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IRREGULAR AND TOTALLY IRREGULAR BIPOLAR HESITANCY FUZZY GRAPHS AND SOME OF ITS PROPERTIES

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Abstract

The concepts of irregular bipolar Hesitancy Fuzzy Graphs, neighbourly irregular bipolar Hesitancy Fuzzy Graphs, neighbourly totally irregular bipolar Hesitancy Fuzzy Graphs, highly irregular bipolar Hesitancy Fuzzy Graphs and highly totally irregular bipolar Hesitancy Fuzzy Graphs are introduced and investigated. Also define the relation between neighbourly irregular and highly irregular hesitancy Fuzzy Graph.

1. Introduction

L. A. Zadeh [2] introduced the notion of fuzzy sets which involves the concept of a membership function defined n on a universal set and the value of the membership function lies in [0, 1]. Kaufmann in 1973 gave the first definition of a fuzzy graph which was based on Zadeh's fuzzy relations. Azriel Rosenfeld [4] introduced the concept of fuzzy graphs in 1975 [0]. It has been growing fast and has numerous applications in various fields. A. Nagoor Gani and S. R. Latha [5] introduced Irregular fuzzy graphs and discussed some of its properties. Krassimir T. Atanassov [6] introduced the intuitionistic fuzzy graph theory. R. Parvathi and M. G. Karunambigai [7] introduced intuitionistic fuzzy graphs as a special case of Atanassov's IFG and discussed

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some properties of regular intuitionistic fuzzy graphs. K. Sankar and D. Ezhilmaran introduced the concept of bipolar intuitionistic fuzzy graphs [9].

In the year 2015, T. Pathinathan and J. Jon Arockiaraj [10] introduced a new fuzzy graph called Hesitancy Fuzzy Graphs and discussed their various theoretical properties and validations. These are motivating us to introduce the concept of irregular bipolar Hesitancy fuzzy graphs, totally irregular bipolar Hesitancy Fuzzy Graph and discussed some of its properties.

2. Preliminaries

This paper is organized as follows. Section 2 focuses on the concept of Basic definitions of Fuzzy sets and Hesitancy Fuzzy Graphs. Section 3 and section 4 introduces the new irregular Bipolar HFG and totally irregular Bipolar HFG and followed by conclusion in Section 4.

Definition 2.1. Let X be a non empty set. A bipolar fuzzy set B in X is an object having the form $B\{x, \mu_1^P(x), \mu_1^N(x)/x \in X\}$ where $\mu_1^P : X \to [0, 1]$ and $\mu_1^N : X \to [-1, 0]$ are mappings.

Definition 2.2. Let X be a non empty set. A bipolar Hesitancy Fuzzy set $B = \{x, \mu_1^P(x), \mu_1^N(x), \gamma_1^P(x), \gamma_1^N(x), \beta_1^P(x), \beta_1^N / x \in X\}$ where $\mu_1^P : X \to [0, 1]$, $\mu_1^N : X \to [-1, 0]$, $\gamma_1^P : X \to [0, 1]$, $\gamma_1^N : X \to [-1, 0]$, $\beta_1^P : X \to [0, 1]$ and $\beta_1^N : X \to [-1, 0]$ are the mappings such that $0 \le \mu_1^P(x) + \gamma_1^P(x) + \beta_1^P(x) \le 1$ and $-1 \le \mu_1^N(x) + \gamma_1^N(x) + \beta_1^N(x) \le 0$.

The positive membership degree $\mu_1^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a Bipolar Hesitancy Fuzzy set B. The negative membership degree $\mu_1^N(x)$ to denote the satisfaction degree of an element x to the property corresponding to a Bipolar Hesitancy Fuzzy set B. The positive non membership degree $\gamma_1^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a Bipolar Hesitancy Fuzzy set B. The negative non membership degree $\gamma_1^N(x)$ to denote the satisfaction degree of an element x to the property corresponding to a Bipolar Hesitancy Fuzzy set B. The positive hesitant degree $\beta_1^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a Bipolar Hesitancy Fuzzy set B. The negative hesitant degree $\beta_1^N(x)$ to denote the satisfaction degree of an element x to the property corresponding to a Bipolar Hesitancy Fuzzy set B.

(i) If $\mu_1^P(x) \neq 0, \mu_1^N(x) = 0, \gamma_1^P(x) = 0, \gamma_1^N(x) = 0, \beta_1^P(x) = 0, \beta_1^N(x) = 0$ then x consider as only the positive membership property of a bipolar Hesitancy Fuzzy set.

(ii) If $\mu_1^P(x) = 0, \mu_1^N(x) \neq 0, \gamma_1^P(x) = 0, \gamma_1^N(x) = 0, \beta_1^P(x) = 0, \beta_1^N(x) = 0$ then x consider as only the negative membership property of a bipolar Hesitancy Fuzzy set.

