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VERIFICATION OF A CONJECTURE ON UNIT FRACTIONS PROPOSED BY DONALD J. NEWMAN

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Dedicated to the memory of the late Prof. Donald J. Newman

Abstract

The numbers $U_1 = 2,\ U_2 = 3,\ U_3 = 7,\ U_4 = 43, \ldots,$ have the property that for $i \geq 2$, each number U_i is the product of all preceding numbers plus one. For all values of k, the sum of k reciprocals is less than one. Among all possible sums less than one consisting of k positive distinct reciprocals, Newman's conjecture states that the sum of k reciprocals of the numbers U_i is the maximal one. In this article, it is shown that this is indeed true. All the results are achieved in an elementary way.

1. Some Preliminaries

In this paper we establish the validity of Newman's conjecture [4] stating that for integers $1 < a_1 < a_2 < \cdots < a_n$ satisfying $S_n \equiv \sum_{i=1}^{n} \frac{1}{a_i} < 1$ and S_n is a maximum, then at each choice one should take the smallest integer satisfying the inequality constraint. In [4], for example when $n = 4$, one would choose $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} < 1$. All the results are obtained in an elementary way.

Authors like L. Brenton - R. Hill, D. R. Curtiss [1, 2] and others have considered the problem and related topics. The problem is also cited in [3].

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Consider the following integers

$$1 < U_1 < U_2 < \cdots < U_n < \cdots$$

which have the following properties:

(i) When $n = 1$, then $U_1 = 2$.

When $n \geq 2$, then $U_n = \prod_{i=1}^{n-1} U_i + 1$ is the smallest integer.

Thus $U_2 = 3$, $U_3 = 7$, $U_4 = 43$, $\cdots$

(ii) $\sum_{i=1}^{n} \frac{1}{U_i} = \frac{\prod_{i=1}^{n} U_i - 1}{\prod_{i=1}^{n} U_i}$ for all values of n.

(iii) Any two integers U_r, U_s satisfy $\gcd(U_r, U_s) = 1$.

The identity

$$\frac{1}{N} = \frac{1}{N+M} + \frac{M}{N(N+M)},$$

where $N \geq 1$, $M \geq 1$ are positive integers shall be utilized and is central to our study. Only values of M which satisfy $M \mid N(N+M)$ are considered in order to obtain two distinct unit fractions in the right-hand side of the identity. Hence, $M = N$ is impossible, and $M \neq N$. When $M > N$, it is easily seen that the same denominators are obtained when $M < N$. Therefore, without loss of generality, it suffices to consider only values $M < N$, i.e., $N + M < \frac{N(N+M)}{M}$. Hence

$$\frac{1}{N} = \frac{1}{N+M} + \frac{1}{\frac{N(N+M)}{M}} \quad 1 \leq M < N, \ \ M \mid N(N+M). \tag{1}$$

2. The Main Results

We begin this section by proving two simple claims on particular sums of positive distinct unit fractions. These will be used in the forthcoming theorems. Next, the four cases which stem from the problem are demonstrated. Finally, three theorems which correspond to the four cases establish the desired result.

Claim 1. *Let* $\frac{a-1}{a}$ *be the sum of t positive distinct unit fractions, and let* $\frac{b-1}{b}$ *be another sum of t positive distinct unit fractions. If* $\frac{b-1}{b} > \frac{a-1}{a}$, *then* $b > a$, *whereas if* $b > a$, *then* $\frac{b-1}{b} > \frac{a-1}{a}$ (*i.e.*: $b > a \leftrightarrow \frac{b-1}{b} > \frac{a-1}{a}$).

Proof. The inequality $\frac{b-1}{b} > \frac{a-1}{a}$ immediately yields $b > a$. When $b > a$, then $\frac{-1}{b} > \frac{-1}{a}$ implies that $1 - \frac{1}{b} > 1 - \frac{1}{a}$ or that $\frac{b-1}{b} > \frac{a-1}{a}$. ◻

In particular, we can now state Claim 2.

