m \leq | D | \leq m(n - m), & \text{if } m \text{ is odd} \\ \frac{m}{2} \leq | D | \leq m(n - m), & \text{if } m \text{ is even} \end{cases} \tag{2}$$

Case 3. Let $D - E_1 \cup E_2$

Then the subgraph $E(\Gamma_{pm}(R)) - D$ edges are all the neighbour of D. Thus D is the edge dominating set.

$$| D | = mC_2 + m(n - m) \tag{3}$$

From (1), (2) and (3), we get

The minimum cardinality of the edge dominating set is

$$\gamma'(\Gamma_{pm}(R)) = \begin{cases} \dfrac{m}{2}, & m \text{ is even} \\ \left\lceil \dfrac{m}{2} \right\rceil, & m \text{ is odd} \end{cases}$$ □

For example, in figure 1, the edge domination number for the product maximal graph $\gamma'(\Gamma_{pm}(Z_9)) = 2$.

The following results are the parameter of the vertex domination of the product maximal graph.

Here $|\overline{M}|$ is the number of elements which are not in maximal ideals.

Theorem 3.5. *If $\Gamma_{pm}(R)$ is the product maximal graph, then the k-fair domination number for $\Gamma_{pm}(R)$ is*

$$\gamma_{kFD}(\Gamma_{pm}(R)) = \begin{cases} m, & \text{if } |\overline{M}| \geq |M| \\ n - m, & \text{if } |\overline{M}| \leq |M| \end{cases}$$

Proof. Let D be the dominating set of $\Gamma_{pm}(R)$. We prove that the dominating set D is the k-fair dominating set.

By [11], the set of all elements which are in the maximal ideals is the connected dominating set. Also, the elements which are not in maximal ideals is an independent dominating set.

If $D = M$, then $|N(v) \cap D| = m$ for every vertex v in $V(\Gamma_{pm}(R)) - D$.

If $D = \overline{M}$, then $|N(v) \cap D| = n - m$ for every vertex v in $V(\Gamma_{pm}(R)) - D$.

It concludes that, the minimum cardinality of the k-fair dominating set depends upon the number of elements which are in the maximal ideals and the non-maximal ideals.

Therefore, $\gamma_{kFD}(\Gamma_{pm}(R)) = \begin{cases} m, & \text{if } |\overline{M}| \geq |M| \\ n - m, & \text{if } |\overline{M}| \leq |M| \end{cases}$ □

For instance, the k-fair domination number for $\Gamma_{pm}(Z_9)$ is 3 from figure 1.

Theorem 3.6. *Let R be a finite commutative ring with unity, then the annihilator domination number of the product maximal graph is* $\gamma_a(\Gamma_{pm}(R)) = m$.

Proof. Assume that $\Gamma_{pm}(R)$ be the product maximal graph, then by [7] it contains complete subgraph K_m and an independent subgraph I_{n-m}.

Let D be the dominating set of $\Gamma_{pm}(R)$.

Claim: D is an annihilator dominating set.

If $D = M$, then the induced subgraph $V(\Gamma_{pm}(R)) - D$ of $\Gamma_{pm}(R)$ is a graph with isolated vertices.

Therefore, the minimum cardinality of an annihilator dominating set is the annihilator domination number.

$$\therefore \gamma_a(\Gamma_{pm}(R)) = m. \quad \square$$

For example, the annihilator domination number of the product maximal graph is $\gamma_a(\Gamma_{pm}(Z_9)) = 3$ from figure 1.

3.3 Coloring, connectivity of the product maximal graph

In this section, we find the edge coloring and connectivity of the product maximal graph.

Generally, the chromatic index for complete graph K_n is either $n-1$ if n is even or n if n is odd. Also the chromatic index for complete bipartite $K_{m,n} = \max\{m, n\}$. Based on this, we prove that the chromatic index of $\Gamma_{pm}(R)$.

Theorem 3.7. *Let* $\Gamma_{pm}(R)$ *be the product maximal graph of a finite commutative ring, then the chromatic index*

$$\chi'(\Gamma_{pm}(R)) = \begin{cases} n, & \text{if } m \text{ is odd} \\ 2m-1, & \text{if } m \text{ is even} \end{cases}.$$

Proof. Let n be the number of vertices in $\Gamma_{pm}(R)$ and m be the number of elements in $M = \bigcup_{i=1}^{r} M_i$ where M_i's are the maximal ideals.

