Advances and Applications in Mathematical Sciences ISSN 0974-6803
Volume 22, Issue 7, May 2023, Pages 1663-1678

INDEPENDENT DOMINATION IN DOUBLE VERTEX GRAPHS

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Abstract

A set of vertices in a graph G is independent if no two vertices in the set are adjacent. A set that is both dominating and independent is called an independent dominating set. The minimum size of an independent dominating set is called independent domination number of G denoted by $i(G)$. The double vertex graph $U_2(G)$ of a graph G of order $n \geq 2$ is the graph whose vertex set consists of all $\binom{n}{2}$ unordered pairs of vertices of G and two vertices $\{a, b\}$ and $\{c, d\}$ are adjacent in $U_2(G)$ if and only if $|\{a, b\} \cap \{c, d\}| = 1$ and if $a = c$ then b and d are adjacent in G. In this paper, bounds on the independent domination number of double vertex graph of G in terms of parameters of G are obtained.

1. Introduction

In this paper, $G = (V, E)$ be a simple, finite, undirected and connected graph of order $n \geq 2$ and size m where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. For undefined graph theoretic terminologies and notations refer to [1].

A set of vertices in a graph G is independent if no two vertices in the set are adjacent. A maximal independent set (MIS) of G is an independent set that is not a proper subset of any other independent set of G. An independent set of vertices in G is maximal independent if and only if it is independent

2020 Mathematics Subject Classification: 05C76, 05C69.
Keywords: double vertex graph, independent set, dominating set.
Received December 31, 2021; Accepted March 15, 2022

dominating set of G. This result has led to the definition of the independent domination number of G as the minimum cardinalities of the independent dominating sets (equivalently, of the maximal independent sets) of G. An independent dominating set has the advantage of both structures (dominating and independent) therefore independent domination finds many applications especially in wireless sensor networks (WSN). WSN are spatially distributed sensors to monitor the data related to physical or environmental conditions like temperature, sound, pressure, etc. and push through the network to a base station. Clustering method is used to extend the network lifetime in a WSN where sensor nodes are grouped into clusters and cluster heads (CHs) are elected for all the clusters. CHs collect the data from particular cluster nodes and forward the collected data to the base station. Energy efficient clustering has been widely to extend lifetime of WSNs.

In clustering schemes, independent sets result in clusterheads that have local control of their cluster without interference. A dominating independent set based clustering scheme ensures that the entire network is covered. In energy-efficient wireless computing, clustering allows some wireless nodes to perform less tasks by delegating them to their respective cluster head. However, the tasks of these cluster heads then result in additional energy consumption. So, using as few clusterheads as possible that is choosing them according to minimum independent dominating set results in energy savings for the network.

The theory of independent domination formalized by Berge and Ore in [2]. The minimum cardinality of a maximal independent set of G is called as independent domination number of G, denoted by $i(G)$. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi in [4].

2. Preliminaries

A set D of vertices in G is called a dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. A dominating set D is a minimal dominating set (MDS) if no proper subset of D is a dominating set of G. The set of all MDSs of G is denoted by $MDS(G)$. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a set in $MDS(G)$. A set S of vertices of G is independent if every two distinct vertices of S are non adjacent. The vertex independence number (or the independence number) $\beta(G)$ of G is the maximum cardinality of an independent set of vertices in G. A subset M of the edge set $E(G)$ of G is called independent if no two edges of M are adjacent in G. A matching in G is a set of independent edges. A matching M of G is maximum if G has no matching M' such that $|M'| > |M|$ and M is maximal if G strictly contains M. The cardinality of a maximum matching is the edge independence number of G denoted by $\beta_1(G)$. The edge degree $d(e)$ of the edge $e = uv$ is defined as the number of neighbours of e, i.e., $|N(u) \cup N(v)| - 2$. The maximum degree among all the edges of G is $\Delta'(G)$.

Following are some of the theorems on independent domination in graphs for our immediate reference.

Theorem 2.1[4]. *An independent set is MIS if and only if it is independent and dominating.*

Theorem 2.2[4]. *Every MIS in G is a MDS of G.*

Theorem 2.3[4]. *For a graph* G, $\gamma(G) \leq i(G) \leq \beta(G)$.

Theorem 2.4[5]. *For a graph G if vertices of degree at least three constitute an independent set then* $i(G) = \gamma(G)$.

Theorem 2.5[7]. *For a graph G of order* $n \geq 2$, $\left\lfloor \frac{n}{2} \right\rfloor \leq \beta(U_2(G))$.

