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A CLASS OF NONPARAMETRIC TESTS FOR SPECIAL TWO SAMPLE LOCATION PROBLEM BASED ON *U*-STATISTICS

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Abstract

Comparing the performances of two packing machines, Shetty and Umrani [12] considered a special two sample location problem. Pandit and Acharya [7] considered the extension of Shetty and Umrani [12] by taking subsample maximum and subsample median, which has some potential applications. In this paper, a new class of test statistics is proposed which is based on median of first sample and maximum of second sample. The performance of members of the new class is evaluated in terms of Pitman asymptotic relative efficiency (ARE) in comparisons with Shetty and Umrani [12] and Pandit and Acharya [7]. It can be concluded that the performance of the new proposed class of tests perform better than the test due to Shetty and Umrani [12] and Pandit and Acharya [7] for those distributions considered for evaluation. Small sample powers have been computed for few members of the class using simulation.

1. Introduction

The problem has applications in many fields like economics, botany, medicine, psychology, etc. A number of distribution-free tests are available for the two-sample location problem, namely, Mann-Whitney's test [4], Mood's

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median (M) test [5], Gastwirth's H and L tests [2], Stephenson and Ghosh [13], Shetty and Govindarajulu [10], Shetty and Bhat [11], a test due to Ahmad [1], Xie and Priebe [14], Kossler and Kumar [6] among others whenever the interest is to compare two populations in which they differ only in location. A special type of two-sample location problem is useful in some situations such as comparing the performance of two measuring devices received less attention from researchers. This type of problems can fit in to special two-sample location setup wherein one wishes to test for the point of symmetry versus an appropriate alternative. Shetty and Umarani [12] and Pandit and Acharya [7] are available in the literature for such a two-sample location problem.

In this paper, we consider a class of distribution free test for the special type of two-sample location problem. The proposed class of test is given in section 2. Section 3 deals with the asymptotic distribution of the statistics and asymptotic relative efficiency comparisons. Section 4 gives the simulation study and Section 5 includes some remarks and conclusions.

2. Proposed Class of Tests

Let $(X_1, X_2, \ldots, X_m)$ and $(Y_1, Y_2, \ldots, Y_n)$ be the two independent random samples from absolutely continuous distribution functions with cdf $F_1(x)$ and $F_2(y)$ respectively, where $F_1(x) = F(x + \theta)$ and $F_2(x) = F(x - \theta)$. The problem is to test $H_o : \theta = 0$ versus the alternative $H_1 : \theta > 0$. It is assumed that $f(0) = \frac{1}{2}$. For testing special two-sample location problem, a class of test procedure based on subsample median and maximum is considered.

The proposed class of test statistics is defined by,

$$PD(p, k) = \frac{1}{\binom{m}{p}\binom{n}{k}} \sum_A h(X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_p}; Y_{\beta_1}, Y_{\beta_2}, \ldots, Y_{\beta k}).$$

Where Σ_A indicates the sum over all subsamples of size p and k drawn without replacement from X and Y samples respectively and the symmetric two-sample kernel defined by

$$h(X_1, X_2, \dots, X_p; Y_1, Y_2, \dots, Y_k) = \begin{cases} 1 & \text{if } M_1 \le 0 \le Y_k \\ 0 & \text{otherwise} \end{cases}$$

with $Y_{(k)} = \max (Y_1, Y_2, \dots, Y_k)$, $M_1 = \text{median} (X_1, X_2, \dots, X_p)$.

Here p is an odd positive integer such that $1 \le p \le m$, and $1 \le k \le n$. Here $PD(p, k)$ is the two sample U-statistics with a kernel of degree (p, k) respectively. The test criterion for testing H_0 versus H_1 is to reject H_0 for large value of $PD(p, k)$. That is, reject H_0 if $PD(p, k) > c$, where c is such that $P(PD(p, k) > c \mid H_0) \le \alpha$ and α is the level of significance.

The mean of $PD(p, k)$ is given by

$$E(PD(p, k)) = E(h(X_1, X_2, \dots, X_p; Y_1, Y_2, \dots Y_k))$$

$$= (1 - \overline{F}^k(\theta))\{\sum_{i=a}^{p} \binom{p}{i} F^i(\theta)\overline{F}^{p-i}(\theta)\}$$

$$= \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^k\right), \text{ under } H_0, \text{ where } a = \frac{p+1}{2}.$$

The exact variance of $PD(p, k)$ is difficult to obtain. However, one can obtain the asymptotic variance of $PD(p, k)$ using generalized U-statistics theorem due to Lehmann [3] which is given in following theorem.