Claim 2. *Let* $\frac{a-1}{a}$ *be the sum of t positive distinct unit fractions, and let* $\frac{b-1}{b}$ *be another sum of t positive distinct unit fractions. If* $\frac{b-1}{b} > \frac{a-1}{a}$, *then*

$$\frac{b-1}{b} + \frac{1}{b+1} > \frac{a-1}{a} + \frac{1}{a+1}.$$

Proof. Both sides of the above inequality yield the respective two equalities

$$\frac{b(b+1)-1}{b(b+1)} = 1 - \frac{1}{b(b+1)} \quad \text{and} \quad \frac{a(a+1)-1}{a(a+1)} = 1 - \frac{1}{a(a+1)}.$$

By Claim 1 it follows that $b > a$, and hence $1 - \frac{1}{b(b+1)} > 1 - \frac{1}{a(a+1)}$.

The assertion now follows. □

Let $X_1 < X_2 < \cdots < X_n$ be positive integers satisfying $\sum_{i=1}^{n} \frac{1}{X_i} < 1.$ In order to show that for every value of n, $\sum_{i=1}^{n} \frac{1}{X_i} < \sum_{i=1}^{n} \frac{1}{U_i}$, we shall assume that there exists at least one value of n say $n = k$ which is the smallest value satisfying $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$, and reach a contradiction.

In the forthcoming discussions, we shall concern ourselves in finding the best possible choice of values of X_k, and also of both X_{k-1} and $X_{k.}$

Let $\alpha \ge 1$ be an integer. Four possible cases then exist which are as follows:

(a) $$\sum_{i=1}^{k-1} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-1} X_i - \alpha}{\prod_{i=1}^{k-1} X_i}, \quad \alpha \ge 1, \quad \gcd\left(\alpha,\ \prod_{i=1}^{k-1} X_i\right) = \alpha.$$

(b) $$\sum_{i=1}^{k-1} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-1} X_i - \alpha}{\prod_{i=1}^{k-1} X_i}, \quad \begin{cases} \alpha > 1, \quad \gcd\left(\alpha,\ \prod_{i=1}^{k-1} X_i\right) = 1, \\ \alpha > 1, \quad \gcd\left(\alpha,\ \prod_{i=1}^{k-1} X_i\right) = \beta,\ 1 < \beta < \alpha. \end{cases}$$

(c) $$\sum_{i=1}^{k-2} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-2} X_i - \alpha}{\prod_{i=1}^{k-2} X_i}, \quad \alpha \ge 1, \quad \gcd\left(\alpha,\ \prod_{i=1}^{k-2} X_i\right) = \alpha.$$

(d) $$\sum_{i=1}^{k-2} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-2} X_i - \alpha}{\prod_{i=1}^{k-2} X_i}, \quad \begin{cases} \alpha > 1, \quad \gcd\left(\alpha,\ \prod_{i=1}^{k-2} X_i\right) = 1, \\ \alpha > 1, \quad \gcd\left(\alpha,\ \prod_{i=1}^{k-2} X_i\right) = \beta,\ 1 < \beta < \alpha. \end{cases}$$

The following Theorem 1 consists of both cases **(a)** and **(b)**, whereas cases **(c)** and **(d)** are considered respectively in Theorems 2 and 3. Each theorem is self-contained.

Theorem 1. *Let* $X_1 < X_2 < \cdots < X_k$ *be positive integers satisfying* $\sum_{i=1}^{k} \frac{1}{X_i} < 1$. *Let* $\alpha \geq 1$ *be an integer, and suppose that* $\sum_{i=1}^{k-1} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-1} X_i - \alpha}{\prod_{i=1}^{k-1} X_i}$. *If*

(i) $\gcd\left(\alpha, \prod_{i=1}^{k-1} X_i\right) = \alpha, \ \alpha \geq 1,$

(ii) $\gcd\left(\alpha, \prod_{i=1}^{k-1} X_i\right) = 1, \ \alpha > 1,$ *or* $\gcd\left(\alpha, \prod_{i=1}^{k-1} X_i\right) = \beta, \ 1 < \beta < \alpha,$

then $\sum_{i=1}^{k} \frac{1}{X_i} < \sum_{i=1}^{k} \frac{1}{U_i}$.