Let E_1 and E_2 be the set of edges which is defined in theorem 3.3 (edge domination theorem) Then the set of edges E_1 form a complete subgraph, so we can color the edges of E_1 with either m colors if m is odd or $m - 1$ colors if m is even (1)

Next, the set of edges E_2 form a complete bipartite subgraph, therefore we can color the edges of E_2 with $\max\{m, n - m\}$ colors (2)

From (1) and (2), we get

The chromatic index of the product maximal graph $\Gamma_{pm}(R)$ is

$$\chi'(\Gamma_{pm}(R)) = \begin{cases} n, & \text{if } m \text{ is odd} \\ 2m - 1, & \text{if } m \text{ is even} \end{cases}. \qquad \square$$

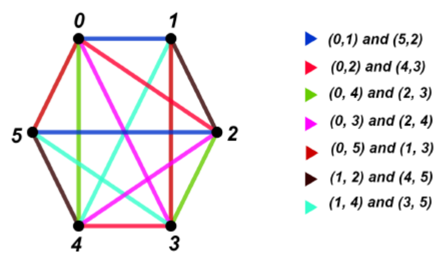

Figure 2. Edge coloring for the product maximal graph $\Gamma_{pm}(Z_6)$.

For example, in figure 2, the chromatic index for the product maximal graph $\gamma'(\Gamma_{pm}(Z_6)) = 7$. By removing the centre of the graph we get the vertex connectivity whose vertices are the minimum eccentricity.

Theorem 3.8. *The vertex connectivity of the product maximal graph is* $\kappa(\Gamma_{pm}(R)) = m$ *where* $m = |M|$.

Proof. If $v_1, v_2, \ldots, v_n$ are the vertices of the product maximal graph, then the eccentricity of v_i to all the other vertices is

$$ecc(v_i) = \begin{cases} 1, & \mathit{if}\ v_i \in M \\ 2, & \mathit{if}\ v_i \notin M \end{cases}$$

We know that the centre of the product maximal graph is the set of all vertices whose eccentricity is minimum. It is clear that the centre $Z(\Gamma_{pm}(R)) = m$.

Now we have the centre of the graph as the set in which every vertex is adjacent to all the other vertices in the graph.

If we delete the centre of $\Gamma_{pm}(R)$, then the graph is disconnected with $(n - m)$-components.

Therefore the vertex connectivity of the product maximal graph $\Gamma_{pm}(R)$ is the number of vertices in the centre of $\Gamma_{pm}(R)$.

$\kappa(\Gamma_{pm}(R)) = m$ where m is the cardinality of M. □

For example, the vertex connectivity of the product maximal graph $\Gamma_{pm}(Z_9)$ is 3 from figure 1. The following result called the edge connectivity of the product maximal graph.

Theorem 3.9. *The edge connectivity of the product maximal graph is* $\kappa'(\Gamma_{pm}(R)) = m$.

Proof. The vertex set of the product maximal graph is $|V(\Gamma_{pm}(R))| = n$. Let u and v be any two vertices in $\Gamma_{pm}(R)$. Since the product maximal graph contains complete subgraph with the elements of maximal ideal M whose degree is $n - 1$ and co-clique with the elements of non-maximal ideal $\overline{M}$ whose degree is m.

If we remove the $n - 1$ edges with respect to a fixed vertex $u \in M$, the graph becomes disconnected as u become isolated.

Also, if we remove the m edges with respect to a fixed vertex $v \in \overline{M}$, the graph becomes disconnected as v become isolated.

Thus the minimum number of edges to be deleted is m.

Hence the edge connectivity of $\Gamma_{pm}(R)$ is $\kappa'(\Gamma_{pm}(R)) = m$.

Since $\kappa'(\Gamma_{pm}(R)) = m$, the product maximal graph $\Gamma_{pm}(R)$ is m-edge connected. □

In particular, the edge connectivity of the product maximal graph is $\kappa'(\Gamma_{pm}(Z_9)) = 3$ by figure 1. By observing theorems 3.7 and 3.8, we conclude that the vertex connectivity is same as the edge connectivity of the product maximal graph $\kappa(\Gamma_{pm}(R)) = \kappa'(\Gamma_{pm}(R)) = m$.

Theorem 3.10. *The minimum degree of the product maximal graph $\Gamma_{pm}(R)$ is equal to its vertex connectivity and edge connectivity. Therefore $\kappa(\Gamma_{pm}(R)) = \kappa'(\Gamma_{pm}(R)) = \delta(\Gamma_{pm}(R)) = m$.*

Proof. The degree of the product maximal graph is already proved in [7] that

$$\deg(v) = \begin{cases} n-1, & \text{if } v \in M_i,\ i = 1, 2, \ldots, r \\ m, & \text{if } v \notin M_i,\ i = 1, 2, \ldots, r \end{cases}$$

where m is the number of elements in maximal ideals and n is the number of vertices in $\Gamma_{pm}(R)$.