Theorem 2.1. *Let* $(X_1, X_2, \dots, X_m)$ *and* $(Y_1, Y_2, \dots, Y_n)$ *denote independent random samples from populations with cdf 's* $F(x)$ *and* $G(y)$ *respectively. Let* $h(\cdot)$ *be a symmetric kernel for an estimable parameter 'γ' of degree* (p, k). *If* $E[h^2(X_1, \dots, X_p; Y_1, \dots, Y_k)] < \infty$ *then* $\sqrt{N}(PD(p, k) - E(PD(p, k)))$ *has a limiting normal distribution with mean zero and variance* $\frac{p^2 \xi_{10}}{\lambda} + \frac{k^2 \xi_{01}}{(1 - \lambda)}$, *provided this variance is positive, where* $0 < \lambda = \lim_{N \to \infty} \frac{m}{N} < 1$ *and* $N = m + n$, *where,*

$$\xi_{10} = Cov\ [h(X_1, X_2, \dots X_p; Y_1, Y_2, \dots, Y_k), h(X_1, X_{p+1}, \dots, X_{2p-1};$$
$$Y_{k+1}, Y_2, \dots Y_{2k})]$$

$$\xi_{01} = \text{Cov}\ [h(X_1, X_2, \ldots X_p; Y_1, Y_2, \ldots, Y_k), h(X_{p+1}, \ldots, X_{2p}; Y_1, \ldots, Y_{k+1}, Y_2, \ldots, Y_{2k-1})].$$

Under H_0,

$$\xi_{10} = C0v_{H_0}[h(X_1, X_2, \ldots, X_p; Y_1, Y_2, \ldots, Y_k), h(X_1, X_{p+1}, X_{2p-1}; Y_{k+1}, Y_2, \ldots, Y_{2k})]$$

$$= \int_{-\infty}^{\infty} [P(med\ (x, x_2, \ldots, X_p) < 0 < \max\ (Y_1, Y_2, \ldots, Y_k))]^2 dF(x) - [E_{H_0}(PD(p, k))]^2. \quad (2.1.1)$$

Now

$$P(\text{med}\ (x, X_2, X_p) < 0 < \max\ (Y_1, Y_2, \ldots, Y_k)) = \binom{p-1}{a-1} d_1 + \binom{p-1}{1}\binom{a-2}{a-1} d_2 + \binom{p-1}{1}\binom{p-2}{a-2} d_3 \quad (2.1.2)$$

with

$$d_1 = P[\max\ (X_2, \ldots, X_a) < x < \min\ (X_{a+1}, X_p), x < 0 < \max\ (Y_1, Y_2, \ldots, Y_k)]$$

$$d_2 = P[\max\ (x, X_2, \ldots, X_{a-1}) < X_a < \min\ (X_{a+1}, \ldots, X_p), X_a < 0 < \max\ (Y_1, Y_2, \ldots, Y_k)]$$

$$d_3 = P[\max\ (X_2, \ldots, X_a) < X_{a+1} < \min\ (x, X_{a+2}, \ldots, X_p), X_{a+1} < 0 < \max\ (Y_1, Y_2, \ldots, Y_k)].$$

Consider,

$$d_1 = P[, \max\ (X_2, \ldots, X_a) < x < \min\ (X_{a+1}, \ldots, X_p), x < 0 < \max\ (Y_1, Y_2, \ldots, Y_k)]$$

$$= P[, \max\ (X_2, \ldots, X_a) < x < \min\ (\min\ (X_{a+1}, \ldots, X_p), 0) < \max\ (Y_1, Y_2, \ldots, Y_k)]$$

$$= d_{11} + d_{12} + d_{13},$$

where

$$d_{11} = P[\max\ (X_2, \ldots, X_a) < x < \min\ (X_{a+1}, \ldots, X_p) < 0 < \max\ (Y_1, Y_2, \ldots, Y_k)]$$

$$d_{12} = P[\max\ (X_2, \ldots, X_a) < x < 0 < \min\ (X_{a+1}, \ldots, X_p) < \max\ (Y_1, Y_2, \ldots, Y_k)]$$

$$d_{13} = P[\max\ (X_2, \ldots, X_a) < x < 0 < \max\ (Y_1, Y_2, \ldots, Y_k) < \min\ (X_{a+1}, \ldots, X_p)].$$

Now,

$$d_2 = P[\max\ (x, X_2, \ldots, X_{a-1}) < X_a < \min\ (X_{a+1}, \ldots, X_p), X_a$$
$$< 0 < \max\ (Y_1, Y_2, \ldots, Y_k)]$$
$$= P\ [\max\ (x, X_2, \ldots, X_{a-1}) < X_a < \min\ (X_{a+1}, \ldots, X_p) < 0 < \max\ (Y_1, Y_2, \ldots, Y_k)]$$
$$+ P\ [\max\ (x, X_2, \ldots, X_{a-1}) < X_a < 0 < \min\ (X_{a+1}, \ldots, X_p) < \max\ (Y_1, Y_2, \ldots, Y_k)]$$
$$+ P\ [\max\ (x, X_2, \ldots, X_{a-1}) < X_a < 0 < \max\ (Y_1, Y_2, \ldots, Y_k) < \min\ (X_{a+1}, \ldots, X_p)]$$
$$= d_{21} + d_{22} + d_{23}$$

with

$$d_{21} = P\ [\max\ (x, X_2, \ldots, X_{a-1}) < X_a < \min\ (X_{a+1}, \ldots, X_p) < 0 < \max\ (Y_1, Y_2, \ldots, Y_k)]$$
$$= P\ [x < \max\ (X_2, \ldots, X_{a-1}) < X_a < \min\ (X_{a+1}, \ldots, X_p) < 0 < \max\ (Y_1, Y_2, \ldots, Y_k)]$$
$$+ P\ [\max\ (X_2, \ldots, X_{a-1}) < x < X_a < \min\ (X_{a+1}, \ldots, X_p) < 0 < \max\ (Y_1, Y_2, \ldots, Y_k)]$$
$$= d_4 + d_5.$$

