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CONNECTED MAJORITY DOM-CHROMATIC NUMBER OF A GRAPH

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Abstract

This paper introduces a connected majority dom-chromatic set of a graph G and a connected majority dom-chromatic number $\gamma_{CM\chi}(G)$. The exact value of $\gamma_{CM\chi}$ for some classes of graphs such as Grid, Cylinder and Corona are determined. Also the characterization theorems on $\gamma_{CM\chi}$ for some graphs are established.

1. Introduction

By a graph $G = (V, E)$, we mean a finite and undirected graph with neither loops nor multiple edges. This article introduces a new parameter namely Connected majority dom-chromatic number of G. A subset D of $V(G)$ is said to be a dominating set [3] of G if every vertex in $(V - D)$ is adjacent to at least one vertex in D. The minimum cardinality of the minimal dominating set of G is called the domination number of G, denoted by $\gamma(G)$. A dominating

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set D is said to be a connected dominating set [8] if the subgraph $\langle D \rangle$ induced by D is connected in G. A connected dominating set is minimal if no proper subset of D is a connected dominating set. The minimum cardinality of the minimal connected dominating set of G is called the connected domination number and denoted by $\gamma_C(G)$.

The majority dominating number $\gamma_M(G)$ [7] of a graph G is the smallest cardinality of a minimal majority dominating set (MD-set) $S \subseteq V(G)$ of vertices and the set S satisfies $|N[S]| \geq \left\| \left\lceil \frac{V(G)}{2} \right\rceil \right\|$. A set $S \subseteq V(G)$ is a connected majority dominating (CMD) set [6] if S is a majority dominating set and the induced subgraph $\langle S \rangle$ is connected in G. A connected majority dominating set is minimal if no proper subset S of G is a connected majority dominating set. The minimum cardinality of the minimal connected majority dominating set S of G is called the connected majority domination number and denoted by $\gamma_{CM}(G)$.

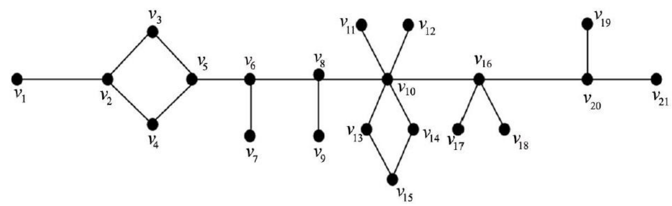
A dominating set $S \subseteq V(G)$ is called the dom-chromatic set [1] and [4] such that the induced subgraph $\langle S \rangle$ satisfies the property $\chi(\langle S \rangle) = \chi(G)$. The minimum cardinality of a dom-chromatic set S is called dom-chromatic number and is denoted by $\gamma_{ch}(G)$ or $\gamma_{\chi}(G)$. A dom-chromatic set S is said to be connected dom-chromatic set if $\langle S \rangle$ is connected. The minimum cardinality of a connected dom-chromatic set S is called connected dom-chromatic number and is denoted by $\gamma_{cch}(G)$ or $\gamma_{C\chi}(G)$.

A subset S of $V(G)$ is majority dominating chromatic set (MDC- set) [5] if (i) S is a majority dominating set and (ii) $\chi(\langle S \rangle) = \chi(G)$. The minimum cardinality of a minimal majority dominating chromatic set is called a majority dominating chromatic number denoted by $\gamma_{M\chi}(G)$.

2. Definitions and Examples

Definition 2.1. A majority dominating chromatic set S is said to be a connected majority dominating chromatic (Connected MDC) set or connected majority dom-chromatic set if the induced subgraph $\langle S \rangle$ is connected in G. The connected MDC set is minimal if no proper subset of S is a connected MDC set. The minimum cardinality of a minimal connected MDC set is called the connected MDC number and is denoted by $\gamma_{CM\chi}(G)$. The maximum cardinality of a minimal connected MDC set is called the upper connected MDC number of G and denoted by $\Gamma_{CM\chi}(G)$.

Example 2.2. Consider the graph G with $p = 21$ vertices.



G : Figure - (i)

For the above graph, $S_1 = \{v_6, v_8, v_{10}\}, S_2 = \{v_8, v_{10}, v_{16}\}$ are the minimal connected MDC sets of G. Hence $\gamma_{CM\chi}(G) = 3$ and $\Gamma_{CM\chi}(G) = 6$. For the graph G, $\gamma_{M\chi}(G) = 3, \gamma_{ch}(G) = 7$ and $\gamma_{M\chi}(G) = 2$.