3. Double Vertex Graphs

Chartrand introduced the term graph-valued function for any kind of rule or procedure which yields a unique graph from given graph. In literature, many graph-valued functions are studied, the earliest being the line graph of a graph. The double vertex graph $U_2(G)$ is one such graph-valued function, introduced by Alavi et al. in [3]. For a G of order $n \geq 2$, $U_2(G)$ is the graph whose vertex set consists of all $\binom{n}{2}$ unordered pairs of vertices from $V(G)$ and two vertices $\{a, b\}$ and $\{c, d\}$ of $U_2(G)$ are adjacent if and only if $|\{a, b\} \cap \{c, d\}| = 1$ and the two distinct vertices in $\{a, b\}$ and $\{c, d\}$ are adjacent in G.

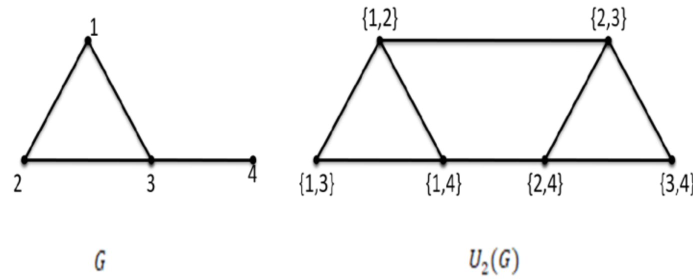

Figure 1. The graph G and $U_2(G)$.

If G has n vertices and m edges then $U_2(G)$ has $\frac{n(n-1)}{2}$ vertices and $m(n-2)$ edges. For each edge of G there are $n-1$ edges of $U_2(G)$. $U_2(G)$ is bipartite if and only if G is bipartite. Every vertex x, y of $U_2(G)$ is either a line a pair or a non line pair. There by $V(U_2(G))$ is partitioned into sets U-the set of line pairs and W-the set of non line pairs. Also, let $V(G) = \{v_i : i = 1, 2, \ldots, n\}$ be the vertex set of G then we partition $V(U_2(G)) = \{\{v_i, v_j\} : i = 1, 2, \ldots, n-1, j = i+1, i+2, \ldots, n\}$. For brevity, we denote $\{v_i, v_j\}$ as $\{i, j\}$.

The vertex independence number and domination related properties of $U_2(G)$ are discussed in [6] and [7].

4. Main Results

In this section, we obtain bounds on the independent domination number of double vertex graph of G in terms of parameters of G. It is noted that the double vertex graph of a graph G has no full degree vertex except for the case when $G \cong K_3, P_3$. So, for graph $G \ncong K_3, P_3$ the independent domination number of $U_2(G)$ is atleast two.

We begin the study of independent domination number of $U_2(G)$ with the following result.

Theorem 4.1. *For any graph G of order atleast four* $\left\lfloor \frac{n}{2} \right\rfloor \leq i(U_2(G))$.

Proof. For any four distinct vertices w, r, s, t of G, there corresponds a set $T = \{\{w, r\}, \{w, s\}, \{w, t\}, \{r, s\}, \{s, t\}\}$ of six vertices of $U_2(G)$ giving rise to three pairs $\{w, r\}, \{r, s\}; \{w, s\}, \{r, t\}$ and $\{w, t\}, \{r, s\}$ of non adjacent vertices in $U_2(G)$. However, for any arbitrary graph G the set T may not have 3-element subset in which every two vertices are mutually non adjacent. So, if $n = 4k$ or $4k+1$ then there are k groups of four vertices where each group gives rise to two non adjacent vertices in $U_2(G)$. On the other hand if $n = 4k+2$ or $4k+3$ there corresponds $2k+1$ vertices where every two vertices are mutually non adjacent. Thus a subset S' of $U_2(G)$ consisting of such vertices is an independent in $U_2(G)$ and of cardinality atleast $\left\lceil \frac{n}{2} \right\rceil$. Then for any maximum independent set S of $U_2(G)$, $|S'| \leq |S|$. Further, if S is maximal then by Theorem 2.2, S is a minimal dominating set and $|S| = i(U_2(G)) = \left\lfloor \frac{n}{2} \right\rfloor$ and if S is not maximal then S is not dominating. Therefore, $i(U_2(G)) > \left\lfloor \frac{n}{2} \right\rfloor$.

In particular for $G \cong K_n, C_4, K_n - (n-2)e, P_4, K_{1,3}$, the set is a maximal independent set and hence minimal dominating set. Thus, $i(U_2(G)) = \left\lfloor \frac{n}{2} \right\rfloor$. Hence the bound is sharp. □

Theorem 4.2. *For a* K_3*-free graph* G, *if* $diam(G) = 2$ *then set of line pairs U is an independent dominating set of* $U_2(G)$.