(iii) If $\mu_1^P(x) = 0, \mu_1^N(x) = 0, \gamma_1^P(x) \neq 0, \gamma_1^N(x) = 0, \beta_1^P(x) = 0, \beta_1^N(x) = 0$ then x consider as only the positive non membership property of a bipolar Hesitancy Fuzzy set.

(iv) If $\mu_1^P(x) = 0, \mu_1^N(x) = 0, \gamma_1^P(x) = 0, \gamma_1^N(x) \neq 0, \beta_1^P(x) = 0, \beta_1^N(x) = 0$ then x consider as only the negative non membership property of a bipolar Hesitancy Fuzzy set.

(v) If $\mu_1^P(x) = 0, \mu_1^N(x) = 0, \gamma_1^P(x) = 0, \gamma_1^N(x) = 0, \beta_1^P(x) \neq 0, \beta_1^N(x) = 0$ then x consider as only the positive hesitancy property of a bipolar Hesitancy Fuzzy set.

(vi) If $\mu_1^P(x) = 0, \mu_1^N(x) = 0, \gamma_1^P(x) = 0, \gamma_1^N(x) = 0, \beta_1^P(x) = 0, \beta_1^N(x) \neq 0$ then x consider as only the negative hesitancy property of a bipolar Hesitancy Fuzzy set.

Definition 2.3. Let X be a non empty set. Then we call a mappings $\mu_2^P : X \times X \to [0, 1], \mu_2^N : X \times X \to [0, 1], \gamma_2^p : X \times X \to [0, 1]$, $\gamma_2^N : X \times X \to [-1, 0], \beta_2^P : X \times X \to [0, 1]$ and $\beta_2^N : X \times X \to [-1, 0]$ are bipolar hesitancy fuzzy relation on X such that $\mu_2^P(x, y) \leq \mu_1^P(x) \wedge \mu_1^P(y)$, $\mu_2^N(x, y) \geq \mu_1^N(x) \vee \mu_1^N(y), \gamma_2^P(x, y) \leq \gamma_1^P(x) \vee \gamma_1^P(y), \gamma_2^N(x, y) \geq \gamma_1^N(x) \vee \gamma_1^N(y)$,

$\beta_2^P(x, y) \leq \beta_1^P(x) \wedge \beta_1^P$ and $\beta_2^N(x, y) \geq \beta_1^N(x) \wedge \beta_1^N(y)$ for all $(x, y) \in X$.

Definition 2.4. A bipolar hesitancy fuzzy relation A on X is called symmetric relation if $\mu_2^P(x, y) = \mu_2^P(x, y), \mu_2^N(x, y) = \mu_2^N(x, y), \gamma_2^P(x, y) = \gamma_2^P(x, y)$, $\gamma_2^N(x, y) = \gamma_2^N(x, y), \beta_2^P(x, y) = \beta_2^P(x)$ and $\beta_2^N(x, y) = \beta_2^N(x, y)$ for all $(x, y) \in X$.

Definition 2.5. A Hesitancy Fuzzy Graphs is of the form $G = (V, E)$, where $V = \{v_1, v_2, \cdots v_n\}$ such that $\mu_1 : V \to [0, 1], \gamma_1 : V \to [0, 1]$ and $\beta_1 : V \to [0, 1]$ denote the degree of membership, non-membership and hesitancy of the vertex $v_i \in V$ respectively and $\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$ for every $v_i \in V$, where $\beta_1(v_i) = 1 - [\mu_1(v_i) + \gamma_1(v_i)]$ and $E \subseteq V \times V$ where $\mu_2 : V \times V \to [0, 1], \gamma_2 : V \times V \to [0, 1]$ and $\beta_2 : V \times V \to [0, 1]$ are such that, $\mu_2(v_i, v_j) \leq \min[\mu_1(v_i), \mu_1(v_j)], \gamma_2(v_i, v_j) \leq \max[\gamma_1(v_i), \gamma_1(v_j)], \beta_2(v_i, v_j)$ $\leq \min[\beta_1(v_i), \beta_1(v_j)]$ and $0 \leq \mu_2(v_i, v_j) + \gamma_2(u_i, v_i) + \beta_2(u_i, v_i) \leq 1$ for every $(v_i, v_j) \in E$.

Definition 2.6. A HFG $H = (V', E')$ is said to be a Hesitancy Fuzzy Subgraph of $G = (V, E)$ if

(i) $V' \subseteq V$, where $\mu_1' = \mu_1, \gamma_1' = \gamma_1, \beta_1' = \beta_1$ for all $v_i \in V', i = 1, 2, \cdots, n$.