Observations 2.3.

(i) If the graph G is vertex color critical graph then $\gamma_{CM\chi}(G) = \gamma_{M\chi}(G) = p$.

(ii) Let G be a triangle free with $\chi(G) \geq 5$. Then $\gamma_{CM\chi}(G) \geq 5$.

(iii) For any bipartite graph with dominating edge, $\gamma_{CM\chi}(G) = 2$.

(iv) If a connected graph G has at least one full degree vertex then $\gamma_C(G) < \gamma_{CM\chi}(G)$.

For example, $G = K_{1,\, p-1}, \gamma_C(G) = 1$ and $\gamma_{CM\chi}(G) = 2$.

(v) Let G be a vertex color critical graph. Then $\gamma_C(G) < \gamma_{CM\chi}(G)$.

(vi) If a connected graph G with at least one MD vertex v then

$\gamma_{CM\chi}(G) = \gamma_C(G)$.

For example, $G = D_{r,\,s},\ r \le s,\ \gamma_C(G) = 2$ and $\gamma_{CM\chi}(G) = 2$.

3. $\gamma_{CM\chi}(G)$ for Some Classes of Graphs

3.1 Results on $\gamma_{CM\chi}(G)$

(i) Let $G = D_{r,\,s},\ K_{1,\,p-1},\ p \ge 2$. Then $\gamma_{CM\chi}(G) = 2$.

(ii) Let $G = K_{m,\,n}$. Then $\gamma_{CM\chi}(G) = 2$.

(iii) For the graph $G = m\overline{K}_2,\ \gamma_{CM\chi}(G) = 3$.

(iv) Let $G = K_p$ be a Complete graph. Then $\gamma_{CM\chi}(G) = p$.

(v) For any Caterpillar graph G, $\gamma_{CM\chi}(G) = \left\lceil \frac{p}{4} \right\rceil - 1$.

Proposition 3.2. *For any Cycle* $G = C_p$,

$$\gamma_{CM\chi}(G) = \begin{cases} p, \text{ if } p \text{ is odd} \\ \left\lceil \frac{p}{4} \right\rceil - 2, \text{ if } p \text{ is even and } p \ge 8. \end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ be the vertex set of G. For a Cycle $G = C_p$,

$$\chi(G) = \begin{cases} 2, \text{ if } p \text{ is even} \\ 3, \text{ if } p \text{ is odd.} \end{cases}$$

Case (i). Let p be odd. Then the graph $G = C_p$ becomes an odd Cycle. Since the graph G is vertex color critical, by the result (4.4) [5], $\gamma_{M\chi}(G) = p$. It implies that $\gamma_{M\chi}(G) = |S| = |\{v_1, v_2, \ldots, v_p\}|$ where S is a MDC set of G. Hence the induced subgraph $\langle S \rangle$ is connected .Therefore S is a connected MDC set of G. Thus $\gamma_{CM\chi}(G) = p$.

Case (ii). Let p be an even. Let S be any set in G and $S = \{v_1, v_2, \ldots, v_t\}$,

$|t| = \left\lceil \frac{p}{2} \right\rceil - 2$ with $d(v_i, v_{i+1}) = 1, i = 1, 2 \ldots, (t-1)$. Then $|N[S]| = \left\lceil \frac{p}{2} \right\rceil - 2 + 2 = \left\lceil \frac{p}{2} \right\rceil$. Since $\chi(G) = 2, (\langle S \rangle) = \chi(G)$. Therefore S is a $\gamma_{M\chi}$-set of G. Since $d(v_i, v_{i+1}) = 1$, the vertices of S are in consecutive. Thus the induced subgraph $\langle S \rangle$ is connected. Hence S is a $\gamma_{CM\chi}$-set of G. It implies that

$$\gamma_{CM\chi}(G) \leq \left\lceil \frac{p}{2} \right\rceil - 2. \tag{1}$$

Let $S' = S - \{v\}$ with $S' = \left\lceil \frac{p}{2} \right\rceil - 3$. Then $|N[S']| = \left\lceil \frac{p}{2} \right\rceil - 3 + 2 = \left\lceil \frac{p}{2} \right\rceil - 1 < \left\lceil \frac{p}{2} \right\rceil$. Hence the set S' will not be a γ_M-set of G. Therefore $\gamma_{CM\chi}(G) > |S| = \left\lceil \frac{p}{2} \right\rceil - 3$ and

$$\gamma_{CM\chi}(G) \geq \left\lceil \frac{p}{2} \right\rceil - 2. \tag{2}$$

Hence from (1) and (2), $\gamma_{CM\chi}(G) = \left\lceil \frac{p}{2} \right\rceil - 2$.