Proof. Since G is K_3-free graph, no two line pairs in $U_2(G)$ are adjacent. Hence the set of line pairs U is an independent set in $U_2(G)$. Further, $diam(G) = 2$, every path of length two in G corresponds to two line pairs and a non line pair in $U_2(G)$. Therefore every non line pair is adjacent to two line pairs thus $U \cup \{\{u, v\}\}$ is not independent for every $\{u, v\} \in W$. Hence U is maximal and by Theorem 2.2. U is dominating set in $U_2(G)$. Hence the theorem. □

Theorem 4.3. *For a bipartite graph G with partite sets X and Y*

$$i(U_2(G)) \leq \begin{cases} pq, & p = p_k,\ q \leq q_k \\ \binom{p}{2} + \binom{q}{2}, & p = p_k,\ q > q_k \\ pq, & p_{k-1} < p < p_k,\ q_{k-2} - i \leq q \leq q_{k-1} + j \\ \binom{p}{2} + \binom{q}{2}, & p_{k-1} < p < p_k,\ q_{k-1} + j \leq q \leq q_{k-2} - i \end{cases}$$

where

$$|X| = p, |Y| = q,\ p_k = \frac{k^2 + 3k + 2}{2},\ q_k = \frac{k^2 + 5k + 6}{2},\ k = 0, 1, 2, 3, \ldots;$$

$$i = k - 1,\ k - 2,\ k - 3,\ \ldots,\ j = 1, 2, 3, \ldots, k.$$

Proof. Let G be bipartite with partite sets X and Y. Then $U_2(G)$ is bipartite and the partite sets of $U_2(G)$ are V_1 and V_2 where $V_1 = \{\{x, y\} : x, y \in X\} \cup \{\{r, s\} : r, s \in Y\}$ and

Now, we find the values of p and q such that

$$pq = \binom{p}{2} + \binom{q}{2} \tag{1}$$

$$\Rightarrow 2pq = p^2 + q^2 - q - p$$

Then $pq = \dfrac{p^2 - p}{2} + \dfrac{q^2 - q}{2} \Rightarrow pq = \dfrac{p^2 + q^2 - q - p}{2}$

$$\Rightarrow p - q = \sqrt{p + q}$$

Then the possible values of p, q which satisfy the Equation 1 are

$$p_0, p_1, p_2, \ldots, q_0, q_1, q_2, \ldots$$

where $p_k = \dfrac{k^2 + 3k + 2}{2}$ and $q_k = \dfrac{k^2 + 5k + 6}{2}$, $k = 0, 1, 2, 3, \ldots,$ we have $pq = \binom{p}{2} + \binom{q}{2}$. Then, for $p_{k-1} < p < p_k$, we assign values for q as depicted in Table 1.

It is observed that, for $i = k-1, k-2, k-3, \ldots, 0,$ $j = 1, 2, 3, \ldots, k$

If $p_{k-1} < p < p_k; q_{k-2} - i \leq q \leq q_{k-1} + j$ then $pq > \binom{p}{2} + \binom{q}{2}$

If $p_{k-1} < p < p_k$, $q_{k-1} + j \leq q \leq q_{k-2} - i$ then $pq < \binom{p}{2} + \binom{q}{2}$.

Hence the Theorem.

Table 1. Values of p and q for which $pq > \binom{p}{2} + \binom{q}{2}$.

p	q	pq	$\binom{p}{2} + \binom{q}{2}$
2	1	2	1
2	2	4	2
2	3	6	4
2	4	8	7
2	5	10	11
2	6	12	16
2	7	14	22
2	8	16	29
2	9	18	37
2	10	20	46
		…	…

Definition 4.4. A crown graph $H_{n,n}$ is a bipartite graph on $2n$ vertices, with $X = \{v_i : i = 1, 2, \ldots, n\}$ and $Y = \{u_j : j = 1, 2, \ldots, n\}$ as partite sets where v_i is adjacent to u_j whenever $i \neq j$.

Theorem 4.5. *For crown graph* $H_{n,n} (n \geq 3)$, $i(U_2(H_{n,n})) \leq 2n-1$.