(ii) $E' \subseteq E$, where $\mu_2' = \mu_2, \gamma_2' = \gamma_2, \beta_2' = \beta_2$ for all $(v_i, v_j) \in E', i, j = 1, 2, \cdots, n$.

Definition 2.7. Hesitancy fuzzy graph $G = (V, E)$ is said to be regular if every vertex of G have same degree.

3. Bipolar Hesitancy Fuzzy Graph

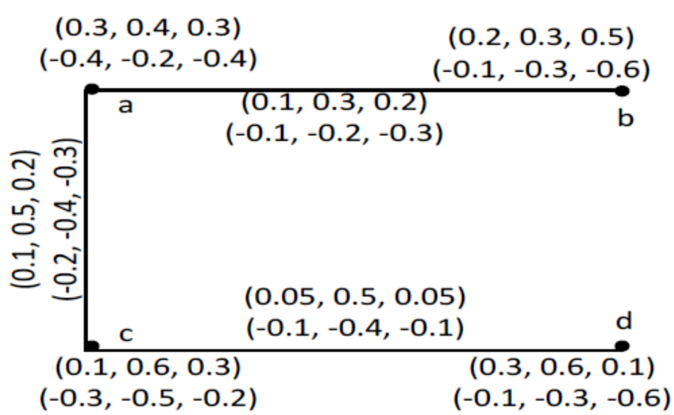
Definition 3.1. A bipolar hesitancy fuzzy graph is of the form $G = (V, E)$, where

(i) $V = \{v_1, v_2, \cdots v_n\}$ such that $\mu_1^P : V \to [0,1], \gamma_1^P : V \to [0,1], \beta_1^P : V \to [0,1]$ denote the degree of positive membership, positive non membership and positive hesitancy of the vertex $v_i \in V$ respectively, $\mu_1^N : V \to [-1, 0]$,

$\gamma_1^N : V \to [-1, 0]$, $\beta_1^N : V \to [-1, 0]$, denote the degree of negative membership negative non membership and negative hesitancy of a vertex $v_i \in V$ and $\mu_1^P(v_i) + \gamma_1^P(v_i) + \beta_1^P(v_i) = 1$, $\mu_1^N(v_i) + \gamma_1^N(v_i) + \beta_1^N(v_i) = -1$ for every $v_i \in V$, where $\beta_1^P(v_i) = 1 - [\mu_1^P(v_i) + \gamma_1^P(v_i)]$, $\beta_1^N(v_i) = -1 - [\mu_1^N(v_i) + \gamma_1^N(v_i)]$.

(ii) $E \subseteq V \times V$ where $\mu_2^P : V \times V \to [0, 1]$, $\gamma_2^P : V \times V \to [0, 1]$, $\beta_2^P : V \times V \to [0, 1]$, $\mu_2^N : V \times V \to [-1, 0]$, $\gamma_2^N : V \times V \to [-1, 0]$, $\beta_2^N : V \times V \to [-1, 0]$ are such that $\mu_2^P(v_i, v_j) \le \min[\mu_1^P(v_i), \mu_1^P(v_j)]$, $\mu_2^N(v_i, v_j) \ge \max[\mu_1^N(v_i), \mu_1^N(v_j)]$, $\gamma_2^P(v_i, v_j) \le \max[\gamma_1^P(v_i), \gamma_1^P(v_j)]$, $\gamma_2^N(v_i, v_j) \ge \min[\gamma_1^N(v_i), \gamma_1^N(v_j)]$, $\beta_2^P(v_i, v_j) \le \min[\beta_1^P(v_i), \beta_1^P(v_j)]$, $\beta_2^N(v_i, v_j) \ge \max[\beta_1^N(v_i), \beta_1^N(v_j)]$ denote the degree of positive, negative membership, degree of positive, negative non membership and degree of positive, negative Hesitancy of the edge $(v_i, v_j) \in E$ respectively and $0 \le \mu_2^P(v_i, v_j) + \gamma_2^P(v_i, v_j) + \beta_2^P(v_i, v_j) \le 1$, $-1 \le \mu_2^N(v_i, v_j) + \gamma_2^N(v_i, v_j) + \beta_2^N(v_i, v_j) \le 0$ for every $(v_i, v_j) \in E$.

Example 3.2.