The quantities d_4 and d_5 are given by

$$d_4 = P\ [x < \max\ (X_2, \ldots, X_{a-1}) < X_a < \min\ (X_{a+1}, \ldots, X_p)$$
$$< 0 < \max\ (Y_1, Y_2, \ldots, Y_k)]$$

$$d_5 = P\ [\max\ (X_2, \ldots, X_{a-1}) < x, < X_a < \min\ (X_{a+1}, \ldots, X_p)$$
$$< 0 < \max\ (Y_1, Y_2, \ldots, Y_k)].$$

Further,

$$d_{22} = P[\max(x, X_2, \ldots, X_{a-1}) < X_a < 0 < \min(X_{a+1}, \ldots, X_p)$$

$$< \max(Y_1, Y_2, \ldots, Y_k)]$$

$$= P[x < \max(X_2, \ldots, X_{a-1}) < X_a < 0 < \min(X_{a+1}, \ldots, X_p) < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$+ P[\max(X_2, \ldots, X_{a-1}) < x < X_a < 0 < \min(X_{a+1}, \ldots, X_p) < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$= d_6 + d_7.$$

Here d_6 and d_7 are

$$d_6 = P[x < \max(X_2, \ldots, X_{a-1}) < X_a < 0 < \min(X_{a+1}, \ldots, X_p) < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$d_7 = P[\max(X_2, \ldots, X_{a-1}) < x < X_a < 0 < \min(X_{a+1}, \ldots, X_p) < \max(Y_1, Y_2, \ldots, Y_k)].$$

Now

$$d_{23} = P[\max(x, X_2, \ldots, X_{a-1}) < X_a < 0 < \max(Y_1, Y_2, \ldots, Y_k)$$

$$< \min(X_{a+1}, \ldots, X_p)]$$

$$= d_8 + d_9,$$

where

$$d_8 = P[x < \max(X_2, \ldots, X_{a-1}) < X_a < 0 < \max(Y_1, Y_2, \ldots, Y_k) < \min(X_{a+1}, \ldots, X_p)]$$

$$d_9 = P[\max(X_2, \ldots, X_{a-1}) < x < X_a < 0 < \max(Y_1, Y_2, \ldots, Y_k) < \min(X_{a+1}, \ldots, X_p)].$$

On similar lines d_3 is evaluated as

$$d_3 = P[\max(X_2, \ldots, X_a) < X_{a+1} < \min(x, X_{a+2}, \ldots, X_p), X_{a+1} < 0 < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$= P[\max(X_2, \ldots, X_a) < X_{a+1} < \min(\min(x, X_{a+2}, \ldots, X_p), 0) < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$= P[\max(X_2, \ldots, X_a) < X_{a+1} < \min(x, X_{a+2}, \ldots, X_p) < 0 < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$+ P[\max(X_2, \ldots, X_a) < X_{a+1} < 0 < \min(x, X_{a+2}, \ldots, X_p) < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$+ P[\max(X_2, \ldots, X_a) < X_{a+1} < 0 < \max(Y_1, Y_2, \ldots, Y_k) < \min(x, X_{a+2}, \ldots, X_p)]$$

$$= d_{31} + d_{32} + d_{33},$$

where

$$d_{31} = P[\max(X_2, \ldots, X_a) < X_{a+1} < \min(x, X_{a+2}, \ldots, X_p)$$

$$< 0 < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$= d_{10} + d_{11} + d_{12} + d_{13} + d_{14} + d_{15}$$

with

$$d_{10} = P[\max(X_2, \ldots, X_a) < X_{a+1} < x < \min(X_{a+2}, \ldots, X_p)$$

$$< 0 < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$d_{11} = P[\max(X_2, \ldots, X_a) < X_{a+1} < x < 0 < \min(X_{a+2}, \ldots, X_p)$$

$$< \max(Y_1, Y_2, \ldots, Y_k)]$$

$$d_{12} = P[\max(X_2, \ldots, X_a) < X_{a+1} < x < 0 < \max(Y_1, Y_2, \ldots, Y_k)$$

$$< \min(X_{a+2}, \ldots, X_p)]$$

$$d_{13} = P[\max(X_2, \ldots, X_a) < X_{a+1} < \min(X_{a+2}, \ldots, X_p)$$

$$< x < 0 < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$d_{14} = P[\max(X_2, \ldots, X_a) < X_{a+1} < \min(X_{a+2}, \ldots, X_p)$$

$$< 0 < x < \max(Y_1, Y_2, \ldots, Y_k)]$$

and

$$d_{15} = P[\max(X_2, \ldots, X_a) < X_{a+1} < \min(X_{a+2}, \ldots, X_p)$$

$$< 0 < \max(Y_1, Y_2, \ldots, Y_k) < x].$$

Next the evaluation of d_{32} is presented. That is,

$$d_{32} = P[\max(X_2, \ldots, X_a) < X_{a+1} < 0 < \min(x, X_{a+2}, \ldots, X_p)$$