Proof. Since $H_{n,n}$ is bipartite $U_2(H_{n,n})$ is bipartite and the partite sets of $U_2(H_{n,n})$ are $V_1 = \{\{h, k\} : h = 1, 2, \ldots, n-1, k = h+1, h+2, \ldots, n\} \cup \{\{r, s\} : r = 1, 2, \ldots, n-1, s = r+1, r+2, \ldots, n\}$ and $V_2 = \{\{p, q\}, p = 1, 2, \ldots, n, q = 1, 2, \ldots, n\}$. Further partition V_1 into sets $V_1' = \{\{h, k\} : h = 1, 2, \ldots, n-1, k = h+1, h+2, \ldots, n\}$ and $V_1'' = \{\{r, s\} : r = 1, 2, \ldots, n-1, s = r+1, r+2, \ldots, n\}$. It is seen that each edge of $U_2(H_{n,n})$ has one end vertex in V_2 and other either in V_1' or V_1''. Therefore, we construct a MDS, D' of $U_2(H_{n,n})$ which contains vertices from both sets V_2 and V_1' (or V_1''). We construct D' as described below.

Case 1. n is even

Consider the subset $D_1 = \{\{v_i, u_j\} : i = 1, 3, 5, \ldots, n-1, j = i+1\} \cup \{\{v_i, u_j\}, i = 2, 4, 6, \ldots, n-1, j = i-1\}$ of V_2. By construction, D_1 is independent and dominates $V_1' - \{\{v_i, v_j\} : i = 1, 3, 5, \ldots, n-1, j = i+1\}$ and $V_1'' - \{\{u_i, u_j\} : i = 1, 3, 5, \ldots, n-1, j = i+1\}$. Further, the subsets $\{\{v_i, v_j\} : i = 1, 3, 5, \ldots, n-1, j = i+1\}$ and $\{\{u_i, u_j\} : i = 1, 3, 5, \ldots, n-2, j = i+1\}$ of V_1' and V_1'' respectively dominate the $V_2 - D_1$. Therefore $D' = D_1 \cup \{\{v_i, v_j\} : i = 1, 3, 5, \ldots, n-1, j = i+1\} \cup \{\{u_i, u_j\} : i = 1, 3, 5, \ldots, n-2, j = i+1\}$ is a dominating set of $U_2(H_{n,n})$ and by construction D' independent. Therefore, by Theorem 2.1, D' is a MIS of $U_2(H_{n,n})$ and so a MDS of $U_2(H_{n,n})$. Therefore $\gamma(U_2(H_{n,n})) \leq |D'| = \frac{n}{2} + \frac{n}{2} - 1 + \frac{n}{2} + \frac{n}{2} = 2n-1$.

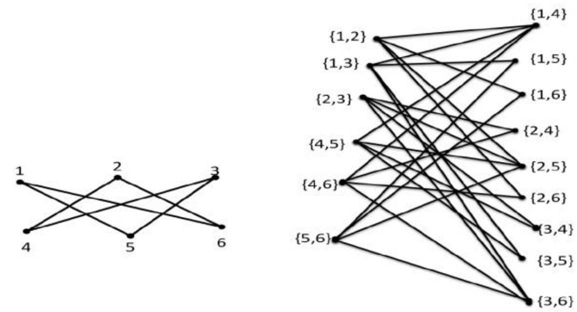

Figure 2. The crown graph $H_{3,3}$ and $U_2(H_{3,3})$.

Case 2. n is odd

As discussed in case 1, the subset $D_1 = \{\{v_i, v_j\} : i = 1, 3, 5, \ldots, n-2, j = i+1\} \cup \{\{v_n, u_n\}\} \cup \{\{v_i, u_j\} : i = 2, 4, 6, \ldots, n-1, j = i-1\}$ of V_2 and subset $\{\{v_i, v_j\} : i = 1, 3, 5, \ldots, n-2, j = i-1\} \cup \{\{u_i, u_j\} : i = 1, 3, 5, \ldots, n-2, j = i-1\}$ of V_1 together form a dominating set of $U_2(H_{n,n})$. Then $D' = D_1 \cup \{\{v_i, v_j\} : i = 1, 3, 5, \ldots, n-2, j = i+1\} \cup \{\{u_i, u_j\} : i = 1, 3, 5, \ldots, n-2, j = i+1\}$. By construction D' is an independent set in $U_2(H_{n,n})$. Therefore, D' is a MIS of $U_2(H_{n,n})$ and so a MDS of $U_2(H_{n,n})$. Also, Therefore $\gamma(U_2(H_{n,n})) \leq |D'| = \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n+1}{2} = 2n-1$. From both the cases we have t the MDS-D' of $U_2(H_{n,n})$ is the disjoint union of three independent subsets D_1, D_2 and D_3 of $V(U_2(H_{n,n}))$. Also each vertex of the sets D_1, D_2 and D_3 is of degree atleast three. Therefore D' is an independent set containing vertices of degree atleast three. Then by Theorem 2.4, $i(U_2(H_{n,n})) = \gamma(U_2(H_{n,n})) \leq |D'| = 2n-1$.

