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ON POLYNOMIAL CENTROSYMMETRIC MATRICES

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Abstract

The basic concepts of polynomial centrosymmetric matrices are introduced. Some properties and characterization for polynomial centrosymmetric matrices are obtained with examples.

I. Introduction

Matrix algebra is the chief mathematical tool used in the multiple factor analysis of psychometrics. Matrix have invaded the business world and such subjects as linear programming utilize matrix notation and problems.

For a number of decades symmetric matrices over the real field have been studied intently by every beginning linear algebra. In [1] Ann lee defines a centrosymmetric matrix P and this definition coincides with the definition given by Graybill of a cross-symmetric matrix [3].

The investigation of Centrosymmetric matrices by I. J. Good [4] and Ray [6] was motivated by the study of certain toeplitz matrices Cruse [2] encountered the group of $n \times n.$ Centrosymmetric permutation matrices in his study of problems from combinational theory.

A matrix $A(\lambda)$ is said to be a Polynomial matrix if all entries of $A(\lambda)$ are Polynomials. Polynomials and Polynomial matrices arise naturally as modelling tools in several areas of science and engineering.

In this paper we will discuss about the basic properties and theorems on

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Polynomials Centrosymmetric matrices, also we will discuss some results on Polynomial centrosymmetric matrices.

II. Preliminaries and Notations

P is Polynomial Centrosymmetric matrix, P^T is called Transpose of P.

Definition 2.1. A Square matrix $A = [a_{ij}]_{n \times n}$ is said to be Symmetric if $A = A^T$ (i.e.,) $a_{ij} = a_{ji} \forall i, j$.

Definition 2.2. A Square matrix which is Symmetric about the centre of its array of elements is called Centrosymmetric thus $C = [c_{ij}]_{n \times n}$ Centrosymmetric if $c_{ij} = c_{n-i+1, n-j+1}$.

Definition 2.3. A matrix $A(\lambda)$ is said to be a Polynomial matrix if all entries of $A(\lambda)$ are Polynomials.

Definition 2.4. A square polynomial matrix $P(\lambda)$ is said to be Polynomial Centrosymmetric if $P(\lambda) = P(\lambda)^T$. In other words, all the coefficient matrices of $P(\lambda)$ are Centrosymmetric.

III. Polynomial Centrosymmetric Matrix

Definition 3.1. A Polynomial Centrosymmetric matrix is a centrosymmetric matrix whose coefficients matrices are centrosymmetric matrix.

Example 3.2. Let

$$P(\lambda) = \begin{bmatrix} 2 - \lambda + 2 & 4 + 3\lambda + 3\lambda^2 & 1 + 2\lambda - \lambda^2 \\ 4 + 3\lambda + 3\lambda^2 & 7 + 6\lambda + 4\lambda^2 & 4 + 3\lambda + 3\lambda^2 \\ 1 + 2\lambda - \lambda^2 & 4 + 3\lambda + 3\lambda^2 & 2 - \lambda + 2 \end{bmatrix} = P_0 + P_1\lambda + P_2\lambda^2$$

$$P_0 = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 7 & 4 \\ 1 & 4 & 2 \end{bmatrix} P_1 = \begin{bmatrix} -1 & 3 & 2 \\ 3 & 6 & 3 \\ 2 & 3 & -1 \end{bmatrix} P_2 = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

$$P_0P_0^T = P_0^TP_0 = \begin{bmatrix} 21 & 40 & 20 \\ 40 & 81 & 40 \\ 20 & 40 & 21 \end{bmatrix}$$

Hence, $P_0P_0^T = P_0^TP_0$

$$P_1P_1^T = P_1^TP_1 = \begin{bmatrix} 14 & 21 & 5 \\ 21 & 54 & 21 \\ 5 & 21 & 14 \end{bmatrix}$$

Hence, $P_1P_1^T = P_1^TP_1$

$$P_2P_2^T = P_2^TP_2 = \begin{bmatrix} 14 & 15 & 5 \\ 15 & 30 & 15 \\ 5 & 15 & 14 \end{bmatrix}$$

Hence, $P_2P_2^T = P_2^TP_2$.

Theorem 3.3. *A polynomial centrosymmetric matrix is always Centrosymmetric.*

Proof. Let $P(\lambda) = P_0 + P_1\lambda + P_2\lambda^2 + \ldots + P_n\lambda^n$ be a Polynomial centrosymmetric matrix.

Here the coefficient matrices P_i's are Polynomial centrosymmetric matrices.

Since, polynomial centrosymmetric matrices are centrosymmetric, the coefficient matrices of $P(\lambda)$ are all Centrosymmetric.