$$< \max(Y_1, Y_2, \ldots, Y_k)]$$

$$= d_{16} + d_{17} + d_{18} + d_{19},$$

where

$$d_{16} = P[\max(X_2, \ldots, X_a) < X_{a+1} < 0 < x < \min(X_{a+2}, \ldots, X_p)$$

$$< \max(Y_1, Y_2, \ldots, Y_k)]$$

$$d_{17} = P[\max(X_2, \ldots, X_a) < X_{a+1} < 0 < x < \max(Y_1, Y_2, \ldots, Y_k)$$

$$< \min(X_{a+2}, \ldots, X_p)]$$

$$d_{18} = P[\max(X_2, \ldots, X_a) < X_{a+1} < 0 < \min(X_{a+2}, \ldots, X_p)$$

$$< x < \max(Y_1, Y_2, \ldots, Y_k)]$$

and

$$d_{19} = P[\max(X_2, \ldots, X_a) < X_{a+1} < 0 < \min(X_{a+2}, \ldots, X_p)$$

$$< \max(Y_1, Y_2, \ldots, Y_k) < x].$$

Now,

$$d_{33} = P[\max(X_2, \ldots, X_a) < X_{a+1} < 0 < \max(Y_1, Y_2, \ldots, Y_k)$$

$$< \min(x, X_{a+2}, \ldots, X_p)]$$

$$= d_{20} + d_{21}$$

with

$$d_{20} = P[\max(X_2, \ldots, X_a) < X_{a+1} < 0 < \max(Y_1, Y_2, \ldots, Y_k)$$

$$< x < \min(X_{a+2}, \ldots, X_p)]$$

and

$$d_{21} = P[\max(X_2, \ldots, X_a) < X_{a+1} < 0 < \max(Y_1, Y_2, \ldots, Y_k)]$$

$$< \min(X_{a+2}, \ldots, X_p) < x.$$

Now addition of terms corresponding to $x < 0$ yields,

$$A = \binom{p-1}{a-1}(d_{11} + d_{12} + d_{13}) + \binom{p-1}{1}\binom{p-2}{a-1}(d_4 + d_5 + d_6 + d_7 + d_8 + d_9)$$

$$+\binom{p-1}{1}\binom{p-2}{a-2}(d_{10}+d_{11}+d_{12}+d_{13})$$

and addition of terms corresponding to $x > 0$ yields,

$$B=\binom{p-1}{1}\binom{p-2}{a-2}(d_{14}+d_{15}+d_{16}+d_{17}+d_{18}+d_{19}+d_{20}+d_{21}).$$

Now from [2.1.1] and [2.1.2] we get,

$$\xi_{10}=\int_{-\infty}^{\infty}[P(\text{med }(x, X_2, \dots, X_p)<0<\max(Y_1, Y_2, \dots, Y_k))]^2 dF(x)$$

$$-[E_{H_0}PD(p, k)]^2=\int_{-\infty}^{0}[A]^2 dF(x)+\int_{0}^{\infty}[B]^2 dF(x)-\left[\left(\frac{1}{2}\right)^{k+1}\right]^2 \quad [2.1.3]$$

$$P(\text{med }(X_1, X_2, \dots, X_p)<0<\max(y, Y_2, \dots, Y_k))$$

$$=b_1+b_2+b_3+b_4+b_5+b_6 \quad [2.1.4]$$

where,

$$b_1=P[\text{med }(X_1, \dots, X_p)<0<y<\max(Y_2, \dots, Y_k)]$$

$$b_2=P[\text{med }(X_1, \dots, X_p)<0<\max(Y_2, \dots, Y_k)<y]$$

$$b_3=P[\text{med }(X_1, \dots, X_p)<\max(Y_2, \dots, Y_k)<0<y]$$

$$b_4=P[\max(Y_2, \dots, Y_k)<\text{med }(X_1, \dots, X_k)<0<y]$$

$$b_5=P[\text{med }(X_1, \dots, X_p)<y<0<\max(Y_2, \dots, Y_k)]$$

$$b_6=P[y<\text{med }(X_1, \dots, X_p)<0<\max(Y_2, \dots, Y_k)].$$

Now from equation [2.1.4],

$$\xi_{01}=\text{Cov}_{H_0}[h(X_1, X_2, \dots, X_p; Y_1, Y_2, \dots Y_k), h(X_{p+1}, \dots, X_{2p}; Y_1, Y_{k+1}, \dots, Y_{2k-1})]$$

$$=\int_{-\infty}^{\infty}[P(med\ (X_1, X_2, \dots, X_p)<0<\max(y, Y_2, \dots, Y_k))]^2 dF(x)$$

$$-[E_{H_0}(PD(p, k))]^2.$$

Thus $\sqrt{N}(PD(p, k)-E(PD(p, k)))$ has asymptotically normal distribution with mean zero and variance $\sigma^2 = \frac{p^2 \xi_{10}}{\lambda} + \frac{k^2 \xi_{01}}{(1-\lambda)}$ when H_0 is true.