Definition 3.3. A Bipolar Hesitancy Fuzzy Graph $H = (V', E')$ is said to be Bipolar Hesitancy Fuzzy Subgraph of $G = (V, E)$ if (i) $V' \subseteq V$ where

$\mu_1^P(v_i') = \mu_1^P(v_i)$, $\mu_1^N(v_i') = \mu_1^N(v_i)$, $\gamma_1^P(v_i') = \gamma_1^P(v_i) = \gamma_1^N(v_i') = \gamma_1^N(v_i)\,\beta_1^P(v_i')$ $= \beta_1^P(v_i)\beta_1^N(v_i') = \beta_1^N(v_i)$ for all $v_i' \in V'$ and $v_i' = v_i \in V$, $i = 1, 2, \cdots, n$. (ii) $E' \subseteq E$ where $\mu_2^P(v_i', v_j') = \mu_2^P(v_i, v_j)$, $\mu_2^N(v_i', v_j') = \mu_2^N(v_i, v_j)$, $\gamma_2^P(v_i', v_j') = \gamma_2^P(v_i, v_j)$, $\gamma_2^N(v_i', v_j') = \gamma_2^N(v_i, v_j)$, $\beta_2^P(v_i', v_j') = \beta_2^P(v_i', v_j')$ and $\beta_2^N(v_i', v_j' = \beta_2^N(v_i, v_j)$ for all $(v_i', v_j') \in E'$ and $(v_i', v_j') = (v_i, v_j) \in Ei, j = 1, 2, 3, \cdots, n$.

Definition 3.4. A Bipolar Hesitancy Fuzzy Graph $G = (V, E)$ is said to be complete Bipolar Hesitancy Fuzzy Graph if $\mu_2^P(v_i, v_j) = \mu_1^P(v_i,) \wedge \mu_1^P(v_j)$, $\mu_2^N(v_i, v_j) = \mu_1^N(v_i,) \vee \mu_1^N(v_j)$, $\gamma_2^P(v_i, v_j) = \gamma_1^P(v_i) \vee \gamma_1^P(v_j)$, $\gamma_2^N(v_i, v_j) = \gamma_1^N(v_i,)$ $\wedge\, \gamma_1^N(v_j)$, $\beta_2^P(v_i, v_j) = \beta_1^P(v_i) \wedge \beta_1^P(v_j)$, $\beta_2^N(v_i, v_j) = \beta_1^N(v_i) \vee \beta_1^N(v_j)$ for every $v_i, v_j \in V$.

Definition 3.5. A Bipolar Hesitancy Fuzzy Graph $G = (V, E)$ is said to be strong Bipolar Hesitancy Fuzzy Graph if $\mu_2^P(v_i, v_j) = \mu_1^P(v_i) \wedge \mu_1^P(v_j)$, $\mu_2^N(v_i, v_j) = \mu_1^N(v_i) \vee \mu_1^N(v_j)$, $\gamma_2^P(v_i, v_j) = \gamma_1^P(v_i) \vee \gamma_1^P(v_j)$, $\gamma_2^N(v_i, v_j) = \gamma_1^N(v_i) \wedge$ $\gamma_1^P(v_j)$, $\beta_2^P(v_i, v_j) = \beta_1^P(v_i) \wedge \beta_1^P(v_j)$, $\beta_2^N(v_i, v_j) = \beta_1^N(v_i) \vee \beta_1^N(v_j)$ for every $(v_i, v_j) \in E$.

Definition 3.6. The complement of a Bipolar Hesitancy Fuzzy Graph $G = (V, E)$ is a Bipolar Hesitancy Fuzzy Graph $G = (\overline{V}, \overline{E})$ where

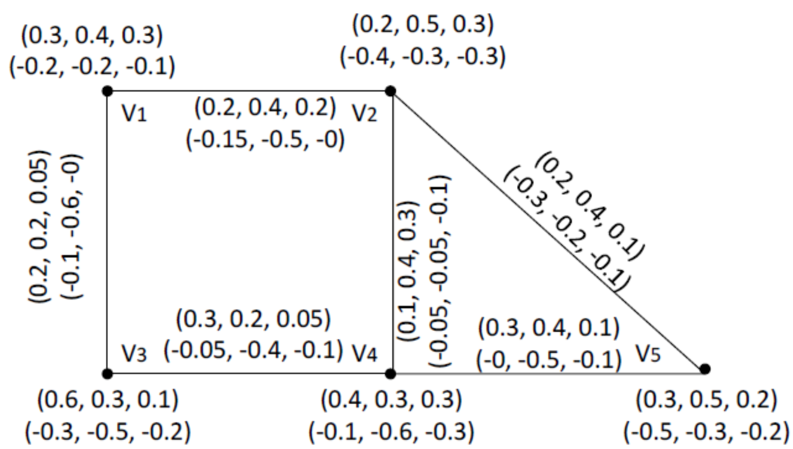
(i) $\overline{V} = V$ that is, $\overline{\mu}_1^P(v_i) = \mu_1^P(v_i)$, $\overline{\mu}_1^N(v_i) = \mu_1^N(v_i)\,\overline{\gamma}_1^P(v_i) = \gamma_1^P(v_i)$, $\overline{\gamma}_1^N(v_i)$ $= \gamma_1^N(v_i)$, $\overline{\beta}_1^P(v_i) = \beta_1^P(v_i)$, $\overline{\beta}_1^N(v_i) = \beta_1^N(v_i)$ for all $v_i \in \overline{V}$ and