Hence, $P(\lambda)$ is a centrosymmetric matrix.

Example 3.4. Consider the polynomial centrosymmetric matrix

Let

$$P(\lambda) = \begin{bmatrix} 2 - 4\lambda & 5 + 4\lambda + 6\lambda^2 & 1 + 2\lambda - 3\lambda^2 \\ 5 + 4\lambda + 6\lambda^2 & 7 + 6\lambda + 4\lambda^2 & 5 + 4\lambda + 6\lambda^2 \\ 1 + 2\lambda - 3\lambda^2 & 5 + 4\lambda + 6\lambda^2 & 2 - 4\lambda \end{bmatrix} = P_0 + P_1\lambda + P_2\lambda^2,$$

Where,

$$P_0 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & 7 & 5 \\ 1 & 5 & 2 \end{bmatrix} P_1 = \begin{bmatrix} -4 & 4 & 2 \\ 4 & 6 & 4 \\ 2 & 4 & -4 \end{bmatrix} P_2 = \begin{bmatrix} 0 & 6 & -3 \\ 6 & 4 & 6 \\ -3 & 6 & 0 \end{bmatrix}.$$

Here, P_0, P_1, P_2 are Polynomial centrosymmetric matrices.

$$P_0 = P_0^T = \begin{bmatrix} 2 & 5 & 1 \\ 5 & 7 & 5 \\ 1 & 5 & 2 \end{bmatrix}$$

$$P_1 = P_1^T = \begin{bmatrix} -4 & 4 & 2 \\ 4 & 6 & 4 \\ 2 & 4 & -4 \end{bmatrix} P_2 = P_2^T = \begin{bmatrix} 0 & 6 & -3 \\ 6 & 4 & 6 \\ -3 & 6 & 0 \end{bmatrix}$$

i.e., P_0, P_1, P_2 all are Centrosymmetric.

Also, $P_0 = P_0^T$, $P_1 = P_1^T$, $P_2 = P_2^T$.

Hence, $P(\lambda)$ is a centrosymmetric.

Theorem 3.5. *If $P(\lambda)$ is a $n \times n$ Polynomial Centrosymmetric matrix, then all its coefficients matrices are Polynomial Centrosymmetric matrix.*

Proof. Let $P(\lambda) = P_0 + P_1 + P_2\lambda^2 + \dots P_n\lambda^n$ be a Polynomial centrosymmetric matrix. Here the coefficient matrices Pi's are Polynomial centrosymmetric matrices.

i.e., Since, Polynomial Centrosymmetric matrices are Centrosymmetric.

i.e., $P_i = P_i^T$ for $i = 0, 1, \dots, n$. Hence $P_i P_i^T = P_i^T P_i = P_i^2$. Hence proved.

Example 3.6. Consider the Polynomial Centrosymmetric matrix.

$$\text{Let } P(\lambda) = \begin{bmatrix} 3\lambda & -1-2\lambda & 4-5\lambda \\ -1-2\lambda & 3+6\lambda & -1-2\lambda \\ 4-5\lambda & -1-2\lambda & 3\lambda \end{bmatrix} = P_0 + P_1\lambda$$

$$P_0 = \begin{bmatrix} 0 & -1 & 4 \\ -1 & 3 & -1 \\ 4 & -1 & 0 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 3 & -2 & -5 \\ -2 & 6 & -2 \\ -5 & -2 & 3 \end{bmatrix}$$

$$P_0 P_0^T = P_0^T P_0 = \begin{bmatrix} 17 & -7 & 1 \\ -7 & 11 & -7 \\ 1 & -7 & 17 \end{bmatrix}$$

$$P_0^2 = \begin{bmatrix} 17 & -7 & 1 \\ -7 & 11 & -7 \\ 1 & -7 & 17 \end{bmatrix}$$

Hence, $P_0 P_0^T = P_0^T = P_0^T P_0 = P_0^2$

$$P_1 P_1^T = P_1^T P_1 = \begin{bmatrix} 38 & -8 & -26 \\ -8 & 36 & -8 \\ -26 & -8 & 38 \end{bmatrix}$$

$$P_1^2 = \begin{bmatrix} 38 & -8 & -26 \\ -8 & 36 & -8 \\ -26 & -8 & 38 \end{bmatrix}.$$

Hence, $P_1 P_1^T = P_1^T = P_1^T P_1 = P_1^2$. Hence all coefficients matrices P_0, P_1 are Polynomial centrosymmetric matrices.

