Advances and Applications in Mathematical Sciences
Volume 22, Issue 6, April 2023, Pages 1199-1207

ISSN 0974-6803

ANALYSIS OF SOLUTION OF DIFFERENTIAL EQUATION OF CHARGING IN LCR CIRCUIT BY TECHNIQUE OF REMOVING FIRST ORDER DERIVATIVE

RAHUL DRAVID[1], ASHOK KUMAR[1], SACHIN KHER[1] and KRISHNA DAHAKE[1]

[1]Department of Mathematics
Medi-Caps University
Indore-453331, India
Email: rahul.dravid@medicaps.ac.in
ashok.kumar@medicaps.ac.in
sachin.kher@medicaps.ac.in
krishna.dahake@medicaps.ac.in

Abstract

In this paper, we provide a general solution to the matter of the charging of capacitor in LCR circuit. The solution is the second order differential equation supported by removal of first order derivative in second order linear differential equation. This is much closed relative in the study of elementary physics or mechanics.

1. Introduction

The foremost vital problem of charging a capacitor in LCR circuit is used in electrical network. Its solution is given in classical manner and in several literatures [1-3], there are different formal methods. In this paper, we provide a general solution of charging of capacitor connected in LCR circuit. Here we consider applied potential difference force is any time dependent function.

In LCR circuit; Inductance L, Capacitor of capacity C and Resistance R are connected in series as shown in Figure 1.

2020 Mathematics Subject Classification: 34 (34-00, 34-01, 34A12, 34A30).
Keywords: concept of LCR circuit, ordinary differential equation and solution.
Received July 7, 2022; Accepted December 22, 2022

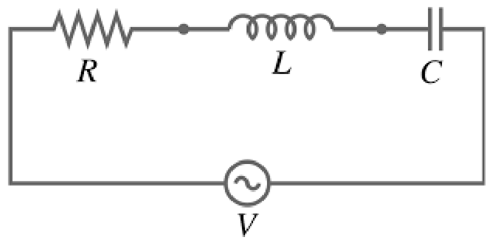

Figure 1. LCR Circuit.

2. Mathematical Analysis

Let applied potential difference in LCR circuit is $V(t) = V_0 \sin \omega t,$ the equation of the charging of capacitor is given by [4]:

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{q}{C} = V_0 \sin \omega t \tag{1}$$

On dividing equation (1) by L, we get

$$\frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{Q}{LC} = \frac{V_0}{L} \sin \omega t \tag{2}$$

where L is the inductance, R is resistance and C is capacitance of capacitor.

$V(t)$ is time dependent function (could also be Sinusoidal, Cosine). Equation (2) can be written as

$$\frac{d^2q}{dt^2} + 2b \frac{dq}{dt} + n^2 q = \frac{V(t)}{L}, \tag{3}$$

where $V(t) = V_0 \sin \omega t,$

$$\frac{d^2q}{dt^2} + 2b \frac{dq}{dt} + n^2 q = \frac{V(t)}{L} \tag{4}$$

where $2b = R/L, b$ is damping factor, $n^2 = 1/LC, n = \frac{1}{\sqrt{LC}},$ is resonant frequency.

Comparing the equation (4) with second order linear differential equation given by [5-6]

$$\frac{d^2x}{dt^2} + P\frac{dx}{dt} + Qx = R, \tag{5}$$

We have $P = 2b,\ Q = n^2,\ R = \frac{V(t)}{L}$.

Now remove the first derivative from given equation.

We choose

$$u = e^{-\int \frac{1}{2}Pdt} = u = e^{-bt}$$

Let complete solution be $y = uw$

Then w is given by the traditional equation by equation (7)

$$\frac{d^2w}{dt^2} + Iw = R/u,$$

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dt},\ n^2 - b^2$

$$\frac{d^2w}{dt^2} + (n^2 - b^2)w = \frac{V(t)}{L}/e^{-bt} \tag{6}$$

Let $\frac{d}{dt} = D$ then equation (2) are going to be

$$[D^2 - (b^2 - n^2)]w = \frac{V(t)}{L}e^{bt}$$

Putting $D = m$ we find auxiliary equation and find roots of the equation are $m = \pm\sqrt{(b^2 - n^2)}$ then complementary solution

$$\text{C.F.} = Ae^{\sqrt{(b^2-n^2)}t} + Be^{-\sqrt{(b^2-n^2)}t} = Ae^{pt} + Be^{-pt}$$

where $p = \sqrt{(b^2 - n^2)}$ as $b \to n,\ p \to 0$.