By theorems 3.7 and 3.8, $\kappa(\Gamma_{pm}(R)) = \kappa'(\Gamma_{pm}(R)) = m$.

Hence $\kappa(\Gamma_{pm}(R)) = \kappa'(\Gamma_{pm}(R)) = \delta(\Gamma_{pm}(R)) = m$. □

3.4 Planarity and Hamiltonian of the product maximal graph

In this section, the product maximal graph that can be drawn in a plane without any line crossing and proved that the graph $\Gamma_{pm}(R)$ is Hamiltonian.

Theorem 3.11. *Let R be a finite commutative ring with unity. Then the product maximal graph is planar if $n \le 4$.*

Proof. Let R be a finite commutative ring with unity. We know that the size of the product maximal graph is $mC_2 + m(n-m)$, so that the graph contains complete sub graph.

Claim: $\Gamma_{pm}(R)$ is planar.

If $n > 4$, then the complete subgraph is more than K_4. Also each independent vertex is adjacent to all the vertices in the complete subgraph. So, it is not possible to draw the edges without intersect. Therefore, $\Gamma_{pm}(R)$ is planar if $n \le 4$. □

Theorem 3.12. *If $\Gamma_{pm}(R)$ is the product maximal graph then $\Gamma_{pm}(R)$ is Hamiltonian if* $|M| \geq |\overline{M}|$.

Proof. Let R be the finite commutative ring with unity and $M_1, M_2, \ldots, M_r$ be the maximal ideals of R.

By [7], the graph $\Gamma_{pm}(R)$ can be split into two subgraphs namely complete subgraph K_m whose vertices are in maximal ideal M_i, $i = 1, 2, \ldots, r$ and an independent subgraph I_{n-m} whose vertices are not in maximal ideals.

If $|\overline{M}| > |M|$, then there is no cycle which includes every vertex of $\Gamma_{pm}(R)$. Therefore, $\Gamma_{pm}(R)$ is Hamiltonian if $|M| \geq |\overline{M}|$. □

4. Distance Based Topological Indices for the Product Maximal Graph

A numerical value related to a graph that is invariant under graph automorphism is a topological index. The Wiener index is one of the most studied topological indices, both from a theoretical point of view and applications. This index was the first topological index to be used in chemistry. The following theorem based on topological indices using distance of vertices for $\Gamma_{pm}(R)$.

Theorem 4.1. *The wiener index for the product maximal graph is* $W(\Gamma_{pm}(R)) = mC_2 + (n-m)(n-1)$.

Proof. Let R be a finite commutative ring with unity. Let n be the number of vertices of the product maximal graph and m be the number of elements in the maximal ideals of R. In $\Gamma_{pm}(R)$, the distance matrix $D(\Gamma_{pm}(R)) = [d_{ij}]$ is the real symmetric matrix where

$$d_{ij} = \begin{cases} 1, & \text{if } v_i \text{ or } v_j \in M \\ 2, & \text{if } v_i \text{ and } v_j \notin M \ \ i, j = 1, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

The wiener index of $\Gamma_{pm}(R)$ is

$$W(\Gamma_{pm}(R)) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}$$

$$= \frac{1}{2}[2mC_2 + 2m(n-m) + 2(n-m)(n-m-1)]$$

$$= mC_2 + (n-m)(n-1)$$ □

For example, the wiener index of the product maximal graph $\Gamma_{pm}(Z_6)$ is $W(\Gamma_{pm}(Z_6)) = 16$ by figure 2. The result of the theorem 4.1 can be speculate the hyper-Weiner index for $\Gamma_{pm}(R)$.

Theorem 4.2. *The hyper wiener index of the product maximal graph is* $WW(\Gamma_{pm}(R)) = mC_2 + (n-m)(2(n-1)-m)$.

Proof. The distance matrix of $\Gamma_{pm}(R)$ is $[d_{ij}]$ which is defined in the theorem 4.1. Then the hyper Wiener index is

$$WW(\Gamma_{pm}(R)) = \frac{1}{2}[\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} + d_{ij}^2]$$

$$= \frac{1}{2}[2mC_2 + 2m(n-m) + 4(n-m)(n-m-1)]$$

$$= mC_2 + (n-m)(n-1)$$ □

For example, the hyper wiener index of the product maximal graph $\Gamma_{pm}(Z_6)$ is $WW(\Gamma_{pm}(Z_6)) = 18$. The famous resistance distance based topological indices as the Kirchhoff index. The following theorem shows the Kirchhoff index for $\Gamma_{pm}(R)$. In [12], they discussed some relation between resistance and Laplacian matrices.