Proof. We shall assume that k is the smallest possible value which satisfies $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$, and reach a contradiction.

Denote $\prod_{i=1}^{k-1} X_i = L$, $\sum_{i=1}^{k-1} \frac{1}{U_i} = \frac{U-1}{U}$, and $U_k = U + 1$.

Suppose (i), i.e., $\gcd\left(\alpha, \prod_{i=1}^{k-1} X_i\right) = \alpha$ where $\alpha \geq 1$. We shall distinguish two cases, namely $\alpha = 1$ and $\alpha > 1$.

Suppose that $\alpha = 1$.

Then

$$\sum_{i=1}^{k-1} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-1} X_i - 1}{\prod_{i=1}^{k-1} X_i} = \frac{L-1}{L}.$$

By our assumption, it now follows that

$$\sum_{i=1}^{k-1} \frac{1}{X_i} = \frac{L-1}{L} < \sum_{i=1}^{k-1} \frac{1}{U_i} = \frac{U-1}{U},$$

and by Claim 1 $L < U$. The maximal value of $\frac{1}{X_k}$ is $\frac{1}{L+1}$. Then Claim 2 implies that

$$\sum_{i=1}^{k} \frac{1}{X_i} = \frac{L-1}{L} + \frac{1}{L+1} < \frac{U-1}{U} + \frac{1}{U+1} = \sum_{i=1}^{k} \frac{1}{U_i},$$

a contradiction.

Our assumption that $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$ is therefore false, and the assertion follows.

Suppose that $\alpha > 1$.

Since $\gcd\left(\alpha, \prod_{i=1}^{k-1} X_i\right) = \alpha$, let

$$\sum_{i=1}^{k-1} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-1} X_i - \alpha}{\prod_{i=1}^{k-1} X_i} = \frac{\frac{L}{\alpha} - 1}{\frac{L}{\alpha}} = \frac{T-1}{T}.$$

Hereafter, one may now proceed entirely as in the former case of $\alpha = 1$. This then yields the same contradiction as before, and the desired result is achieved.

Suppose (ii), i.e., $\gcd\left(\alpha, \prod_{i=1}^{k-1} X_i\right) = 1$, where $\alpha > 1$.

Since $\prod_{i=1}^{k-1} X_i = L,$ then by our supposition $\gcd(\alpha, L) = 1,$ and $\frac{L}{\alpha}$ is not an integer. Denote $\frac{L}{\alpha} = B + \varepsilon$ where B is an integer and $0 < \varepsilon < 1.$ Thus,

$$\sum_{i=1}^{k-1} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-1} X_i - \alpha}{\prod_{i=1}^{k-1} X_i} = \frac{L - \alpha}{L} = \frac{\frac{L}{\alpha} - 1}{\frac{L}{\alpha}} = \frac{B + \varepsilon - 1}{B + \varepsilon}.$$

It is observed, that if our assumption holds when $X_k = B + V$ where $V > 1$ is an integer, then it certainly holds for $X_k = B + 1$ the smallest possible such value. Hence, the choice of $X_k = B + 1$ is clearly justified.

Already mentioned before, $\sum_{i=1}^{k-1} \frac{1}{U_i} = \frac{U - 1}{U}$ and $U_k = U + 1.$ Hence $\sum_{i=1}^{k} \frac{1}{U_i} = \frac{U(U + 1) - 1}{U(U + 1)}.$ By our assumption we have that $\frac{L - \alpha}{L} < \frac{U - 1}{U}$ which implies that $L < \alpha U$ or $B + \varepsilon = \frac{L}{\alpha} < U.$ Therefore, $B < B + \varepsilon < U$ yields $B + 1 < U + 1.$ With $X_k = B + 1,$ we obtain

$$\sum_{i=1}^{k} \frac{1}{X_i} = \sum_{i=1}^{k-1} \frac{1}{X_i} + \frac{1}{X_k} = \frac{B + \varepsilon - 1}{B + \varepsilon} + \frac{1}{B + 1} = \frac{(B + \varepsilon)(B + 1) - 1}{(B + \varepsilon)(B + 1)} + \frac{\varepsilon}{(B + \varepsilon)(B + 1)}.$$