Particular integral,

$$P.I = \frac{1}{[D^2 - (b^2 - n^2)]}\frac{V(t)}{L}e^{bt},$$

Here we assume external force is sine function with angular frequency ω, then

$$V(t) = V_0 \sin \omega t = \frac{V_0}{L} \sin \omega t = v \sin \omega t, \text{ where } v = \frac{V_0}{L}$$

$$P.I = \frac{1}{[D^2 - (b^2 - n^2)]} ve^{bt} \sin \omega t = v \left[\frac{1}{[D^2 - (b^2 - n^2)]} e^{bt} \sin \omega t \right]$$

$$P.I. = ve^{bt} \frac{1}{[D^2 + 2bD + n^2]} \sin \omega t = ve^{bt} \frac{1}{[D^2 + b^2 + 2bD - b^2 + n^2]} \sin \omega t$$

$$P.I. = ve^{bt} \frac{1}{[D^2 + 2bD + n^2]} \sin \omega t = ve^{bt} \frac{1}{[-\omega^2 + 2bD + n^2]} \sin \omega t$$

$$P.I. = ve^{bt} \frac{1}{[2bD - (n^2 - \omega^2)]} \sin \omega t$$

On rationalization we get

$$P.I. = -ve^{bt} \left[\frac{2b\omega \cos \omega t - (n^2 - \omega^2) \sin \omega t}{[4b^2\omega^2 + (n^2 - \omega^2)^2]} \right]$$

On solving this by method to find P.I. then by solution

$$w(t) = C.F + P.I$$

$$w(t) = Ae^{pt} + Be^{-pt} - ve^{bt} \left[\frac{2b\omega \cos \omega t - (n^2 - \omega^2) \sin \omega t}{[4b^2\omega^2 + (n^2 - \omega^2)^2]} \right]$$

Solution of differential equation of charging of capacitor connected in LCR circuit is given by:

$$Q(t) = uw = e^{-bt} * \left[Ae^{pt} + Be^{-pt} - ve^{bt} \left\{ \frac{2b\omega \cos \omega t - (n^2 - \omega^2) \sin \omega t}{[4b^2\omega^2 + (n^2 - \omega^2)^2]} \right\} \right]$$

$$Q(t) = e^{-bt} * [Ae^{pt} + Be^{-pt}] - v \left\{ \frac{2b\omega \cos \omega t - (n^2 - \omega^2) \sin \omega t}{[4b^2\omega^2 + (n^2 - \omega^2)^2]} \right\} \tag{7}$$

Now by given initial condition we can find the value of A and B then finally capacitor get charged at any time 't'.

The first part represented by (7) is complementary solution which decreases exponentially with time and after sometime this term vanishes, hence it is also known as transient charging of capacitor and charging with frequency other than frequency of applied potential difference, after a long time $t >> \tau$ (relaxation time) capacitor starts charging with frequency of applied potentials called steady state of charging of capacitor. When capacitor is fully charged then start discharge through inductance and resistance.

Since current in the circuit is $I = \frac{dQ}{dt}$

$$I = \frac{d}{dt}\left[e^{-bt} * [Ae^{pt} + Be^{-pt}] - v\left\{\frac{2b\omega Cos\omega t - (n^2 - \omega^2)\sin\omega t}{[4b^2\omega^2 + (n^2 - \omega^2)^2]}\right\}\right]$$

$$I = [-be^{-bt}(Ae^{pt} + Be^{-pt}) + e^{-bt}(pAe^{pt} - pBe^{-pt})]$$

$$-v\left\{\frac{[4b^2\omega^2 + (n^2 - \omega^2)^2](-2b\omega^2\sin\omega t - (n^2 - \omega^2)\omega\cos\omega t)}{[4b^2\omega^2 + (n^2 - \omega^2)^2]^2}\right\}$$

$$I = e^{(-b+p)t}[(-b + p)A] - e^{(-b+p)t}[(b + p)B]$$

$$-v\left\{\frac{[4b^2\omega^2 + (n^2 - \omega^2)^2](-2b\omega^2\sin\omega t - (n^2 - \omega^2)\omega\cos\omega t)}{[4b^2\omega^2 + (n^2 - \omega^2)^2]^2}\right\} \tag{8}$$