Corollary 4.6. *For a bipartite graph G,*

(i) *If* $G \cong P_n$, $n \geq 4$, $i(U_2(P_n)) \leq \begin{cases} \left(\frac{n-1}{2}\right)^2, & n \textit{ is odd} \\ \left(\frac{n-2}{2}\right)\frac{n}{2}, & n \textit{ is even} \end{cases}$

(ii) *If* $G \cong C_n$, $n \geq 4$, n *is even,* $i(U_2(C_n)) \leq \left(\frac{n-2}{2}\right)\frac{n}{2}$.

Theorem 4.7. *For a cycle graph* C_n, $n \geq 5$, n *is odd*

$$i(U_2(C_n)) \leq \begin{cases} \left(\frac{n-3}{4}\right)n, & \left\lfloor \frac{n}{2} \right\rfloor \text{ is odd} \\ \frac{n^2-3n-2}{4}, & \left\lfloor \frac{n}{2} \right\rfloor \text{ is even} \end{cases}$$

Proof. We partition $V(U_2(C_n))$ into $n-1$ disjoint sets $V_l = \{\{i, i+l\} : l = 1, 2, \ldots, n-1, i = 1, 2, \ldots, n-1\}$. Clearly, $U = V_1 \cup V_{n-1}$ and $W = V_2 \cup V_3 \cup V_4 \cup \ldots \cup V_{n-2}$. Then $V_1 \cup V_{n-1}$ is an independent set in $U_2(C_n)$. Further, it is seen that each line pair is adjacent to a vertex of the subset $V_2 \cup V_{n-2}$ of W, therefore $V_2 \cup V_{n-2}$ dominates U and is also an independent set in $U_2(C_n)$. Now, we construct a MDS, I of $U_2(C_n)$ that contains the set $V_2 \cup V_{n-2}$ as described below:

Case 1. $\left\lfloor \frac{n}{2} \right\rfloor$ is odd

Here, each V_l is an independent set except for $l = \left\lfloor \frac{n}{2} \right\rfloor$ in $U_2(C_n)$. We define the subset $I = V_2 \cup V_{n-2} \cup V_4 \cup V_{n-4} \cup V_6 \cup \ldots \cup V_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \cup V_{\left\lfloor \frac{n}{2} \right\rfloor + 2} \cup V_{\left\lfloor \frac{n}{2} \right\rfloor + 4} \cup \ldots \cup V_{n-6}$ of W. By construction, I is independent. Also, each vertex of $V(U_2(C_n)) - I$ is adjacent to a vertex of I therefore I is a dominating set of $U_2(C_n)$. By Theorem 2.1, I is a maximal independent set and therefore by Theorem 2.2, I is a MDS of $U_2(C_n)$. Hence, $i(U_2(C_n)) \leq |I|$.

Since $I = V_2 \cup V_{n-2} \cup V_4 \cup V_{n-4} \cup V_6 \cup \ldots \cup V_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \cup V_{\left\lfloor \frac{n}{2} \right\rfloor + 2} \cup V_{\left\lfloor \frac{n}{2} \right\rfloor + 4} \cup \ldots \cup V_{n-6}$

$$\Rightarrow |I| = |V_2| + |V_2| + \ldots + |V_{\left\lfloor \frac{n}{2} \right\rfloor - 1}| + |V_{\left\lfloor \frac{n}{2} \right\rfloor + 2}| + \ldots + |V_{\left\lfloor \frac{n}{2} \right\rfloor + 4}| + \ldots$$
$$+ |V_{n-4}| + |V_{n-2}| \qquad (2)$$

where $|V_2| + |V_4| + \ldots + |V_{\left\lfloor \frac{n}{2} \right\rfloor - 1}| = n - 2 + n - 4 + \ldots + n - \left(\frac{n-3}{2}\right)$

On simplification,

$$n-2+n-4+\ldots+n-\left(\frac{n-3}{2}\right)=\left(\frac{3n-1}{4}\right)\left(\frac{n-3}{4}\right) \tag{3}$$

and $|V_{\lfloor \frac{n}{2} \rfloor+2}|+|V_{\lfloor \frac{n}{2} \rfloor+4}|+\ldots+|V_{n-4}|+|V_{n-2}|$

$$=n-\left(\frac{n+3}{2}\right)+n-\left(\frac{n+7}{2}\right)+n-\left(\frac{n+11}{2}\right)\ldots+n-\left(\frac{n+(n-4)}{2}\right)$$