Proposition 3.3. *Let G be a Path P_p, $p \geq 7$. Then* $\gamma_{CM\chi}(G) = \left\lceil \frac{p}{2} \right\rceil - 2$.

Proof. From the similar arguments of the case (ii) of proposition (3.2), $\gamma_{CM\chi}(G) = \left\lceil \frac{p}{2} \right\rceil - 2$.

Proposition 3.4. *For a Grid graph* $G = P_3 \times P_j$, $j \geq 4$,

$$\gamma_{CM\chi}(G) = \begin{cases} \left\lfloor \frac{p}{6} \right\rfloor, \text{ if } j \text{ is odd} \\ \frac{p}{6}, \text{ if } j \text{ is even.} \end{cases}$$

Proof. Let the graph $G = P_3 \times P_j$, $j \geq 4$. Let $V(G) = \{v_{11}, v_{12}, \ldots, v_{1j}, v_{21}, v_{22}, \ldots, v_{2j}, v_{31}, v_{32}, \ldots, v_{3j}\}$ be the vertex set of first, second and third row respectively and $|V(G)| = p = 3j$, $j \geq 4$. For $G = P_3 \times P_j$, $\chi(G) = 2$.

Case (i). Let j be odd. Consider the set $S \subseteq V(G)$, $S = \{v_{22}, v_{23}, \ldots, v_{2t}\}$ with $|S| = \left\lfloor \frac{p}{6} \right\rfloor$ and $|t| = \left\lceil \frac{p}{6} \right\rceil$.

$$|N[S]| = \sum_{i=2}^{t} d(v_{2i}) - \{|S| - 2\} = \frac{p}{2} + 2 > \left\lceil \frac{p}{2} \right\rceil.$$

It implies that $\frac{p}{2} = \frac{3j}{2}$ and S is a majority dominating set of G. Since every vertex in S is of distance one, $\chi(\langle S \rangle) = 2 = \chi(G)$ and the induced subgraph $\langle S \rangle$ of G is connected. Therefore the set S is a $\gamma_{CM\chi}$-set of G.

Hence $\gamma_{CM\chi}(G) \leq |S| = \left\lfloor \frac{p}{6} \right\rfloor$. (1)

Now, let $S' = S - \{v_{2j}\}$ with $|S'| = \left\lfloor \frac{p}{6} \right\rfloor - 1$. Then

$$|N[S']| = \sum_{i=1}^{\left\lceil \frac{p}{6} \right\rceil - 1} d(v_{2i}) - \{|S'| - 1\} = \frac{p}{2} - 1 < \left\lceil \frac{p}{2} \right\rceil.$$

Hence S' could not be a majority dominating set of G. Therefore $\gamma_{CM\chi}(G) > |S'| = \left\lfloor \frac{p}{6} \right\rfloor - 1$. Then $\gamma_{CM\chi}(G) \geq \left\lfloor \frac{p}{6} \right\rfloor$. (2)

Hence from (1) and (2), $\gamma_{CM\chi}(G) = \left\lfloor \frac{p}{6} \right\rfloor$, if j is odd.

Case (ii). Let j be even. Let $S = \{v_{21}, v_{22}, \ldots, v_{2t}\}$ with $|S| = \frac{p}{6}$ be the subset of G. Now,

$$|N[S]| = \sum_{i=1}^{t} d(v_{2i}) - \{|S| - 2\} = 3\left(\frac{p}{6}\right) + 2 > \left\lceil \frac{p}{2} \right\rceil.$$

It implies that S is a majority dominating set of G and the induced subgraph $\langle S \rangle$ of G is connected. Therefore the set S is a $\gamma_{CM\chi}$-set of G.