Since $(B + \varepsilon)(B + 1) < U(U + 1),$ it follows that $\frac{-1}{(B + \varepsilon)(B + 1)} < \frac{-1}{U(U + 1)},$ or equivalently, $1 - \frac{1}{(B + \varepsilon)(B + 1)} < 1 - \frac{1}{U(U + 1)}$ and

$$\frac{(B + \varepsilon)(B + 1) - 1}{(B + \varepsilon)(B + 1)} < \frac{U(U + 1) - 1}{U(U + 1)}.$$

It is observed that the term $\frac{\varepsilon}{(B+\varepsilon)(B+1)}$ is quite negligible as may be seen for instance by Example 2 in Section 3. Therefore, it follows by the above inequality that $\sum_{i=1}^{k} \frac{1}{X_i} < \sum_{i=1}^{k} \frac{1}{U_i}$.

Our assumption that $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$ is therefore false, and the assertion follows.

Suppose (ii), i.e., $\gcd\left(\alpha, \prod_{i=1}^{k-1} X_i\right) = \beta$, where $1 < \beta < \alpha$.

Denote $\frac{L}{\beta} = G$ and $\frac{\alpha}{\beta} = H$. Since $\prod_{i=1}^{k-1} X_i = L$, then by our supposition

$$\sum_{i=1}^{k-1} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-1} X_i - \alpha}{\prod_{i=1}^{k-1} X_i} = \frac{L-\alpha}{L} = \frac{\frac{L}{\beta} - \frac{\alpha}{\beta}}{\frac{L}{\beta}} = \frac{G-H}{G},$$

where $\gcd(H, G) = 1$, and $\frac{G}{H}$ is not an integer.

The remaining part of the case now proceeds entirely as the former case when $\gcd\left(\alpha, \prod_{i=1}^{k-1} X_i\right) = 1$ and $\alpha > 1$. This is due to the fact that the only difference is that the values α, L used before are now replaced by the values $\frac{\alpha}{\beta} = H$ and $\frac{L}{\beta} = G$. One will then arrive at the same contradiction as before, and the assertion follows.

This concludes the proof of Theorem 1. □

Theorem 2. *Let* $X_1 < X_2 < \cdots < X_k$ *be positive integers satisfying* $\sum_{i=1}^{k} \frac{1}{X_i} < 1$. *Let* $\alpha \geq 1$ *be an integer, and suppose that* $\sum_{i=1}^{k-2} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-2} X_i - \alpha}{\prod_{i=1}^{k-2} X_i}$.

If $\gcd\left(\alpha, \prod_{i=1}^{k-2} X_i\right) = \alpha$, *then* $\sum_{i=1}^{k} \frac{1}{X_i} < \sum_{i=1}^{k} \frac{1}{U_i}$.

Proof. We shall assume that k is the smallest possible value which satisfies $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$, and reach a contradiction.

Two cases will be considered, namely $\alpha = 1$ and $\alpha > 1$.

Suppose that $\alpha = 1$.

Denote $\prod_{i=1}^{k-2} X_i = A$, $\sum_{i=1}^{k-2} \frac{1}{U_i} = \frac{U-1}{U}$, and $U_{k-1} = U + 1$. Then

$$\sum_{i=1}^{k-2} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-2} X_i - 1}{\prod_{i=1}^{k-2} X_i} = \frac{A-1}{A}.$$