3. Asymptotic Relative Efficiency

Pitman [9] defined the asymptotic relative efficiency (ARE) of one test P relative to another test Q as the limiting ratio of sample sizes required to obtain the same limiting power for a sequence of alternatives converging to null hypothesis. By Noether's theorem it follows that

$$ARE(P, Q) = \left[\frac{eff(P)}{eff(Q)}\right]^2,$$

where

$$eff(P) = \lim_{N \to \infty} \frac{\frac{d}{d\theta} E(P)|_{\theta=\theta_0}}{\sqrt{NVar_{H_0}(P)}} = \frac{\frac{d}{d\theta} E(P)|_{\theta=\theta_0}}{\frac{p^2 \xi_{10}}{\lambda} + \frac{k^2 \xi_{01}}{(1-\lambda)}}.$$

Under the assumption that $f(x)$ is differentiable at 0 with pdf $f(0) \neq 0$, we have $\frac{d}{d\theta} E(PD(p, k))|_{\theta=\theta_0} = f(0)\sum_{i=0}^{p-1}\binom{p}{i}(0.5)^{p-1}[(1-(0.5)^k(1+k))]$.

The performance of $PD(p, k)$ is compared with the test due to Pandit and Acharya [7] $(W(k, p))$ in terms of Pitman ARE. The values of ARE's $PD(p, k)$ relative $W(k, p)$ for various underlying probability distributions like Logistic, Cauchy, Uniform, Laplace, Triangular and Parabolic are given in Tables from 1 to 6, for $p = 3$ and $p = 5$.

Table 1. ARE of $PD(3, k)(PD(5, k))$ relative to $W(k, 3)$ for Normal Distribution.

$\lambda \downarrow \rightarrow k$	1	2	3	4	5	6
0.1	1.4895 (1.5569)	1.8910 (1.8134)	2.4835 (2.1958)	2.5608 (2.2281)	2.6380 (2.1532)	2.7548 (1.9986)
0.2	1.3505	1.6278	1.6083	1.5908	1.5866	1.4992

	(1.2619)	(1.4868)	(1.3452)	(1.2394)	(1.1543)	(0.9985)
0.3	1.2382 (1.1552)	1.2803 (1.0251)	1.2574 (0.9023)	1.1743 (0.8213)	1.0957 (0.8084)	0.9823 (0.7615)
0.4	1.1421 (0.8583)	1.0063 (0.7666)	0.9208 (0.6696)	0.8280 (0.5904)	0.7538 (0.5341)	0.7029 (0.4968)
0.5	0.9989 (0.7618)	0.8614 (0.7109)	0.8163 (0.5858)	0.7724 (0.5153)	0.7264 (0.5081)	0.6638 (0.4972
0.6	0.9432 (0.6941)	0.8284 (0.6432)	0.7533 (0.5637)	0.6864 (0.4952)	0.6573 (0.4573)	0.5805 (0.4313)
0.7	0.8836 (0.6123)	0.7212 (0.5312)	0.6743 (0.4983)	0.5894 (0.4527)	0.5364 (0.4363)	0.5138 (0.3844)
0.8	0.8423 (0.5816)	0.7018 (0.5123)	0.6362 (0.4513)	0.5304 (0.4045)	0.4824 (0.3689)	0.4530 (0.3491)
0.9	0.7023 (0.5058)	0.6672 (0.4869)	0.5552 (0.4134)	0.4849 (0.3676)	0.4404 (0.3394)	0.4141 (0.3011)

Table 2. ARE of $PD(3, k)(PD(5, k))$ relative to $W(k, 3)$ for Logistic Distribution.

$\lambda \downarrow \rightarrow k$	1	2	3	4	5	6
0.1	1.9562 (2.0868)	2.2871 (2.3627)	2.6808 (2.4421)	2.7505 (2.4853)	2.7913 (2.4049)	2.7719 (2.3551)
0.2	1.9197 (1.8218)	2.0972 (1.8918)	2.3260 (1.8593)	2.3854 (1.7985)	2.1376 (1.7334)	1.9913 (1.6852)
0.3	1.7937 (1.6869)	1.8547 (1.6473)	1.8277 (1.5577)	1.7697 (1.4921)	1.6958 (1.4282)	1.6389 (1.3857)
0.4	1.7287 (1.6527)	1.6623 (1.5492)	1.6155 (1.4321)	1.5386 (1.3546)	1.4716 (1.2845)	1.4273 (1.2066)
0.5	1.6965 (1.5318)	1.6453 (1.4247)	1.5746 (1.3507)	1.4901 (1.2747)	1.4238 (1.2185)	1.3823 (1.1841)

0.6	1.6636 (1.4485)	1.5509 (1.3876)	1.4657 (1.2914)	1.3784 (1.1082)	1.3139 (1.0251)	1.2757 (0.9936)
0.7	1.6106 (1.4122)	1.5723 (1.3492)	1.4981 (1.2073)	1.3804 (1.1023)	1.2681 (1.0524)	1.1914 (0.9732)
0.8	1.5627 (1.3179)	1.5051 (1.2041)	1.4058 (1.1009)	1.3188 (0.9911)	1.2087 (0.9737)	1.0245 (0.8664)
0.9	1.5191 (1.2654)	1.4473 (1.2029)	1.1048 (0.9955)	1.0594 (0.8931)	1.001 (0.8697)	0.9681 (0.8994)

Table 3. ARE of $PD(3, k)(PD(5, k))$ relative to $W(k, 3)$ for Cauchy Distribution.