Hence $\gamma_{CM_\chi}(G) \leq |S| = \frac{p}{6}$. (3)

Applying the same arguments as in case (i),

we get, $\gamma_{CM_\chi}(G) \geq \frac{p}{6}$. (4)

Thus, from (3) and (4), $\gamma_{CM_\chi}(G) \geq \frac{p}{6}$, if j is even.

Proposition 3.5. *Let* $G = C_4 \times P_j$, $j \geq 4$, *a Cylinder. Then*

$$\gamma_{CM_\chi}(G) = \begin{cases} \frac{p}{6}, \text{ if } j \equiv 0 \pmod 3 \\ \left\lceil \frac{p}{6} \right\rceil, \text{ if } j \equiv 1, 2 \pmod 3. \end{cases}$$

Proof. Let $G = C_4 \times P_j$, $j \geq 4$. Let $V(G) = \{v_{11}, v_{12}, \ldots, v_{1j}, v_{21}, v_{22}, \ldots, v_{2j}, v_{31}, v_{32}, \ldots, v_{3j}, v_{41}, v_{42}, \ldots, v_{4j}\}$ be the vertex set of first, second, third and fourth sets of G respectively. For $G = C_4 \times P_j$, $\chi(G) = 2$.

Case (i). Let $j \equiv 0 \pmod 3$. Let $S = \{v_{11}, v_{12}, v_{1\left(\frac{p}{6}\right)}\} \subseteq V(G)$ with $|S| = \frac{p}{6}$. Now, $|N[S]| = 3|S| + 1 = 3\left(\frac{p}{6}\right) + 1 = \frac{p}{2} + 1 \geq \left\lceil \frac{p}{2} \right\rceil$. Therefore S is a majority dominating set of G. Since every vertex of S is of distance one, $\chi(\langle S \rangle) = 2 = \chi(G)$ and the induced subgraph $\langle S \rangle$ of G is connected. Therefore the set S is a γ_{CM_χ}-set of G.

Hence $\gamma_{CM_\chi}(G) \leq |S| = \frac{p}{6}$. (1)

Suppose, let $S' = S - \{v_{1j}\}$ with $|S'| = \frac{p}{6} - 1$. Then $|N[S']| = 3|S'| - 3 = 3\left(\frac{p}{6} - 1\right) - 3 < \left\lceil \frac{p}{2} \right\rceil$. It implies that S' could not be majority dominating set of G and $\gamma_{CM_\chi}(G) > |S'| = \frac{p}{6} - 1$. Therefore $\gamma_{CM_\chi}(G) \geq \frac{p}{6}$. (2)

From (1) and (2), $\gamma_{CM_\chi}(G) = \frac{p}{6}$, if $j \equiv 0 \pmod 3$.

Case (ii). Let $j \equiv 1, 2 \pmod 3$. Let $p = 2(\mathrm{mod}\, 6)$ such that p is divided by 4. Let $S = \{v_{11}, v_{12}, \ldots, v_{1t}\}$ with $|S| = \left\lceil \frac{p}{6} \right\rceil - 1 = t$ be the subset of $V(G)$. Now, $|N[S]| = 3|S| + 1 = 3\left\{\left\lceil \frac{p}{6} \right\rceil - 1\right\} + 1$. Let $p = 6r + 1$. Then $|N[S]| = 3\left[\frac{6r+2}{6} - 1\right] + 1 = \frac{6r+2}{2} - 2 = 3r - = 1 = 3\left(\frac{p-2}{6}\right) - 1 \geq \left\lceil \frac{p}{2} \right\rceil$. Let $p = 4(\mathrm{mod}\, 6)$ such that p is divided by 4. Let $S = \{v_{11}, v_{12}, \ldots, v_{1t}\}$ with $|S| = \left\lceil \frac{p}{6} \right\rceil - 1 = t$ be the subset of $V(G)$. Now, $|N[S]| = 3|S| + 1 = 3\left\{\left\lceil \frac{p}{6} \right\rceil - 1\right\} + 2$. Let $p = 6r + 4$. Then $|N[S]| = 3\left[\frac{6r+4}{6} - 1\right] + 2 = \frac{6r+4}{2} - 1 = 3r - 1 = 3\left(\frac{p-2}{6}\right) - 1 \geq \left\lceil \frac{p}{2} \right\rceil$. It implies that S be a majority dominating set of G.