On simplification,

$$n-\left(\frac{n+3}{2}\right)+n-\left(\frac{n+7}{2}\right)+n-\left(\frac{n+11}{2}\right)\ldots+n-\left(\frac{n+(n-4)}{2}\right)$$

$$=\left(\frac{n+1}{4}\right)\left(\frac{n-3}{4}\right)$$

From (2) and (3) we have $|I|=\left(\frac{3n-1}{4}\right)\left(\frac{n-3}{4}\right)+\left(\frac{n+1}{4}\right)\left(\frac{n-3}{4}\right)$ $=n\left(\frac{n-3}{4}\right)$

Therefore, $i(U_2(C_n))\leq|I|=n\left(\frac{n-3}{4}\right)$.

Case 2. $\lfloor \frac{n}{2} \rfloor$ is even

As in case 1, we construct the subset $I=V_2\cup V_4\cup V_6\cup\ldots\cup V_{\lfloor \frac{n}{2} \rfloor-2}$ $\cup V_{\lfloor \frac{n}{2} \rfloor+1}\cup V_{\lfloor \frac{n}{2} \rfloor+3}\cup\ldots\cup V_{n-2}$ of $U_2(C_n)$. By construction I is independent and also each vertex of $V(U_2(C_n))-I$ is adjacent to a vertex of I implies I is a dominating set of $U_2(C_n)$. Therefore, I is a MIS and hence a MIDS of $U_2(C_n)$. So, $i(U_2(C_n))\leq|I|$.

And $I=V_2\cup V_4\cup V_6\cup\ldots\cup V_{\lfloor \frac{n}{2} \rfloor-2}\cup V_{\lfloor \frac{n}{2} \rfloor+1}\cup V_{\lfloor \frac{n}{2} \rfloor+3}\cup\ldots\cup V_{n-2}$

$$\Rightarrow|I|=|V_2|+|V_4|+\ldots+|V_{\lfloor \frac{n}{2} \rfloor-2}|+|V_{\lfloor \frac{n}{2} \rfloor+1}|+|V_{\lfloor \frac{n}{2} \rfloor+3}|+\ldots+|V_{n-2}| \tag{4}$$

where $|V_2|+|V_4|+\ldots+|V_{\lfloor \frac{n}{2}\rfloor -2}| = n-2+n-4+\ldots+n-\left(\frac{n-5}{2}\right)$

On simplification,

$$n-2+n-4+\ldots+n-\left(\frac{n-5}{2}\right)=\left(\frac{3n+1}{4}\right)\left(\frac{n-5}{4}\right) \tag{5}$$

and $$|V_{\lfloor \frac{n}{2}\rfloor +1}|+|V_{\lfloor \frac{n}{2}\rfloor +3}|+\ldots+|V_{n-2}| = n-\left(\frac{n+1}{2}\right)+n-\left(\frac{n+5}{2}\right)$$

$$+n-\left(\frac{n+9}{2}\right)\ldots+n-\left(\frac{n+(n-4)}{2}\right)$$

On simplification,

$$n-\left(\frac{n+1}{2}\right)+n-\left(\frac{n+5}{2}\right)+n-\left(\frac{n+9}{2}\right)\ldots+n-\left(\frac{n+(n-4)}{2}\right)$$

$$=\left(\frac{n-1}{4}\right)\left(\frac{n+3}{4}\right)$$

From (4) and (5) we have $|I'|=\left(\frac{3n+1}{4}\right)\left(\frac{n-5}{4}\right)+\left(\frac{n-1}{4}\right)\left(\frac{n+3}{4}\right)$

$$=\frac{n^2-3n-2}{4}$$

Therefore, $i(U_2(C_n))\leq |I'| = \frac{n^2-3n-2}{4}$. □

Theorem 4.8. *For a wheel* $W_{1,n}$, $n \geq 5$

$$i(U_2(W_{1,n}))\leq \begin{cases} \frac{n^2-1}{8}, & n \text{ is odd} \\ \frac{n^2-2n+16}{8} & n \text{ is even.} \end{cases}$$

Proof. Consider the partition of $V(U_2(W_{1,n}))$ into sets $V(U_2(C_n)) = \{\{i, j\} : i = 1, 2, \ldots, n-1,\ j = i+1, i+2, \ldots, n\}$ and $X = \{\{i, n+1\} : i = 1, 2, \ldots, n\}$. For each vertex $\{i, n+1\}$ of X there exists at least one vertex $\{h, k\}$ in $V(U_2(C_n))$ not adjacent to $\{i, n+1\}$. Therefore, we construct a MDS-I of

$U_2(W_{1,n})$ which contains vertices of both X and $V(U_2(C_n))$ as described as below:

Case 1. n is odd, $n \geq 5$

For $n = 3$, $W_{1,3} \cong K_4$. Then $i(U_2(W_{1,3})) = i(U_2(K_4)) = 2$.