Theorem 4.3. *The Kirchhoff's index for the product maximal graph of the finite commutative ring with unity is* $Kf(\Gamma_{pm}(R)) = \frac{1}{m}[2C_2 - m(n-m)]$.

Proof. For the product maximal graph, the Laplacian matrix is $L(\Gamma_{pm}(R)) = Deg(\Gamma_{pm}(R)) - A(\Gamma_{pm}(R))$ where $Deg(\Gamma_{pm}(R))$ is the degree matrix and $A(\Gamma_{pm}(R))$ is the adjacency matrix.

By [7], the degree matrix $Deg(\Gamma_{pm}(R)) = [\deg_{ij}]$ where

$$\deg_{ij} = \begin{cases} n-1, & \text{if } v_i \in M \text{ and } i = j \\ m, & \text{if } v_j \notin M \text{ and } i = j \text{ where } 1 \le i, j \le n \\ 0, & \text{if } i \ne j \end{cases}$$

where m and n are the cardinalities of M and R respectively.

Also by theorem 3.1, the adjacency matrix $A(\Gamma_{pm}(R)) = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \in M \text{ or } v_j \in M \text{ and } i \ne j \\ 0, & \text{otherwise} \end{cases}, 1 \le i, j \le n.$$

Therefore, the Laplacian matrix $L(\Gamma_{pm}(R)) = [L(i, j)]$ where

$$L(i, j) = \begin{cases} \deg_{ij}, & \text{if } i = j \\ -1, & \text{if } v_j \text{ is ad jacent to } v_j \text{ and } i \ne j \; 1 \le i, j \le n \\ 0, & \text{otherwise} \end{cases}$$

The resistance distance from the vertex v_i to v_j become

$$\Omega_{ij} = \frac{\det L(i, j)}{\det L[i, i]}$$

where $\det L(i, j)$ is to deleting the i^{th} row and column and j^{th} row and column. $\det L[i, i]$ is to delete the i^{th} row and column.

So we get the resistance distance matrix for the product maximal graph $W(\Gamma_{pm}(R)) = [\Omega_{ij}]$ where

$$\Omega_{ij} = \begin{cases} \dfrac{2n}{nm}, & \text{if } v_i, v_j \notin M \\ \dfrac{2m}{nm}, & \text{if } v_i, v_j \in M \\ \dfrac{n + m - 1}{nm}, & \text{if } v_i \text{ or } v_j \in M \\ 0, & \text{otherwise} \end{cases}$$

Now the Kirchhoff's index for the product maximal graph $\Gamma_{pm}(R)$

$$Kf = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\Omega_{ij})$$

$$= \frac{1}{2nm}[2n(n-m)(n-m-1) + 2m(m)(m-1) + (n+m-1)(2m)(n-m)]$$

$$= \frac{1}{2}[2C_2 - m(n-m)] \qquad \square$$

For example, the Kirchhoff's index of the product maximal graph $\Gamma_{pm}(R)$ of the finite commutative ring with unity is $Kf(\Gamma_{pm}(Z_6)) = \frac{11}{2}$ from figure 2.

The following statement is the Kirchhoff's sum index for $\Gamma_{pm}(R)$ by applying both distance and resistance distance matrix.

Theorem 4.4. *The Kirchhoff's sum index for the product maximal graph* $KSf(\Gamma_{pm}(R)) = \frac{1}{m}[nC_2 + mC_2]$.

Proof. By theorem 4.1 and theorem 4.3, we can find the distance matrix and resistance distance matrix. So the Kirchhoff's sum index for $\Gamma_{pm}(R)$

$$KSf = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Omega_{ij}/D_{ij}$$

$$= \frac{1}{2nm}\left[2m^2(m-1) + \frac{2n}{2}(n-m) + (n+m-1)(2m)(n-m)\right]$$

$$= \frac{1}{2}[nm^2 - nm + n^3 - n^2]$$

$$= \frac{1}{m}[nC_2 + mC_2] \qquad \square$$

For example, The Kirchhoff's Sum index of the product maximal graph $\Gamma_{pm}(Z_6)$ is $KSf(\Gamma_{pm}(Z_6)) = \frac{9}{4}$ from figure 2.

Conclusion

In this paper, we begin with the definition of the product maximal graph, then we analyse some aspects of $\Gamma_{pm}(R)$. Initially, some algebraic invariants such as rank, nullity, girth are explored. Subsequently, domination for the graph are discussed especially edge dominating set, k-fair dominating set and annihilator dominating set. In addition, we found coloring and connectivity for $\Gamma_{pm}(R)$. Furthermore, planar and Hamiltonian are interpreted. Finally, the topological indices using electrical network theory as the application for the product maximal graph are discussed.

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