(ii) $\overline{\mu}_2^P(v_i, v_j) = [\mu_1^P(v_i) \wedge \mu_1^P(v_j)] - \mu_2^P(v_i, v_j)$, $\overline{\mu}_2^N(v_i, v_j) = [\mu_1^N(v_i) \vee \mu_1^N(v_j)]$ $- \mu_2^N(v_i, v_j)$, $\overline{\gamma}_2^P(v_i, v_j) = [\gamma_1^P(v_i) \vee \gamma_1^P(v_j)] - \gamma_2^P(v_i, v_j)$, $\overline{\gamma}_2^N(v_i, v_j) = [\gamma_1^N(v_i) \wedge$ $\gamma_1^N(v_j)] - \gamma_2^N(v_i, v_j)$, $\overline{\beta}_2^P(v_i, v_j) = [\beta_1^P(v_i) \wedge \beta_1^P(v_j)] - \beta_2^P(v_i, v_j)$ and $\overline{\beta}_2^N(v_i, v_j)$ $= [\beta_1^N(v_i) \vee \beta_1^N(v_j)] - \beta_2^N(v_i, v_j)$ for every $v_i, v_j \in \overline{V}$.

Definition 3.7. Let G be a Bipolar Hesitancy Fuzzy Graph. The neighbourhood of a vertex x in G is defined by $N(x) = (N_\mu^P(x), N_\mu^P(x), N_\gamma^P(x), N_\gamma^P(x), N_\beta^P(x), N_\beta^P(x))$ where $N_\mu^P(x) = \{y \in V / \mu_2^P(x, y) \leq \mu_1^P(x) \wedge \mu_1^P(x)\}$, $N_\mu^N(x) = \{y \in V / \mu_2^N(x, y) \geq \mu_1^N(x) \vee \mu_1^N(x)\}$, $N_\gamma^P(x) = \{y \in V / \gamma_2^P(x, y) \leq \gamma_1^P(x) \vee \gamma_1^P(x)\}$, $N_\gamma^N(x) = \{y \in V / \gamma_2^N(x, y) \geq \gamma_1^N(x) \wedge \gamma_1^N(x)\}$, $N_\beta^N(x) = \{y \in V / \beta_2^P(x, y) \leq \beta_1^P(x) \wedge \beta_1^P(x)\}$ and $N_\beta^N(x) = \{y \in V / \beta_2^N(x, y) \geq \beta_1^N(x) \vee \beta_1^N(x)\}$.

Definition 3.8. Let G be a Bipolar Hesitancy Fuzzy Graph. The neighbourhood degree of a vertex x in G is defined by $\deg(x) = [\deg \mu^P(x), \deg \mu^N(x), \deg \gamma^P(x), \deg \gamma^N(x), \deg \beta^P(x), \deg \beta^N(x)]$ where $\deg \mu^P(x) = \sum_{y \in N(x)} \mu_1^P(y), \deg \mu^N(x) = \sum_{y \in N(x)} \mu_1^N(y), \deg \gamma^P(x) = \sum_{y \in N(x)} \gamma_1^P(y)$, $\deg \gamma^N(x) = \sum_{y \in N(x)} \gamma_1^N(y), \deg \beta^P(x) = \sum_{y \in N(x)} \beta_1^P(y), \deg \beta^N(x) = \sum_{y \in N(x)} \beta_1^N(y)$.

Definition 3.9. Let G be a Bipolar Hesitancy Fuzzy Graph on G^*. If there is a vertex which is adjacent to vertices with distinct neighbourhood degrees, then G is called an irregular bipolar Hesitancy fuzzy graph.