For $n \geq 5$, define a subset $I_1 = \{\{i, n+1\} : i = 4, 6, 8, \ldots, n-1\}$, $|I_1| = \frac{n-3}{2}$ of X. Then I_1 dominates the vertices $\{\{i, n+1\} : i = 3, 5, 7, 9, \ldots, n\}$ and $\{\{i, k\} : k = 1, 2, \ldots, n\}$ of X and $V(U_2(C_n))$ respectively. By construction I is an independent subset of X. Further, the vertices $\{1, 3\}$ and $\{2, n\}$ of $V(U_2(C_n))$ dominates the vertices $\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 2\}, \{1, n\}, \{2, n-1\}$, $\{3, n+1\}$ and $\{1, n+1\}, \{2, n+1\}, \{3, n+1\}, \{n, n+1\}$ of $V(U_2(C_n))$ and X respectively. Also, no vertex of I_1 is dominated by the vertices $\{v_1, v_3\}, \{v_2, v_n\}$. Therefore $I_1 \cup \{\{v_1, v_3\}, \{v_2, v_n\}\}$ is an independent set in $U_2(W_{1,n})$.

Now in $V(U_2(C_n))$, we are left with the vertices $Z = \{\{v_i, v_j\} : i = 1, 2, 3, j = 5, 7, 9, \ldots, n-2\} \cup \{\{v_h, v_k\} : h = 5, 7, 9, \ldots, n-2, k = h+2, h+4, h+6, \ldots, n\}$. Clearly, Z is not an independent set and also no vertex of Z is adjacent to a vertex of $I_1 \cup \{\{v_1, v_3\}, \{v_2, v_n\}\}$. Therefore, we define a subset $I_2 = \{\{v_2, v_j\} : j = 5, 7, 9, \ldots, n-2\} \cup \{\{v_h, v_k\} : h = 5, 7, 9, \ldots, n-3, k = h+2, h+4, h+6, n-1\}$ of Z consisting of $\frac{n-5}{2} + \binom{\frac{n-3}{2}}{2}$ vertices. By construction I_2 is independent and dominating set of Z and no vertex of $I_1 \cup \{\{v_1, v_3\}, \{v_2, v_n\}\}$ is adjacent to a vertex of I_2. Thus $I = I_1 \cup \{\{v_1, v_3\}, \{v_2, v_n\}\} \cup I_2$ is an independent set of $U_2(W_{1,n})$ and each vertex of $V(U_2(W_{1,n})) - I$ is adjacent to a vertex of I therefore I is a dominating set of $U_2(W_{1,n})$. By Theorem 2.1, I is a MIS of $U_2(W_{1,n})$ and so a MDS of $U_2(W_{1,n})$. Hence, $i(U_2(W_{1,n})) \leq |I|$.

$$\Rightarrow i(U_2(W_{1,n})) \leq |I_1| + 2 + |I_2| = \frac{n-3}{2} + 2 + \frac{n-5}{5} + \binom{\frac{n-3}{2}}{2} = \frac{n^2-1}{8}$$

$$\Rightarrow i(U_2(W_{1,n})) \leq \frac{n^2-1}{8}.$$

Case 2. n is even, $n \geq 6$,

For $n = 4$, the minimum independent dominating set of $U_2(W_{1,4})$ is $\{\{v_1, v_3\}, \{v_2, v_5\}\}$ and $i(U_2(W_{1,4})) = 2$.

For $n \geq 6$, as discussed in Case 1, we define a subset $I_2 = \{\{v_i, v_{n+1}\} : i = 5, 7, 9, \ldots, n-1\}, |I_1| = \frac{n-4}{2}$ of V_1, the subset $I_2 = \{\{v_i, v_j\} : i = 1, 3 : j = 6, 8, \ldots, n-2\} \cup \{\{v_h, v_k\} : h = 6, 8, 10, \ldots, n-2$, $k = h+2, h+4, h+6, \ldots, n-2\} \cup \{\{v_3, v_4\}, \{v_4, v_n\}\} \cup \{\{v_1, v_3\}, \{v_2, v_n\}\}$ of $V(U_2(C_n))$. By construction, $I = I_1 \cup I_2$ is an independent dominating set of $U_2(W_{1,n})$ and so I is a maximal independent set of $U_2(W_{1,n})$. By Theorem 2.2, I is a MIDS of $U_2(W_{1,n})$. Hence, $i(U_2(W_{1,n})) \leq |I| \Rightarrow i(U_2(W_{1,n}))$

$$\leq |I_1| + |I_2| = \frac{n-4}{2} + 2 + n - 6 + \binom{\frac{n-6}{2}}{2} + 2 = \frac{n^2 - 2n + 16}{8}. \qquad \square$$