From (1), when N is substituted by A and M is substituted by B, we obtain

$$\frac{1}{A} = \frac{1}{A+B} + \frac{B}{A(A+B)} \quad 1 \leq B < A, \;\; B \mid A(A+B),$$

which yields for $B = 1$

$$\frac{1}{A} = \frac{1}{A+1} + \frac{1}{A(A+1)},$$

whereas when $B > 1$

$$\frac{1}{A} = \frac{1}{A+B} + \frac{1}{\frac{A(A+B)}{B}}.$$

To prove that $X_{k-1} = A + 1$ and $X_k = A(A+1) + 1$ are the best possible values, it suffices to show for all values $B > 1$, that the inequality

$$\frac{1}{A+B} + \frac{1}{\frac{A(A+B)}{B} + 1} < \frac{1}{A+1} + \frac{1}{A(A+1)+1} \tag{2}$$

holds.

Simplifying (2) yields the inequality

$$0 < (B-1)(B(A+1)^2 + A^2)$$

which is valid for all values of $B > 1$.

Having established (2), the best possible values of X_{k-1}, X_k are now $X_{k-1} = A + 1$, $X_k = A(A+1) + 1$, and

$$\sum_{i=1}^{k} \frac{1}{X_i} = \frac{A-1}{A} + \frac{1}{A+1} + \frac{1}{A(A+1)+1} < 1.$$

By our assumption $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$, therefore $\sum_{i=1}^{k-2} \frac{1}{X_i} = \frac{A-1}{A}$ $< \sum_{i=1}^{k-2} \frac{1}{U_i} = \frac{U-1}{U}$ and Claim 1 implies that $A < U$. We now show that even with the best possible values $X_{k-1} = A + 1$ and $X_k = A(A+1) + 1$, $\sum_{i=1}^{k} \frac{1}{X_i} < \sum_{i=1}^{k} \frac{1}{U_i}$. By Claim 2, we have

$$\sum_{i=1}^{k-1} \frac{1}{X_i} = \frac{A-1}{A} + \frac{1}{A+1} = \frac{A^2+A-1}{A^2+A} < \frac{U-1}{U} + \frac{1}{U+1} = \frac{U^2+U-1}{U^2+U} = \sum_{i=1}^{k-1} \frac{1}{U_i}. \tag{3}$$

Applying Claim 2 to (3), now yields

$$\sum_{i=1}^{k} \frac{1}{X_i} = \frac{A^2+A-1}{A^2+A} + \frac{1}{A^2+A+1} < \frac{U^2+U-1}{U^2+U} + \frac{1}{U^2+U+1} = \sum_{i=1}^{k} \frac{1}{U_i},$$

and confirms our assertion. Hence, the assumption that $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$ is false.

Suppose that $\alpha > 1$.

Denote $\prod_{i=1}^{k-2} X_i = \alpha K$. Hence, when $A = \alpha K$

$$\sum_{i=1}^{k-2} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-2} X_i - \alpha}{\prod_{i=1}^{k-2} X_i} = \frac{\alpha K - \alpha}{\alpha K} = \frac{K-1}{K}.$$

Hereafter, we shall proceed as in the case $\alpha = 1$, where $\frac{K-1}{K}$ is identified with $\frac{A-1}{A}$ in the former case. In (1), substitute the value N by K and M by B. The values of X_{k-1}, X_k are therefore $X_{k-1} = K + 1$ and $X_k = K(K+1) + 1$. The same contradiction as in the case $\alpha = 1$ is now achieved. Our assumption that $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$ is therefore false, and the assertion follows.

This completes our proof. □

Theorem 3. *Let* $X_1 < X_2 < \cdots < X_k$ *be positive integers satisfying* $\sum_{i=1}^{k} \frac{1}{X_i} < 1$. *Let* $\alpha > 1$ *be an integer, and suppose that* $\sum_{i=1}^{k-2} \frac{1}{X_i} = \frac{\prod_{i=1}^{k-2} X_i - \alpha}{\prod_{i=1}^{k-2} X_i}$. *If* $\gcd\left(\alpha, \prod_{i=1}^{k-2} X_i\right) = 1$, *or if* $\gcd\left(\alpha, \prod_{i=1}^{k-2} X_i\right) = \beta$, *and* $1 < \beta < \alpha$, *then* $\sum_{i=1}^{k} \frac{1}{X_i} < \sum_{i=1}^{k} \frac{1}{U_i}$.