Since all vertices of S are of distinct one, the vertex set of S is connected. Hence S is connected MDC set of G. Therefore $\gamma_{CM\chi}(G) \leq \left\lceil \frac{p}{6} \right\rceil - 1$. (3)

Now, suppose $S' = S - \{v_{1j}\}$ with $|S'| = \left\lceil \frac{p}{6} \right\rceil - 1$. Then $|N[S']| = 3|S| - 3 = 3\left(\left\lceil \frac{p}{6} \right\rceil - 1\right) - 3 < \left\lceil \frac{p}{2} \right\rceil$. It implies that S' could not be a majority dominating set of G and $\gamma_{CM\chi}(G) > |S'| = \left\lceil \frac{p}{6} \right\rceil - 2$. Therefore

$$\gamma_{CM\chi}(G) \geq \left\lceil \frac{p}{6} \right\rceil - 1. \tag{4}$$

Hence from (3) and 4), $\gamma_{CM\chi}(G) = \left\lceil \frac{p}{6} \right\rceil - 1$ if $j = 1, 2(\mathrm{mod}\, 6)$.

Proposition 3.6. *Let $G = C_t \circ K_j$, $t = 6$ and $j \geq 2$ be a Corona graph with a Cycle C_t and a Complete graph K_j. Then $\gamma_{CM\chi}(G) = \left\lceil \frac{p}{6} \right\rceil + 2$.*

Proof. Let $V(G) = \{v_1, v_{11}, v_{12}, \ldots, v_{1j}, v_2, v_{21}, v_{22}, \ldots, v_{2j}, \ldots, v_6, v_{61}, \ldots, v_{6j}\}$, where $v_i \in C_t$ and $v_{ij} \in K_j$, $i = 1, \ldots, 6$, $j \geq 2$. Let $S = \{v_1, v_{11}, v_{12}, \ldots, v_{1\left(\frac{p}{6}-1\right)}, v_2, \ldots, v_{\frac{t}{2}}\} \in V(G)$ with $|S| = \frac{p}{6} - 1 + 3 = \left\lceil \frac{p}{6} \right\rceil + 2$. Now,

$$|N[S]| \geq \sum_{i=1}^{6} d(v_{ij}) + \sum_{i=1}^{\frac{t}{2}} d(v_i) \geq 3\left\lceil \frac{p}{6} \right\rceil + 2 \geq \left\lceil \frac{p}{2} \right\rceil = 3(j+1).$$

It implies that S is a majority dominating set of G. Since G contains a Complete graph K_j, $j = \frac{p}{6} - 1$, $\chi(\langle S \rangle) = \frac{p}{6} - 1 = \chi(G)$. All the vertices of S are connected, the S is a γ_{CM_χ}-set of G. Hence $\gamma_{CM_\chi}(G) \leq \left\lceil \frac{p}{6} \right\rceil + 2$. (1)

Now, suppose $S' = S - \{v_i\}$ with $|S'| = \left\lceil \frac{p}{6} \right\rceil + 2 - 1$. Then $|N[S']| = d(v_i)d(v_i) + 2 - d(v_i) = \frac{p}{3} + 2 < \left\lceil \frac{p}{2} \right\rceil$. It implies that S' could not be a majority dominating set of G. Hence $\gamma_{CM_\chi}(G) < |S'| = \left\lceil \frac{p}{2} \right\rceil$. Therefore $\gamma_{CM_\chi}(G) \geq \left\lceil \frac{p}{6} \right\rceil + 2$. (2)

Thus, from (1) and (2), we get $\gamma_{CM_\chi}(G) = \left\lceil \frac{p}{6} \right\rceil + 2$.

Proposition 3.7. *Let G be a vertex color critical graph of t vertices and H be any graph with order $s \leq t$. Then $\gamma_{CM_\chi}(G) = \gamma_{CM_\chi}(G') = t$ where $G' = G^\circ H$, a Corona graph.*

Corollary 3.8. *For a Corona graph $G = K_t^\circ K_{m,n}, m \leq n$, $\gamma_{CM_\chi}(G) = t$, where K_t and $K_{m,n}$ are Complete and Complete bipartite graphs respectively.*

Theorem 3.9. *Let G be a connected graph. If $\chi(G) = \gamma_{M_\chi}(G)$ then*

$\gamma_{CM\chi}(G) = \gamma_{M\chi}(G)$.