Theorem 3.7. *If* $P(\lambda)$ *is a polynomial centrosymmetric matrix iff* $[P(\lambda)]^T$ *is a polynomial centrosymmetric matrix.*

Proof. Necessary Part:

Let $P(\lambda) = P_i(\lambda^i) = P_0 + P_1\lambda + P_2\lambda^2 + \ldots + P_n\lambda^n$ be a Polynomial centrosymmetric matrix. Here the coefficient matrices Pi's are Polynomial centrosymmetric matrices.

i.e.,

$$\left.\begin{aligned} P_0 P_0^T &= P_0^T P_0 \\ P_1 P_1^T &= P_1^T P_1 \\ &\ldots \\ P_n P_n^T &= P_n^T P_n \end{aligned}\right\} \to *$$

To prove $[P_i(\lambda^i)]^T$ is a Polynomial Centrosymmetric matrix

From above equation

$$[P_i(\lambda^i]^T = [P_0^T + P_1^T \lambda + \ldots + P_n^T \lambda^n].$$

We know that coefficient matrices P_i's are Polynomial centrosymmetric matrices. Hence, $[P_i(\lambda^i)]^T$ is a Polynomial Centrosymmetric matrix.

i.e., $[P(\lambda)]^T$ is a Polynomial Centrosymmetric matrix.

Sufficient Part:

Let $[P(\lambda)]^T = [P_i(\lambda^i)]^T$ is a Polynomial Centrosymmetric matrix.

Here $[P_i(\lambda^i)]^T = \sum_{i=0}^{n} [P_i(\lambda^i)]$

i.e., $[P_i(\lambda^i)]^T = [P_0^T + P_1^T \lambda + \ldots + P_n^T \lambda^n]$

where P_i's are Polynomial centrosymmetric matrices.

To prove $P_i(\lambda^i)$ is a Polynomial Centrosymmetric matrix.

i.e., $P_0 P_0^T = P_0^T P_0$

$P_1 P_1^T = P_1^T P_1$

$\ldots$

$P_n P_n^T = P_n^T P_n$

i.e., $P_0 P_0^T = P_0^T P_0 \Rightarrow P_0 = P_0^T$

…

$$P_n P_n^T = P_n^T P_n \Rightarrow P_n = P_n^T$$

i.e., $[P_i(\lambda^i)]^T = [P_0^T + P_1^T\lambda + \ldots + P_n^T\lambda^n]$ is a polynomial centrosymmetric matrices.

Therefore, $[P_i(\lambda^i)]$ is a Polynomial Centrosymmetric matrix.

i.e., $[P_i(\lambda^i)]^T = [P_i(\lambda^i)] = [P(\lambda)]$

i.e., $[P(\lambda)]$ is a Polynomial Centrosymmetric matrix.

Hence proved.

Example 3.8. Let

$$P(\lambda) = \begin{bmatrix} -1 + 2\lambda - \lambda^2 & 4 + 3\lambda + 2\lambda^2 & 2 + \lambda + \lambda^2 \\ 4 + 3\lambda + 2\lambda^2 & 3 + 4\lambda + 5\lambda^2 & 4 + 3\lambda + 2\lambda^2 \\ 2 + \lambda + \lambda^2 & 4 + 3\lambda + 2\lambda^2 & -1 + 2\lambda - \lambda^2 \end{bmatrix} = P_0 + P_1\lambda + P_2\lambda^2$$

$$[P(\lambda)]^T = \begin{bmatrix} -1 + 2\lambda - \lambda^2 & 4 + 3\lambda + 2\lambda^2 & 2 + \lambda + \lambda^2 \\ 4 + 3\lambda + 2\lambda^2 & 3 + 4\lambda + 5\lambda^2 & 4 + 3\lambda + 2\lambda^2 \\ 2 + \lambda + \lambda^2 & 4 + 3\lambda + 2\lambda^2 & -1 + 2\lambda - \lambda^2 \end{bmatrix} = P_0 + P_1\lambda + P_2\lambda^2$$

Here, $P(\lambda) = [P(\lambda)]^T$.

Now, $P(\lambda) = \begin{bmatrix} -1 & 4 & 2 \\ 4 & 3 & 4 \\ 2 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 3 \\ 1 & 4 & 2 \end{bmatrix}\lambda + \begin{bmatrix} -1 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & -1 \end{bmatrix}\lambda^2$

$$P_0 = P_0^T = \begin{bmatrix} -1 & 4 & 2 \\ 4 & 3 & 4 \\ 2 & 4 & -1 \end{bmatrix}$$

$$P_1 = P_1^T = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$P_2 = P_2^T = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & -1 \end{bmatrix}.$$

Hence, $P_0 = P_0^T$, $P_1 = P_1^T$, $P_2 = P_2^T$.

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