Proof. In both cases, we shall assume that k is the smallest possible value satisfying $\sum_{i=1}^{k}\frac{1}{X_i} > \sum_{i=1}^{k}\frac{1}{U_i}$, and derive a contradiction.

First, we shall establish the best possible choice of values X_{k-1} and X_k. With these values, it will be shown that $\sum_{i=1}^{k}\frac{1}{X_i} < \sum_{i=1}^{k}\frac{1}{U_i}$ as asserted, contrary to our assumption.

Suppose that $\gcd\left(\alpha,\ \prod_{i=1}^{k-2} X_i\right) = 1.$

Denote $\prod_{i=1}^{k-2} X_i = Q$. Hence $\sum_{i=1}^{k-2}\frac{1}{X_i} = \frac{Q-\alpha}{Q}$ and $\gcd(\alpha,\ Q) = 1$.

From (1) when N is substituted by Q, we obtain

$$\frac{1}{Q} = \frac{1}{Q+M} + \frac{M}{Q(Q+M)}, \qquad 1 \le M < Q, \qquad M \mid Q(Q+M),$$

which implies that

$$\frac{\alpha}{Q} = \frac{\alpha}{Q+M} + \frac{\alpha M}{Q(Q+M)},$$

or equivalently,

$$\frac{\alpha}{Q} = \frac{1}{\frac{Q+M}{\alpha}} + \frac{1}{\frac{Q(Q+M)}{\alpha M}}. \tag{4}$$

The right-hand side of (4) will consist of two unit fractions, if $\alpha \mid (Q+M)$, and $\alpha M \mid Q(Q+M)$. Hence, $X_{k-1} = \frac{Q+M}{\alpha}$, and $X_k = \frac{Q(Q+M)}{\alpha M} + 1$.

We shall distinguish two cases, namely $M = 1$ and $M > 1$.

Suppose that $M = 1$.

Then (4) yields $\frac{\alpha}{Q} = \frac{1}{\frac{Q+1}{\alpha}} + \frac{1}{\frac{Q(Q+1)}{\alpha}}$, and $\alpha \mid (Q+1)$. We now obtain

$$\sum_{i=1}^{k-1} \frac{1}{X_i} = \sum_{i=1}^{k-2} \frac{1}{X_i} + \frac{1}{X_{k-1}} = \frac{Q-\alpha}{Q} + \frac{1}{\frac{Q+1}{\alpha}} = \frac{Q(Q+1)-\alpha}{Q(Q+1)} = \frac{\frac{Q(Q+1)}{\alpha} - 1}{\frac{Q(Q+1)}{\alpha}}.$$

By our assumption that k is the smallest possible value for which $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$, it therefore follows that the above form of $\sum_{i=1}^{k-1} \frac{1}{X_i}$ satisfies $\sum_{i=1}^{k-1} \frac{1}{X_i} < \sum_{i=1}^{k-1} \frac{1}{U_i}$. Then, with the respective two smallest possible values, namely $X_k = \frac{Q(Q+1)}{\alpha} + 1$ and U_k, we have by Claim 2 that

$$\sum_{i=1}^{k} \frac{1}{X_i} = \sum_{i=1}^{k-1} \frac{1}{X_i} + \frac{1}{X_k} < \sum_{i=1}^{k-1} \frac{1}{U_i} + \frac{1}{U_k} = \sum_{i=1}^{k} \frac{1}{U_i},$$

contrary to our assumption.

Suppose that $M > 1$.

From (1) we have that $M \mid Q(Q+M)$ and hence $M \mid Q^2$. Denote $\gcd(M, Q) = a > 1$. Then $M = ab$, $Q = ac$, $\gcd(b, c) = 1$. Moreover, $M \mid Q^2$ implies that $b \mid a$. Certainly, there exists a value $M = M'$, such that $\alpha \mid (Q + M')$ implying that $\alpha \mid (b+c)$ since $\gcd(\alpha, a) = 1$. Therefore

$$\frac{Q(Q+M')}{\alpha M'} = \frac{(ac)(ac+ab)}{\alpha(ab)}$$

is an integer.