Theorem 4.9. *For a non-bipartite graph G, $i(U_2(G)) \leq \binom{\beta}{2} + \left\lfloor \frac{n-\beta}{2} \right\rfloor$.*

Proof. Let S be the maximum independent set of a non bipartite graph G. Consider the following partition of $V(U_2(G)) = S' \cup X \cup T$ where $S' = \{\{x, y\} : x \in S, y \in V(G) - S\}$, $X = \{\{x, y\} : x, y \in S\}$ and $T = \{\{x, y\} : x, y \in V(G) - S\}$. Here S' is dominating and not independent, X is independent and T is not dominating and not independent. So, we find superset Y of X which is independent and dominating in $U_2(G)$.

Since S is a maximum independent set, each $y \in S$ is adjacent to at least one $y' \in V(G) - S$. Therefore, for each vertex $\{x, y\} \in X$ there is at least one $\{x, y'\} \in S'$ which is adjacent to $\{x, y\}$ therefore X dominates S' and

$X \cup \{\{x, y\}/\{x, y\} \in S'\}$ is not independent. Further, the sets X and T are vertex disjoint and T is not independent, by Theorem 2.5, we have a maximum independent set T'' of T which has at least $\left\lfloor \frac{n-\beta}{2} \right\rfloor$ vertices such that $Y = X \cup T'$ is independent in $U_2(G)$. Clearly, Y is a dominating set of $U_2(G)$ which is also independent. Therefore $X \cup T'$ is a independent dominating set of $U_2(G)$. Further, for a minimum independent dominating set of $U_2(G)$,

$$i(U_2(G)) \leq |X \cup T'| = |X| + |T'| = \binom{\beta}{2} + \left\lfloor \frac{n-\beta}{2} \right\rfloor \Rightarrow i(U_2(G)) \leq \binom{\beta}{2} + \left\lfloor \frac{n-\beta}{2} \right\rfloor.$$

□

Theorem 4.10. *For a graph* G, $\beta_1(G) \leq i(U_2(G))$.

Proof. Let E' be the maximum edge independent set of G. Then in $U_2(G)$, the line pairs which are due to E' form an independent subset E'' of U. If E'' dominates $V(U_2(G)) - E''$ then E'' is an independent dominating set of $U_2(G)$ and $\beta_1(G) \leq i(U_2(G))$. If E'' is not dominating then construct a set $I = E'' \cup J$ where $J \subseteq V(U_2(G)) - E''$ and $J \cap N[E''] = \emptyset$ then I is an independent dominating set in $U_2(G)$. Then for any minimum independent dominating set of $U_2(G)$, $\beta_1(G) \leq i(U_2(G))$.

The bound is attained for $G \cong K_n$, $K_n - e$, triangle with a tail. □

Theorem 4.11. *For a graph* G, $i(U_2(G)) \leq \binom{n}{2} - \Delta'(G)$.

Proof. Let xy be an edge of maximum degree $\Delta'(G)$ in G then the corresponding line pair $\{x, y\}$ in $U_2(G)$ dominates $\Delta'(G)$ vertices in $U_2(G)$ which means $\binom{n}{2} - \Delta'(G) - 1$ vertices of $U_2(G)$ are not adjacent to $\{x, y\}$. Let I' be a subset of $U_2(G)$ consisting of $\binom{n}{2} - \Delta'(G) - 1$ vertices of $U_2(G)$ that are not adjacent to $\{x, y\}$. We construct an independent subset I'' of I' such that $I'' \cup \{\{x, y\}\}$ forms an independent dominating set of $U_2(G)$. Then for any minimum independent dominating set of $U_2(G)$, $i(U_2(G)) \leq |I'' \cup \{\{x, y\}\}| \Rightarrow i(U_2(G)) \leq \binom{n}{2} - \Delta'(G)$. The bound is sharp for $G \cong C_4$. □

5. Conclusion

In this paper, we have obtained bounds on the independent domination number of double vertex graph of a graph G in terms of parameters of G. Also, we have obtained bounds on the independent domination number of double vertex graph of some classes of graphs.

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