Proof. Let $\chi(G) = \gamma_{M\chi}(G)$ and S be a $\gamma_{M\chi}$-set of G. Suppose $\chi(G) = \chi(\langle S \rangle) = k$ then $\langle S \rangle = K_k$. It implies that $\gamma_{M\chi}(G) = |S| = k$. The induced subgraph $\langle S \rangle$ is connected and $\gamma_{CM\chi}(G) = k$. Hence $\gamma_{CM\chi}(G) = \gamma_{M\chi}(G)$.

Proposition 3.10. *For a connected graph G which contains all its vertices of degree* $d(v_i) < \left\lceil \frac{p}{2} \right\rceil - 1$, $\gamma_{CM\chi}(G) < \gamma_C(G)$.

4. Characterization theorems on $\gamma_{CM\chi}$

Theorem 4.1. *Let G be a connected graph* $p \geq 2$. *Then G is vertex color critical if and only if* $\gamma_{CM\chi}(G) = p$.

Proof. Let G be a vertex color critical graph. Then by the result (4.4) [5], $\gamma_{M\chi}(G) = |S| = p$. Since $\gamma_{M\chi}(G) = p$, all vertices are in consecutive. It implies that S is a connected MDC set of G and $\gamma_{CM\chi}(G) = p$. The converse is obvious.

Theorem 4.2. *Let G be a tree. Then* $\gamma_{CM\chi}(G) = \gamma_C(G)$ *if and only if* $diam(G) = 3$, *where* $\gamma_C(G)$ *is the connected domination number*.

Proof. Let $\gamma_{CM\chi}(G) = \gamma_C(G)$. (1)

Let S and S' be the $\gamma_{CM\chi}$-set and γ_C-set of G.

Case (i). If $diam(G) = 1$ then the graph structures becomes $G = K_2$. Then by result (iv) [3.1], $\gamma_{CM\chi}(G) = 2$. The γ_C-set of G is $S' = \{v\}$. It implies that $\gamma_C(G) = |S'| = 1$. It is a contradiction to the assumption (1).

Case (ii). If $diam(G) = 2$ then the graphs are like $G = P_3$ and $G = K_{1,\, p-1}$. For $G = P_3$, the $\gamma_{CM\chi}$-set is $S = \{v_1, v_2\}$ and $\gamma_{CM\chi}(G) = 2$. The γ_C-set is $S' = \{v_2\}$ and $\gamma_C(G) = 1 < \gamma_{CM\chi}(G)$. It is a contradiction to (1).

Case (iii). Let $diam(G) = 3$. Then the graph G becomes P_4 and $D_{r,s}$, a double star. The $\gamma_{CM\chi}$-set and γ_C-set of P_4 is $\{v_2, v_3\}$. It implies that $\gamma_{CM\chi}(G) = 2 = \gamma_C(G)$. Therefore the condition (1) holds. For $G = D_{r,s}$, $\gamma_{CM\chi}$-set and γ_C-set is $\{u, v\}$, where u, v are central vertices. Then $\gamma_{CM\chi}(G) = 2 = \gamma_C(G)$. Therefore the condition (1) holds for $diam(G) = 3$. Suppose $diam(G) \geq 4$. Then the graph structures being $G = P_p$, $p \geq 5$ and $\gamma_{CM\chi}(G) = \left\lceil \frac{p}{2} \right\rceil - 2 < \gamma_C(G) = p - 2$. Hence the condition (1) is not true for $diam(G) \geq 4$.

Conversely, if $diam(G) = 3$ then the graph G has a dominating edge $e = uv$ and both u and v have some pendants. Let $S = \{u, v\} \subseteq V(G)$ with $d(u, v) = 1$. Then $\chi(\langle S \rangle) = 2 = \chi(G)$ and $\langle S \rangle$ is connected. Clearly $|N[S]| = p > \left\lceil \frac{p}{2} \right\rceil$, then S is both γ_C-set and $\gamma_{CM\chi}$-set of G. Hence, $\gamma_{CM\chi}(G) = 2 = \gamma_C(G)$.