$\lambda \downarrow \rightarrow k$	1	2	3	4	5	6
0.1	1.5293 (1.7293)	2.3793 (1.9889)	2.8845 (2.4966)	2.9202 (2.5883)	2.9603 (2.2546)	2.9992 (2.1906)
0.2	1.4532 (1.4183)	1.6592 (1.5667)	1.5992 (1.5261)	1.5506 (1.4396)	1.4826 (1.3545)	2.4513 (1.2893)
0.3	1.4375 (1.3053)	1.5921 (1.2540)	1.4532 (1.2364)	1.3773 (1.2080)	1.2653 (0.9956)	1.1200 (0.9164)
0.4	1.4195 (1.2532)	1.3683 (1.2269)	1.3212 (1.1099)	1.2844 (0.9906)	1.2508 (0.8354)	1.0923 (0.7976)
0.5	1.3623 (0.9498)	1.2519 (0.9056)	1.1643 (0.8546)	1.0529 (0.8436)	0.9979 (0.7287)	0.9849 (0.6978)
0.6	0.9933 (0.9043)	0.8685 (0.8599)	0.7537 (0.7408)	0.6668 (0.7086)	0.6017 (0.6598)	0.5930 (0.6318)
0.7	0.9343 (0.8968)	0.8876 (0.8138)	0.8209 (0.7386)	0.5892 (0.5429)	0.5226 (0.4071)	0.5073 (0.3889)
0.8	0.9219 (0.7518)	0.8218 (0.6283)	0.7506 (0.5514)	0.5205 (0.4608)	0.46705 (0.3887)	0.4578 (0.3593)
0.9	0.8359 (0.6610)	0.7676 (0.5161)	0.6554 (0.4139)	0.5843 (0.3680)	0.5633 (0.3591)	0.4694 (0.3414)

Table 4. ARE of $PD(3, k)(PD(5, k))$ relative to $W(k, 3)$ for Laplace Distribution.

$\lambda \downarrow \rightarrow k$	1	2	3	4	5	6
0.1	1.8489 (1.3234)	1.9778 (1.8745)	2.5827 (2.4932)	2.9784 (2.5701)	3.2176 (2.2121)	3.0285 (2.0020)
0.2	1.7630 (1.2924)	1.6069 (1.4657)	1.5774 (1.3233)	1.5395 (1.2376)	1.5051 (1.1336)	1.4912 (1.0885)
0.3	1.5564 (1.0048)	1.4301 (0.9946)	1.3772 (0.9657)	1.2366 (0.8774)	1.1953 (0.8086)	1.0337 (0.7611)
0.4	1.4416 (0.9979)	1.3560 (0.9463)	1.2805 (0.8931)	1.1377 (0.8401)	1.0036 (0.7339)	0.9726 (0.7066)
0.5	0.9981 (0.9483)	0.9811 (0.9010)	0.9642 (0.8456)	0.9023 (0.7753)	0.8663 (0.7486)	0.8232 (0.6476)
0.6	0.9719 (0.9233)	0.9684 (0.8995)	0.9532 (0.8305)	0.8661 (0.7383)	0.8072 (0.6570)	0.7700 (0.6317)
0.7	0.9576 (0.9164)	0.8890 (0.8814)	0.8504 (0.7983)	0.8088 (0.6827)	0.7359 (0.6408)	0.7032 (0.6243)
0.8	0.9353 (0.8941)	0.8414 (0.8283)	0.7563 (0.7509)	0.6801 (0.6007)	0.6029 (0.5687)	0.5980 (0.5289)
0.9	0.9054 (0.8054)	0.8075 (0.7858)	0.7351 (0.7136)	0.6540 (0.6677)	0.5403 (0.5390)	0.5240 (0.5010)

Table 5. ARE of $PD(3, k)(PD(5, k))$ relative to $W(k, 3)$ for Uniform Distribution.

$\lambda \downarrow \rightarrow k$	1	2	3	4	5	6
0.1	1.8489 (1.6282)	2.3778 (2.0973)	2.9834 (2.3951)	3.4301 (2.8719)	3.6543 (3.0521)	3.0285 (2.9010)
0.2	1.2500 (1.5123)	1.4569 (1.3157)	1.5577 (1.3012)	1.5394 (1.2317)	1.4655 (1.1532)	1.3914 (1.0885)
0.3	1.6363	1.5903	1.4572	1.3763	1.3554	1.2331

	(1.3448)	(1.2947)	(1.2657)	(1.1974)	(1.0987)	(0.9987)
0.4	1.4233 (1.1224)	1.3762 (1.0993)	1.3505 (1.0424)	1.2974 (1.0102)	1.2636 (0.9959)	1.2011 (0.9566)
0.5	1.2616 (1.0925)	1.1223 (1.0723)	1.0982 (1.0423)	1.0343 (1.0156)	0.9987 (0.9521)	0.9635 (0.9072)
0.6	1.0829 (0.9838)	0.9982 (0.8495)	0.9532 (0.7604)	0.8661 (0.6988)	0.7500 (0.5575)	0.6700 (0.5314)
0.7	0.9934 (0.8963)	0.9870 (0.8214)	0.8704 (0.7083)	0.7988 (0.6526)	0.7359 (0.5306)	0.6402 (0.5043)
0.8	0.9313 (0.8415)	0.9113 (0.8083)	0.8663 (0.6509)	0.7301 (0.6006)	0.6822 (0.5187)	0.5951 (0.5009)
0.9	0.9143 (0.7954)	0.8971 (0.6958)	0.8552 (0.6136)	0.7240 (0.5677)	0.6403 (0.5091)	0.5740 (0.4913)

Table 6. ARE of $PD(3, k)(PD(5, k))$ relative to $W(k, 3)$ for Triangular Distribution.