Rewriting (4), we obtain

$$\frac{\alpha}{Q} = \frac{1}{\frac{Q+M'}{\alpha}} + \frac{1}{\frac{Q(Q+M')}{\alpha M'}},$$

where $X_{k-1} = \frac{Q+M'}{\alpha}$ and $X_k = \frac{Q(Q+M')}{\alpha M'} + 1$.

Then

$$\sum_{i=1}^{k-1} \frac{1}{X_i} = \sum_{i=1}^{k-2} \frac{1}{X_i} + \frac{1}{X_{k-1}} = \frac{Q-\alpha}{Q} + \frac{1}{\frac{Q+M'}{\alpha}} = \frac{Q(Q+M') - \alpha M'}{Q(Q+M')}$$

$$= \frac{\frac{Q(Q+M')}{\alpha M'} - 1}{\frac{Q(Q+M')}{\alpha M'}}.$$

Our assumption that k is the smallest possible value satisfying $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$ implies therefore that $\sum_{i=1}^{k-1} \frac{1}{X_i} < \sum_{i=1}^{k-1} \frac{1}{U_i}$. Applying Claim 2 to the above form of $\sum_{i=1}^{k-1} \frac{1}{X_i}$, and to $\sum_{i=1}^{k-1} \frac{1}{U_i}$ together with their respective smallest possible values $X_k = \frac{Q(Q+M')}{\alpha M'} + 1$ and U_k results in

$$\sum_{i=1}^{k} \frac{1}{X_i} = \sum_{i=1}^{k-1} \frac{1}{X_i} + \frac{1}{X_k} < \sum_{i=1}^{k-1} \frac{1}{U_i} + \frac{1}{U_k} = \sum_{i=1}^{k} \frac{1}{U_i}$$

and contradicts our assumption.

Our assumption that there exists a value k for which $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$ is therefore false, and the assertion follows.

Suppose that $\gcd\left(\alpha, \prod_{i=1}^{k-2} X_i\right) = \beta$ where $1 < \beta < \alpha$.

Since $\prod_{i=1}^{k-2} X_i = Q$, denote $\frac{Q}{\beta} = G$, $\frac{\alpha}{\beta} = H$, and $\gcd(H, G) = 1$. The values G, H, M now satisfy

$$\frac{H}{G} = \frac{1}{\frac{G+M}{H}} + \frac{1}{\frac{G(G+M)}{HM}}. \tag{5}$$

The right-hand side of (5) will consist of two unit fractions provided $H \mid (G + M)$, and $HM \mid G(G + M)$. Thus, $X_{k-1} = \frac{G + M}{H}$, and $X_k = \frac{G(G + M)}{HM} + 1$.

Observe that (5) is exactly equality (4) when H is substituted for α, and G is substituted for Q. All other mentioned values in the previous case remain fixed here, and one may consider again the two possibilities of $M = 1$ and of $M > 1$. As before, in both cases of $M = 1$ and of $M > 1$, one would now arrive at the same two contradictions.

Our assumption that there exists a value k for which $\sum_{i=1}^{k} \frac{1}{X_i} > \sum_{i=1}^{k} \frac{1}{U_i}$ is therefore false, and the assertion follows.

This concludes our proof. □

3. Some Examples and Conclusion

The forthcoming five examples relate respectively to Theorems 1, 2 and 3. In particular, they enable us to see the very large difference between $\sum_{i=1}^{k} \frac{1}{X_i}$ and $\sum_{i=1}^{k} \frac{1}{U_i}$ when various values of k are considered.

The following Examples 1 and 2 correspond to Theorem 1 parts (i) and (ii) respectively.