Theorem 4.3. *For even cycle $G = C_p$, $\gamma_{CM\chi}(G) = \gamma_{M\chi}(G)$ if and only if $G = C_p$, $p \leq 10$.*

Proof. Let $\gamma_{CM\chi}(G) = \gamma_{M\chi}(G)$. For even cycle, $\chi(G) = 2$. By the proposition (3.2), $\gamma_{CM\chi}(G) = \left\lceil \frac{p}{2} \right\rceil - 2$, if p is even. The $\gamma_{CM\chi}$-numbers of even cycles with $p \geq 4$ are $2, 2, 2, 3, 4, 5, \ldots$. By the proposition (3.2) [5],

$$\gamma_{M\chi}(G) = \begin{cases} 2, \text{ if } p = 4 \text{ to } 8 \\ 3, \text{ if } p = 9, 10. \end{cases}$$

Also it gives $\gamma_{CM\chi}(G) = \gamma_{M\chi}(G)$ if $p \leq 10$. Conversely, from the above arguments, the proof is obvious.

Theorem 4.4. *For any Path, $\gamma_{CM\chi}(G) = \gamma_{cch}(G)$ if and only if $G = P_p$, $p = 3, 4$ where $\gamma_{cch}(G)$ is connected dom-chromatic number of G.*

Proof. Let $\gamma_{CM_\chi}(G) = \gamma_{cch}(G)$. For a Path, $\gamma_{CM_\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$ and $\gamma_{cch}(G)$ $= p - 2$. Since $\gamma_{CM_\chi}(G) = \gamma_{cch}(G)$, $\left\lceil \frac{p}{2} \right\rceil = p - 2$. It implies that $\frac{p}{2} = p - 2$ or $\frac{p}{2} + 1 = p - 2$. Hence $p = 3$ or $p = 4$. Therefore $G = P_3$ and P_4. The converse is obvious.

Theorem 4.5. *Let T be a tree. Then $\gamma_{CM_\chi}(T) = \gamma_{M_\chi}(T)$ if and only if one of the following conditions holds.*

(i) *T has a vertex of degree $d(u) \geq \left\lceil \frac{p}{2} \right\rceil - 1$*

(ii) *Each non-pendant vertex is adjacent to a pendant vertex*

(iii) $diam(T) \leq 9$.

Proof. Let $\gamma_{CM_\chi}(T) = \gamma_{M_\chi}(T)$. Then S is a γ_{M_χ} and γ_{CM_χ}-set of T with same cardinality. If $diam(T) = 1$ then T becomes K_2 and $S = \{u, v\}$ be the γ_{M_χ} and γ_{CM_χ}-set of T with $d(u) \geq \left\lceil \frac{p}{2} \right\rceil - 1$. If $diam(T) = 2$ then the graph structures like P_3, $K_{1, p-1}$. Let $S = \{u, v\}$ be set with $d(u, v) = 1$ such that $d(u) = p - 1 \geq \left\lceil \frac{p}{2} \right\rceil - 1$. The set S satisfies $\gamma_{CM_\chi}(T) = \gamma_{M_\chi}(T)$. Hence the condition (i) and (ii) holds. Suppose $diam(T) = 3$. Then T has a dominating edge $e = uv$ and $\chi(T) = 2$. Let $S = \{u, v\}$ be the γ_{M_χ} and γ_{CM_χ}-set of T with $d(u) \geq \left\lceil \frac{p}{2} \right\rceil - 1$. Hence the pendant vertices u and v both are adjacent to some pendants at u and v. Hence condition (i) and (ii) holds. Let $4 \leq diam(T) \geq 7$. Then $T = P_p$, $p = 5, 6, 7, 8$. By the result (3.3) [5], $\gamma_{M_\chi}(T) = 2$ and by proposition (3.3), $\gamma_{CM_\chi}(T) = 2$. Hence if $\gamma_{CM_\chi}(T) = \gamma_{M_\chi}(T)$, $4 \leq diam(T) \geq 7$. Now let $diam(T) = 8$ and 9. Let $S = \{v_i, v_j, v_k\}$ be the γ_{M_χ}-set of T such that $d(v_i, v_j) = 1$. Then $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(\langle S \rangle) = 2 = \chi(T)$. It implies that $\gamma_{M_\chi}(T) = |S| = 3$. Since $d(v_i, v_j) = 1 = d(v_j, v_k)$, the induced subgraph $\langle S \rangle$