On giving initial condition we can find current in circuit at any instant 't'.

At, $t = 0$, $q(t) = 0$ and $I = dq/dt = 0$ by equations (7) and (8)

$$A + B = \frac{2b\omega v}{[4b^2\omega^2 + (n^2 - \omega^2)^2]} \tag{i}$$

$$A - B = \frac{-v\omega\{(n^2 - \omega^2) - 2b^2\}}{p[4b^2\omega^2 + (n^2 - \omega^2)^2]} \tag{ii}$$

Solving (i) and (ii) the value of

$$A = \frac{v\omega\{2bp - (n^2 - \omega^2) + 2b^2\}}{2p[4b^2\omega^2 + (n^2 - \omega^2)^2]},\ B = \frac{v\omega\{2bp - (n^2 - \omega^2) + 2b^2\}}{2p[4b^2\omega^2 + (n^2 - \omega^2)^2]}$$

by (7)

$$Q(t) = e^{-bt} * \left[\frac{\frac{V_0}{L} \omega\{2bp - (n^2 - \omega^2) + 2b^2\}}{2p[4b^2\omega^2 + (n^2 - \omega^2)^2]} e^{pt} + \frac{\frac{V_0}{L} \omega\{2bp - (n^2 - \omega^2) + 2b^2\}}{2p[4b^2\omega^2 + (n^2 - \omega^2)^2]} e^{-pt} \right]$$

$$- \frac{V_0}{L} \left\{ \frac{2b\omega \cos \omega t - (n^2 - \omega^2) \sin \omega t}{[4b^2\omega^2 + (n^2 - \omega^2)^2]} \right\}.$$

Now to find Amplitude and Phase difference we assume charging of capacitor is sinusoidal

$$q(t) = q_0 \sin(\omega t + \emptyset)$$

then by equation (7)

$$q_0 \sin(\omega t + \emptyset) = e^{-bt} * [Ae^{pt} + Be^{-pt}] - v \left\{ \frac{2b\omega \cos \omega t - (n^2 - \omega^2) \sin \omega t}{[4b^2\omega^2 + (n^2 - \omega^2)^2]} \right\}$$

$$q_0(\sin \omega t \cos \emptyset + \cos \omega t \sin \emptyset) = e^{-bt} * [Ae^{pt} + Be^{-pt}]$$

$$- v \left\{ \frac{2b\omega \cos \omega t - (n^2 - \omega^2) \sin \omega t}{[4b^2\omega^2 + (n^2 - \omega^2)^2]} \right\}$$

On comparing coefficient of $\sin \omega t$ and $\cos \omega t$ we get

$$q_0 \cos \emptyset = \frac{v(n^2 - \omega^2)}{[4b^2\omega^2 + (n^2 - \omega^2)^2]} \tag{9}$$

and

$$q_0 \sin \emptyset = \frac{2b\omega v}{[4b^2\omega^2 + (n^2 - \omega^2)^2]} \tag{10}$$

On squaring and adding (9) and (10) we get,

$$q_0^2 = \frac{v^2}{[4b^2\omega^2 + (n^2 - \omega^2)^2]}, \; q_0 = \frac{v}{\sqrt{[4b^2\omega^2 + (n^2 - \omega^2)^2]}} \tag{11}$$

On dividing (10) by (9) we get phase difference

$$\tan \theta = \frac{2b\omega}{[n^2 - \omega^2]}, \; \theta = \tan^{-1} \frac{2b\omega}{[n^2 - \omega^2]} \tag{12}$$

If damping $b = 0$ then

$$q_0 = \frac{v}{[n^2 - \omega^2]}$$

and $\theta = 0$ degree.