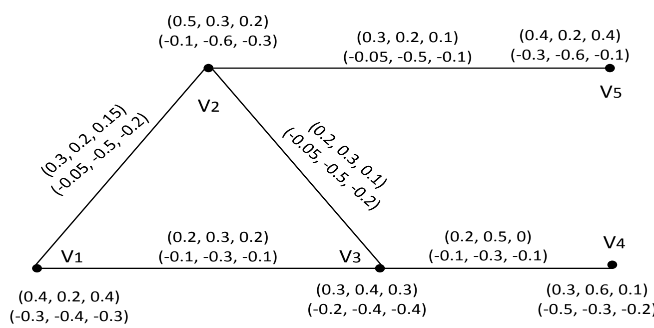
Example 3.10. Here $\deg(v_1) = (0.8 - 0.7, 0.8, -0.8, 0.4 - 0.5)$ $\deg(v_2) = (1.0, -0.8, 1.2, -1.6, 0.8, -0.6)$, $\deg(v_3) = (0.7, -0.3, 0.7, -1.3, 0.6, -0.4)$,

$\deg(v_4) = (1.1 - 1.2, 1.3 - 1.1, 0.6, -0.7)$ and $\deg(v_5) = (0.6, -0.5, 0.8, -0.9, 0.6, -0.6)$. Here all the vertices have distinct neighbourhood degrees. Hence G is an irregular bipolar hesitancy fuzzy graph.

Definition 3.11. Let G be a Bipolar Hesitancy Fuzzy Graph. The closed neighbourhood degree of a vertex x in G is defined by $\deg[x] = [\deg \mu^P[x], \deg \mu^N[x], \deg \gamma^P[x], \deg \gamma^N[x], \deg \beta^P[x], \deg \beta^N[x]]$, where $\deg \mu^P[x] = \deg \mu^P(x) + \mu_1^P(x)$, $\deg \mu^N[x] = \deg \mu^N(x) + \mu_1^N(x)$, $\deg \gamma^P[x] = \deg \gamma^P(x) + \gamma_1^P(x)$, $\deg \gamma^N[x] = \deg \gamma^N(x) + \gamma_1^N(x)$ $\deg \beta^P[x] = \deg \beta^P(x) + \beta_1^P(x)$ and $\deg \beta^N[x] = \deg \beta^N(x) + \beta_1^N(x)$.

Definition 3.12. If there is a vertex which is adjacent to vertices with distinct closed neighbourhood degrees, then G is called a totally irregular Bipolar Hesitancy Fuzzy Graph.

Example 3.13. Here $\deg[v_1] = (1.2, -0.6, 0.9, -1.4, 0.9, -1.0)$, $\deg[v_2] = (1.6, -0.9, 1.1, -2.0, 1.3, -1.1)$, $\deg[v_3] = (1.5, -1.1, 1.5, -1.7, 1.0, -1.2)$, $\deg[v_4] = (0.6, -0.7, 1.0, -0.7, 0.4, -0.6)$ and $\deg[v_5] = (0.9, -0.4, 0.5, -1.2, 0.6, -0.4)$. Hence G is a totally irregular hesitancy fuzzy graph.



Definition 3.14. A connected Bipolar Hesitancy Fuzzy Graph G is said to be a neighbourly irregular Bipolar Hesitancy Fuzzy Graph if any two adjacent vertices of G have distinct open neighbourhood degree.

Definition 3.15. A connected Bipolar Hesitancy Fuzzy Graph G is said to a neighbourly totally irregular Bipolar Hesitancy Fuzzy Graph if every two adjacent vertices of G have distinct closed neighbourhood degree.

Definition 3.16. Let G be a Bipolar Hesitancy Fuzzy Graph. G is called a highly irregular Bipolar hesitancy Fuzzy Graph if every vertex of G is adjacent to vertices with distinct neighbourhood degrees.

Theorem 3.19. *Let G be a Bipolar Hesitancy Fuzzy Graph, then G is highly irregular Bipolar hesitancy Fuzzy Graph and neighbourly irregular Bipolar Hesitancy Fuzzy Graph if and only if the neighbourhood degrees of all the vertices of G are distinct.*

Proof. Let G be a Bipolar Hesitancy Fuzzy Graph with n vertices $v_1, v_2, \cdots v_n$. Assume that G is highly irregular and neighbourly irregular bipolar Hesitancy Fuzzy Graph. To prove that the neighbourhood degrees of all vertices of G are distinct. Let $\deg(v_i) = (a_i, b_i, x_i, y_i, p_i, q_i)$ for all $i = 1, 2, \cdots, n$.

Case (i) If the adjacent vertices of v_1 are $v_2, v_3, \cdots v_n$ with neighbourhood degrees $(a_2, b_2, x_2, y_2, p_2, q_2), (a_3, b_3, x_3, y_3, p_3, q_3), \cdots, (a_n, b_n, x_n, y_n, p_n, q_n)$ respectively. Then $a_2 \neq a_3 \neq \cdots \neq a_n, b_2 \neq b_3 \neq \cdots \neq b_n, x_2 \neq x_3 \neq \cdots \neq x_n, y_2 \neq y_3 \neq \cdots \neq y_n, p_2 \neq p_3 \neq \cdots \neq p_n$ and $q_2 \neq q_3 \neq \cdots \neq q_n$. Given G is highly irregular bipolar Hesitancy Fuzzy Graph. Also we have $a_1 \neq a_2 \neq \cdots \neq a_n, b_1 \neq b_2 \neq \cdots \neq b_n, x_1 \neq x_2 \neq \cdots \neq x_n, y_1 \neq y_2 \neq \cdots y_n, p_1 \neq p_2 \neq \ldots \neq p_n$ and $q_1 \neq q_2 \neq \cdots \neq q_{n.}$ Given G is neighbourly irregular. hence in this case, neighbourhood degrees of all vertices of G are distinct.