is connected. It implies that S be $\gamma_{CM\chi}$-set of T and $\gamma_{CM\chi}(T) = |S| = 3$. Hence the condition $\gamma_{CM\chi}(T) = \gamma_{M\chi}(T)$ is true. Now suppose $diam(T) \geq 10$. Then by result (3.3)[5], $\gamma_{M\chi}(T) = \left\lceil \frac{p}{6} \right\rceil$ or $\left\lceil \frac{p}{6} \right\rceil + 1$ and by proposition (3.3), $\gamma_{CM\chi}(T) = \left\lceil \frac{p}{2} \right\rceil - 2$. It implies that $\gamma_{CM\chi}(T) > \gamma_{M\chi}(T)$. Hence if $\gamma_{CM\chi}(T) = \gamma_{M\chi}(T)$ then $diam(T) \leq 9$. The converse is obvious.

Theorem 4.6. *For any tree* T, $\gamma_{CM\chi}(T) = 2$ *if and only if* T *has at least two vertices* v_i *with* $d(v_i) \geq \left\lceil \frac{p}{2} \right\rceil - 2$.

Proof. Let $\gamma_{CM\chi}(T) = 2$. Let S be a $\gamma_{CM\chi}$-set of T and $\gamma_{CM\chi}(T) = |S| = 2$. Then $S = \{v_i, v_j\}$ with $d(v_i, v_j) = 1$. To prove that T has at least two vertices v_i with $d(v_i) \geq \left\lceil \frac{p}{2} \right\rceil - 2$. Suppose T has vertices v_i with $d(v_i) \geq \left\lceil \frac{p}{2} \right\rceil - 3$. Then $|N[S]| = d(v_i) + d(v_j) \leq \left\lceil \frac{p}{2} \right\rceil - 3 + \left\lceil \frac{p}{2} \right\rceil - 3 \leq p - 6 < \left\lceil \frac{p}{2} \right\rceil$. It implies that S is not be a majority dominating set of T with $|S| = 2$. It is a contradiction to the assumption that S is a $\gamma_{CM\chi}$-set of T. Hence T has at least two vertices v_i with $d(v_i) \geq \left\lceil \frac{p}{2} \right\rceil - 2$.

Conversely, suppose T has at least two vertices v_i with $d(v_i) \geq \left\lceil \frac{p}{2} \right\rceil - 2$.
(1)

To prove $\gamma_{CM\chi}(T) = 2$. Let $S = \{v_i, v_j\} \subseteq V(T)$ with $d(v_i, v_j) = 1$. By the assumption (1), if $d(v_i) = \left\lceil \frac{p}{2} \right\rceil - 2 = d(v_j)$ then

$$|N[S]| = |N[v_i]| = |N[v_j]| = \left\lceil \frac{p}{2} \right\rceil - 2 + \left\lceil \frac{p}{2} \right\rceil - 2 = p - 4 = \left\lceil \frac{p}{2} \right\rceil.$$

It implies that S is a majority dominating set of T. If $d(v_i) = \left\lceil \frac{p}{2} \right\rceil - 2$ and

$d(v_i) \geq 2$ then $|N[S]| = \left\lceil \frac{p}{2} \right\rceil \Rightarrow S$ is a majority dominating set of T. Since $\chi(T) = 2$, $\chi(\langle S \rangle) = 2$ and $\langle S \rangle$ is connected. Hence S is a connected MDC set of T and $\gamma_{CM\chi}(T) \leq |S| = 2$. Suppose $S' = \{v_i\}$ and $|S'| < |S|$. Then $|N[S]| < \left\lceil \frac{p}{2} \right\rceil$ and S' is not a majority dominating set of T. Since $\chi(T) = 2$, $\chi(\langle S \rangle) = 1 \neq \chi(T)$. Therefore S is not a MDC set of T. Hence $\gamma_{CM\chi}(T) > |S'|$ and $\gamma_{CM\chi}(T) \geq |S| = 2$. Thus, $\gamma_{CM\chi}(T) = 2$.

5. Conclusion

In this article, new type of domination parameter of a graph is introduced. Connected majority dominating chromatic number $\gamma_{CM\chi}(G)$ is defined and it is determined for some families of graphs and product of graphs. Then characterization of connected majority dominating chromatic number for graph is established.

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