If $\omega = n$, and $b \to 0$, then amplitude of charging will be infinite and phase difference will be $\theta = \frac{\pi}{2}$ this is the condition of resonance charging.

Special Cases.

1. When $\omega << n$, i.e. the frequency of potential is very much less

2. When $\omega = n$, i.e. resonance state

Case 1. When $\omega << n$, in low damping case $b \to 0$ then from equation

$$q(t) = \frac{v \sin \omega t}{n^2}$$

$$q_0 = \frac{f}{n^2} = \frac{\frac{V_0}{L}}{1/LC} = V_0 C \text{ and}$$

$$\theta = 0 \text{ Degree.}$$

In this case amplitude of charge does not depend on inductance but only depends only on the V_0 and capacitance C of capacitor.

Case 2. When $\omega = n$ by equation (7)

$$q(t) = e^{-bt} * [Ae^{pt} + Be^{-pt}] - \frac{2bnv \cos \omega t}{4b^2 n^2}$$

$$= e^{-bt} * [Ae^{pt} + Be^{-pt}] - \frac{v \cos \omega t}{2bn}.$$

And, $q_0 = \frac{v}{2bn} = \frac{\frac{V_0}{L}}{\frac{R}{L} * n} = \frac{V_0}{nR}$,

where $2b = R/L$, if $R \to 0$, $q_0 = \infty$, $\theta = \frac{\pi}{2}$.

This is called the condition of resonance. Thus at resonance, the amplitude of charging depends on the resistance.

Result and Conclusion

From the above discussion in the paper, we conclude that charge on capacitor at any instant can be find out with the help of equation (7) and current at any instant can be find out with the help of equation (8). However the solution of differential equation of charging of capacitor can be finding out by using Laplace transformation which is the less time consuming solution. graphical representation of charging of capacitor in LCR circuit is shown in Figure 2 as an example.

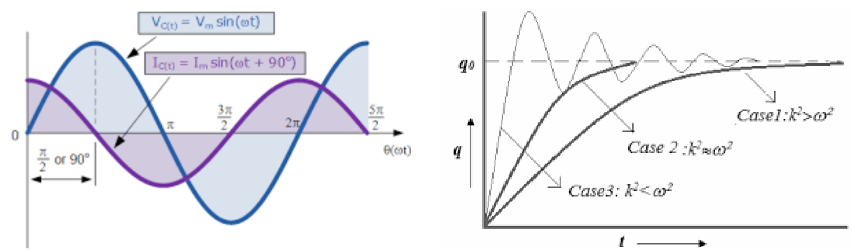

Figure 2. Graphical representation of charging of capacitor in LCR circuit.

Acknowledgement

The authors wish to express their sincere gratitude to the Vice-Chancellor, Pro Vice-Chancellor, and Head of the Department of Mathematics, Medi-Caps University, Indore, for their constant encouragement during the progress of the work. We are thankful to all of those with whom we have had the pleasure to work during this paper.

References

[1] D. Wang, The most energy efficient way to charge the capacitor in a RC circuit, Phys. Ed. 52(6) (2017), 065019.

[2] Teja Aluru and Aaron Osier, Modeling a RLC Circuits with Differential Equations 1-19, May 2014.

[3] Kenny Harwood, Modeling a RLC Circuit's Current with Differential Equations May 17, 2011. http://home2.fvcc.edu/~dhicketh/DiffEqns/Spring11projects/Kenny_Harwood/ACT7/KennyHarwoodFinalProject.pdf

[4] C. Harper, Introduction to Mathematical Physics, New Delhi: Prentice Hall of India Pvt. Ltd., 1999.

[5] D. P. Choudhary and B. Rani, Elementary Ordinary Differential Equation, Alpha Science International, 2005.

[6] H. K. Das, Advance Engineering Mathematics, New Delhi: S Chand, 2018.

[7] P. G. Kshirsagar, Advance Engineering Physics, New Delhi: S Chand, 2017.