Case (ii) If the adjacent vertices of v_2 are $v_1, v_3, v_4, \cdots v_n$ with neighbourhood degrees $(a_1, b_1, x_1, y_1, p_1, q_1), (a_3, b_3, x_3, y_3, p_3, q_3), (a_4, b_4, x_4, y_4, p_4, q_4) \cdots, (a_n, b_n, x_n, y_n, p_n, q_n)$ respectively. Then $a_1 \neq a_3 \neq a_4 \neq \cdots \neq a_n$, $b_1 \neq b_3 \neq b_4 \neq \ldots \neq b_n, x_1 \neq x_3 \neq x_4 \neq \ldots \neq x_n, y_1 \neq y_3 \neq y_4 \neq \ldots \neq y_n, p_1 \neq p_3 \neq p_4 \neq \ldots \neq p_n$ and $q_1 \neq q_3 \neq q_4 \neq \cdots q_n$. Given G is highly irregular bipolar Hesitancy Fuzzy Graph. Also we have $a_1 \neq a_2 \neq a_3 \neq \cdots \neq a_n, b_1 \neq b_2 \neq b_3 \neq \cdots \neq b_n, x_1 \neq x_2 \neq x_3 \neq \cdots \neq x_n$, $y_1 \neq y_2 \; y_3 \neq \cdots \neq y_n, p_1 \neq p_2 \neq p_3 \neq \cdots \neq p_n$ and $q_1 \neq q_2 \neq q_3 \neq \cdots \neq q_n$. Hence all the vertices have distinct neighbourhood degrees. Proceeding like this manner we get, all the vertices of G have distinct neighbourhood degrees.

Theorem 3.20. *A Bipolar Hesitancy Fuzzy Graph G of* G^*, *where* G^* *is a cycle with three vertices is neighbourly irregular and highly irregular BHFG if and only if the positive and negative membership, positive and negative non membership, positive and negative hesitant values of the vertices between every pair of vertices are all distinct.*

Proof. Assume that the positive and negative membership, positive and negative non membership, positive and negative hesitant values of the vertices between every pair of vertices are all distinct.

To prove that G is highly and neighbourly irregular BHFG.

Let $v_i, v_j, v_k \in V$. Since $\mu_1^P(v_i) \neq \mu_1^P(v_j) \neq \mu_1^P(v_k), \mu_1^N(v_i) \neq \mu_1^N(v_j) \neq \mu_1^N(v_k)$, $\gamma_1^P(v_i) \neq \gamma_1^P(v_j) \neq \gamma_1^P(v_k)$, $\gamma_1^N(v_i) \neq \gamma_1^N(v_j) \neq \gamma_1^N(v_k)$, $\beta_1^P(v_i) \neq \beta_1^P(v_j) \neq \beta_1^P(v_k)$ and $\beta_1^N(v_i) \neq \beta_1^N(v_j) \neq \beta_1^N(v_k)$.

We get $\sum_{x \in N(x)} \mu_1^P(v_i) \neq \sum_{x \in N(x)} \mu_1^P(v_j) \neq \sum_{x \in N(x)} \mu_1^P(v_k)$, $\sum_{x \in N(x)} \mu_1^N(v_i) \neq \sum_{x \in N(x)} \mu_1^N(v_j) \neq \sum_{x \in N(x)} \mu_1^N(v_k)$, $\sum_{x \in N(x)} \gamma_1^P(v_i) \neq \sum_{x \in N(x)} \gamma_1^P(v_j) \neq \sum_{x \in N(x)} \gamma_1^P(v_k)$, $\sum_{x \in N(x)} \gamma_1^N(v_i) \neq \sum_{x \in N(x)} \gamma_1^N(v_j) \neq \sum_{x \in N(x)} \gamma_1^N(v_k)$, $\sum_{x \in N(x)} \beta_1^P(v_i) \neq \sum_{x \in N(x)} \beta_1^P(v_j) \neq \sum_{x \in N(x)} \beta_1^P(v_k)$ and $\sum_{x \in N(x)} \beta_1^N(v_i) \neq \sum_{x \in N(x)} \beta_1^N(v_j) \neq \sum_{x \in N(x)} \beta_1^N(v_k)$.