Example 1. Let $k = 4$. The values $X_1 = 2, X_2 = 3, X_3 = 8$ yield $\sum_{i=1}^{3} \frac{1}{X_i} = \frac{46}{48} = \frac{23}{24}$, where $\alpha = 2$ and $\gcd\left(\alpha, \prod_{i=1}^{3} X_i\right) = \alpha$. Hence $\min X_4 = 25$, and $\sum_{i=1}^{4} \frac{1}{X_i} = \frac{599}{600} < \frac{1805}{1806} = \sum_{i=1}^{4} \frac{1}{U_i}$.

Example 2. Let $k = 5$. The values $X_1 = 2, X_2 = 3, X_3 = 11, X_4 = 23$ yield $\sum_{i=1}^{4} \frac{1}{X_i} = \frac{1469}{1518}$, where $\alpha = 49$ and $\gcd\left(\alpha, \prod_{i=1}^{4} X_i\right) = 1$. Thus $\min X_5 = 31$, and $\sum_{i=1}^{5} \frac{1}{X_i} = \frac{47057}{47058} < \frac{3263441}{3263442} = \sum_{i=1}^{5} \frac{1}{U_i}$.

The next example relates to Theorem 2.

Example 3. Let $k = 5$. The values $X_1 = 2, X_2 = 3, X_3 = 10$ yield $\sum_{i=1}^{3} \frac{1}{X_i} = \frac{56}{60} = \frac{14}{15}$, where $\alpha = 4$ and $\gcd\left(\alpha, \prod_{i=1}^{3} X_i\right) = \alpha$. With $\min X_4 = 16$ and $\min X_5 = 241$, $\sum_{i=1}^{5} \frac{1}{X_i} = \frac{57839}{57840} < \frac{3263441}{3263442} = \sum_{i=1}^{5} \frac{1}{U_i}$.

The last two examples correspond to both parts of Theorem 3.

Example 4. Let $k = 5$. The values $X_1 = 2, X_2 = 3, X_3 = 11$ yield $\sum_{i=1}^{3} \frac{1}{X_i} = \frac{61}{66}$, where $\alpha = 5$ and $\gcd\left(\alpha, \prod_{i=1}^{3} X_i\right) = 1$. With $\min X_4 = 14$ and $\min X_5 = 232$, $\sum_{i=1}^{5} \frac{1}{X_i} = \frac{53591}{53592} < \frac{3263441}{3263442} = \sum_{i=1}^{5} \frac{1}{U_i}$.

Example 5. Let $k = 6$. The values $X_1 = 2, X_2 = 3, X_3 = 11, X_4 = 16$ yield $\sum_{i=1}^{4} \frac{1}{X_i} = \frac{1042}{1056} = \frac{521}{528}$, where $\alpha = 14$, and $\gcd\left(\alpha, \prod_{i=1}^{4} X_i\right) = \beta = 2$ and $1 < \beta < \alpha$. With $\min X_5 = 76$ and $\min X_6 = 10033$, $\sum_{i=1}^{6} \frac{1}{X_i}$ $= \frac{100651055}{100651056} < \frac{10650056950805}{10650056950806} = \sum_{i=1}^{6} \frac{1}{U_i}$.

Remark 1. It follows from Theorems 1, 2 and 3 that any k integers which satisfy $1 < X_1 < X_2 < \cdots < X_k$ and $\sum_{i=1}^{k} \frac{1}{X_i} < 1$, also satisfy $\sum_{i=1}^{k} \frac{1}{X_i} < \sum_{i=1}^{k} \frac{1}{U_i}$, and hence $\sum_{i=1}^{k} \frac{1}{U_i}$ is a maximum. The structure of the numbers U_i has already been mentioned in Section 1, and is in accordance with Newman's conjecture.

Thus, the validity of the conjecture is confirmed.

Final Remark. In 1962, while studying for the Master's Degree at Yeshiva University in New York, the author attended a seminar course in unsolved problems in Number Theory given by Prof. Newman. Among several problems, this problem was introduced by Prof. Newman. The author views this article as a personal closure, and a tribute to the memory of Prof. Newman a great mathematician.

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