$\lambda \downarrow \rightarrow k$	1	2	3	4	5	6
0.1	3.2774 (3.1564)	4.2845 (3.0541)	5.3646 (2.9812)	5.9987 (2.8544)	6.5750 (2.8055)	5.9870 (2.5157)
0.2	3.0120 (2.9282)	3.9732 (2.8376)	4.1152 (2.7550)	5.0784 (2.6775)	4.8312 (2.5603)	3.9224 (2.4789)
0.3	2.9798 (2.7094)	3.9380 (2.5393)	4.0156 (2.4316)	4.5301 (2.3537)	4.5902 (2.1172)	3.5674 (2.0922)
0.4	2.9373 (2.6312)	3.5194 (2.3256)	3.8409 (2.1384)	4.0555 (2.0003)	3.5073 (1.9676)	2.8953 (1.9306)
0.5	2.9032 (2.5835)	3.4678 (2.2845)	3.6283 (2.0943)	3.9446 (1.9413)	3.2826 (1.8865)	2.6514 (1.8552)
0.6	2.7778 (2.4275)	3.3644 (2.1643)	3.5648 (1.9989)	3.9271 (1.7976)	3.0144 (1.4965)	2.4343 (1.3675)
0.7	2.6543	3.2057	3.4087	3.7765	2.9784	2.2043

	(2.3217)	(2.0428)	(1.9696)	(1.7653)	(1.3626)	(1.2714)
0.8	2.5625 (2.1820)	3.1421 (2.0016)	3.3168 (1.8673)	3.6602 (1.7231)	2.7344 (1.2972)	2.1032 (1.1932)
0.9	2.3606 (1.9543)	3.0344 (1.7878)	3.2901 (1.6323)	3.5645 (1.5252)	2.3452 (1.1782)	2.0721 (1.0405)

4. Simulation Study

A simulation study is conducted to estimate small sample powers of the tests based on $PD(p, k)$, Pandit and Acharya [7] $(W(k, p))$ Shetty and Umarani [12] $(U_3(p, p))$ for Normal, Cauchy, Logistic, and Laplace distributions by generating 100000 samples of different sizes. The power curves of these tests are presented in the following figures.

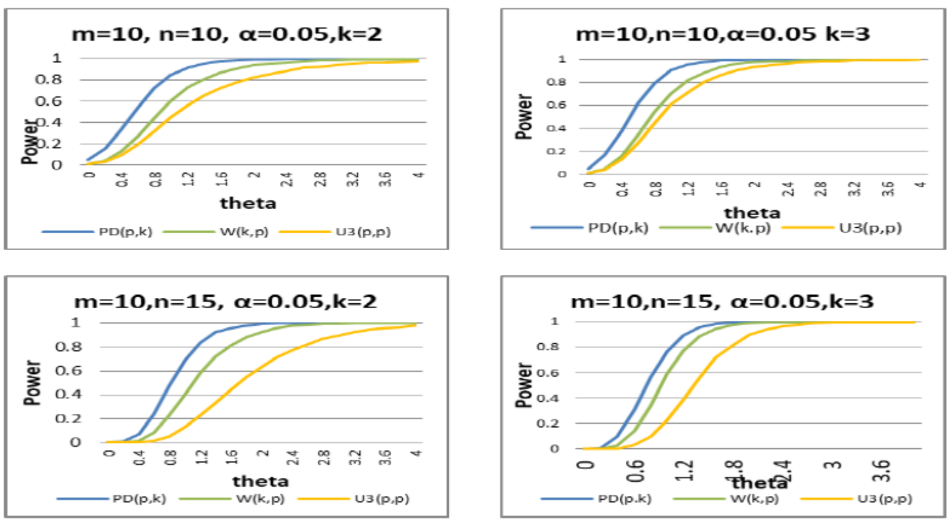

Figure 1. Power curve of $PD(p, k)$, $W(k, p)$ and $U_3(p, p)$ for Normal Distribution at $p = 3$ and 5% level of significance.