That is, $\deg(v_i) \neq \deg(v_j) \neq \deg(v_k)$. Hence by the definition of neighbourly irregular and highly irregular BHFG. Thus G is a neighbourly irregular and highly irregular BHFG. To prove that positive and negative membership, positive and negative non membership positive and negative hesitant values of the vertices are all distinct.

Let $\deg(v_j) = (x_i, y_i, z_i, p_i, q_i, r_i), 1, 2, 3, \ldots$. Suppose assume that positive and negative membership, positive and negative non membership, positive and negative hesitant values of two vertices are same. Let $v_1, v_2 \in V$. Let $\mu_1^P(v_1) = \mu_1^P(v_2)$, $\mu_1^N(v_1) = \mu_1^N(v_2)$, $\gamma_1^P(v_1) = \gamma_1^P(v_2)$, $\gamma_1^P(v_1) = \gamma_1^P(v_2)$, $\beta_1^P(v_1) = \beta_1^P(v_2)$ and $\beta_1^N(v_1) = \beta_1^N(v_2)$. Then $\deg(v_1) = \deg(v_2)$. Hence G is not neighbourly and highly irregular BHFG. Which is a contraction to the fact that G is neighbourly and highly irregular BHFG.

Proposition 3.21. *A neighbourly irregular BHFG may not be a neighbourly totally irregular BHFG.*

Proof. Given that G is a neighbourly irregular BHFG. To prove that G is not a neighbourly totally irregular BHFG. Since any two adjacent vertices of G have distinct open neighbourhood degree. Let $v_1, v_2 \in V$ have distinct open neighbourhood degrees $\deg(v_1) \neq \deg(v_2)$.

That is, $\sum_{x \in N(x)} \mu_1^P(v_1) \neq \sum_{x \in N(x)} \mu_1^P(v_2) \neq \sum_{x \in N(x)} \mu_1^N(v_1) \neq \sum_{x \in N(x)} \mu_1^N(v_2)$, $\sum_{x \in N(x)} \gamma_1^P(v_1) \neq \sum_{x \in N(x)} \gamma_1^P(v_2)$, $\sum_{x \in N(x)} \gamma_1^N(v_1) \neq \sum_{x \in N(x)} \gamma_1^N(v_2)$, $\sum_{x \in N(x)} \beta_1^P(v_1) \neq \sum_{x \in N(x)} \beta_1^P(v_2)$ and $\sum_{x \in N(x)} \beta_1^N(v_1) \neq \sum_{x \in N(x)} \beta_1^N(v_2)$.

It may be we get, $\sum_{x \in N(x)} \mu_1^P + \mu_1^P(v_1) = \sum_{x \in N(x)} \mu_1^P(v_2) + \mu_1^P(v_2)$, $\sum_{x \in N(x)} \mu_1^N + \mu_1^N(v_1) = \sum_{x \in N(x)} \mu_1^N(v_2) + \mu_1^P(v_2)$, $\sum_{x \in N(x)} \gamma_1^P(v_1) + \gamma_1^P(v_1) = \sum_{x \in N(x)} \gamma_1^N(v_2) + \gamma_1^P(v_2)$, $\sum_{x \in N(x)} \gamma_1^N(v_1) + \gamma_1^N(v_1) = \sum_{x \in N(x)} \gamma_1^N(v_2) + \gamma_1^N(v_2)$, $\sum_{x \in N(x)} \beta_1^P(v_1) + \beta_1^P(v_1) = \sum_{x \in N(x)} \beta_1^P(v_2) + \beta_1^P(v_2)$ and $\sum_{x \in N(x)} \beta_1^N(v_1) + \beta_1^N(v_1) = \sum_{x \in N(x)} \beta_1^N(v_2) + \beta_1^N(v_2)$. Hence $\deg[v_1] = \deg[v_2]$. Hence v_1 and v_2 have same closed neighbourhood degrees. Hence G is not neighbourly totally irregular BHFG.

Proposition 3.22. *Let G be a BHFG. If G is neighbourly irregular BHFG and* $(\mu_1^P, \mu_1^N, \gamma_1^P, \gamma_1^N, \beta_1^P, \beta_1^N)$ *is a constant function then G is a neighbourly totally irregular BHFG.*

Proposition 3.23. *If G is neighbourly irregular BHFG, then the bipolar hesitant fuzzy subgraph* $H = (A', B')$ *of G may not be highly irregular BHFG.*

4. Conclusion

In this article we have defined the bipolar hesitancy fuzzy graph with suitable illustration. We introduced the irregular, neighbourly irregular, highly irregular, totally irregular and neighbourly totally irregular bipolar hesitancy fuzzy graph and also discussed the relationship between them with proper illustration.

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