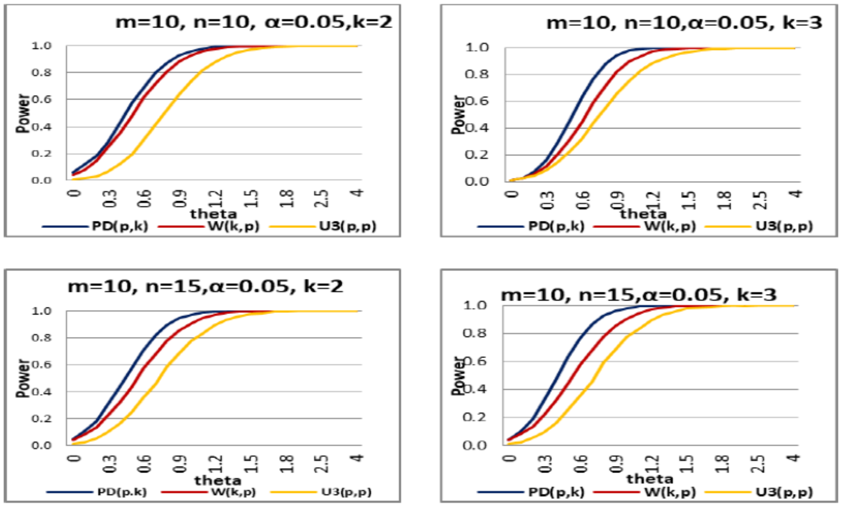

Figure 2. Power curve of $PD(p, k)$, $W(k, p)$ and $U_3(p, p)$ for Cauchy Distribution at $p = 3$ and 5% level of significance.

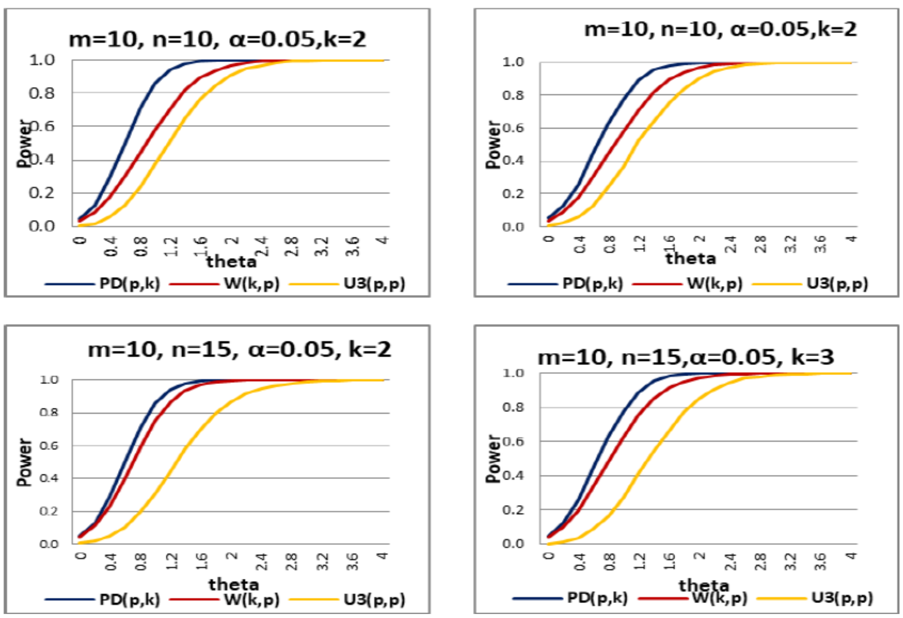

Figure 3. Power curve of $PD(p, k)$, $W(k, p)$ and $U_3(p, p)$ for Logistic Distribution at $p = 3$ and 5% level of significance.

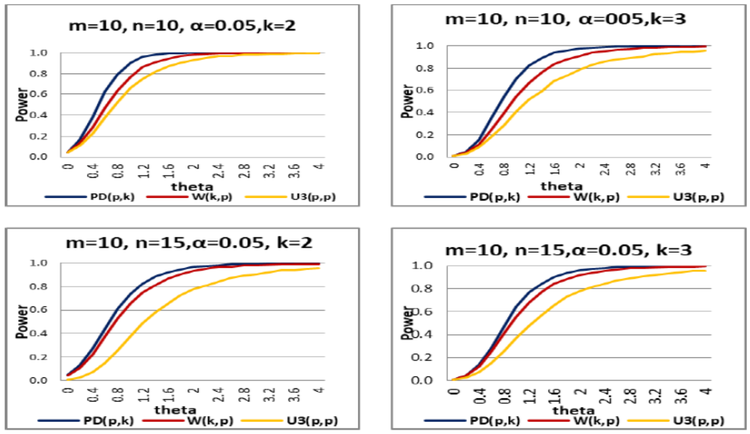

Figure 4. Power curve of $PD(p, k)$, $W(k, p)$ and $U_3(p, p)$ for Laplace Distribution at $p = 3$ and 5% level of significance.

5. Some Remarks and Conclusions

1. A class of test statistics for special two-sample location problem is proposed assuming the underlying distribution of the sample drawn to be symmetric.

2. The performances of few members of the proposed class are evaluated in terms of asymptotic relative efficiencies (AREs).

3. It follows from Lehmann [3] that the test is consistent for testing H_0 against H_1, since the expected value of $PD(p, k)$ under H_1 is greater than its expected value under H_1 and the asymptotic distribution of the test statistic is normal.

4. It is observed that the performance of the proposed test is better than the tests due to Pandit and Acharya [7] for this problem for the distributions Normal, Logistic, Cauchy and Laplace distributions.

5. Simulation experiment is conducted to evaluate small sample powers. It is observed that power increases as the value of theta increases. It is a desirable property for any reasonable test. The power curves are drawn and it is observed that the test proposed in this paper is uniformly better than the tests considered are Pandit and Acharya [7] and Shetty and Umarani [12]

when the underlying sample is from Cauchy, Laplace, Logistic and Normal distributions.

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