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DOM-CHROMATIC NUMBER OF CERTAIN SPLITTING GRAPHS

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Abstract

Graph Theory is a branch of Mathematics which has found a plenty of applications in this modern era. Domination and coloring are the two major concepts in Graph Theory which have seen a major development in the research field. For a given χ-coloring of a graph *G*, a dominating set is said to be a dom-coloring set, if it contains at least one vertex from each color class of *G*. In this paper we have determined the dom-chromatic number of splitting graphs for path and comb graphs.

1. Introduction

Today, Graph theory has seen a major development in various research fields due to its strong connection with Computer Science. It plays an important role in several areas of computer science such as logical design, artificial intelligence, formal languages, information organization and retrieval including the areas of social science, linguistics and communication engineering. In 1958, domination was formalized as a theoretical area in

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graph theory by C. Berge [1]. He referred to the domination number as the introduction co-efficient of external stability [7]. In 1962, Ore was the first to use the term "domination" for undirected graphs and he denoted the domination number by $\delta(G)$ [8]. He also introduced the concepts of minimal and minimum dominating sets of vertices in a graph. In 1977, Cockayne and Hedetniemi introduced the accepted notation $\gamma(G)$ to denote the domination number [4, 5]. Further study on domination, resulted in the introduction of connected domination in graphs which has vast real life applications [12]. Moreover, graph coloring and domination problems are often in relation. Chellali and Volkmann showed some relations between the chromatic number and some domination parameters in the graph [3].

2. Preliminaries

Definition 2.1 [6]**.** Let G be a finite, simple and undirected graph with n vertices. A non-empty subset S of a graph $G(V, E)$ is said to be dom-coloring set if for every vertex v in $V(G) - S,$ there is a vertex u in S such that u is adjacent to v. This set S is said to be a minimum dominating set if it has the least number of vertices. The cardinality of a minimum dominating set is called the domination number of G and is denoted by $\gamma(G)$.

Definition 2.2 [7]**.** Graph coloring is the assignment of colors to the vertices of a graph G such that no two adjacent vertices receive the same color. The least number of colors needed to color the graph G is called the chromatic number and is denoted by $\chi(G)$.

Definition 2.3 [2]**.** For a given χ-coloring of a graph G, a dominating set is said to be dom-colouring set if it contains at least one vertex from each color class of G. The cardinality of the minimum dom-coloring set is called the dom-chromatic number and is denoted by $\gamma_{dc}(G)$.

3. Results

In this section, we have determined the dom-chromatic number of the splitting graph of path P_n and comb graphs P_n^+.

3.1. Dom-chromatic number of splitting graph of path P_n

Definition 3.1 [9]. The splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$, where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' respectively in $S'(G)$. See figure 1

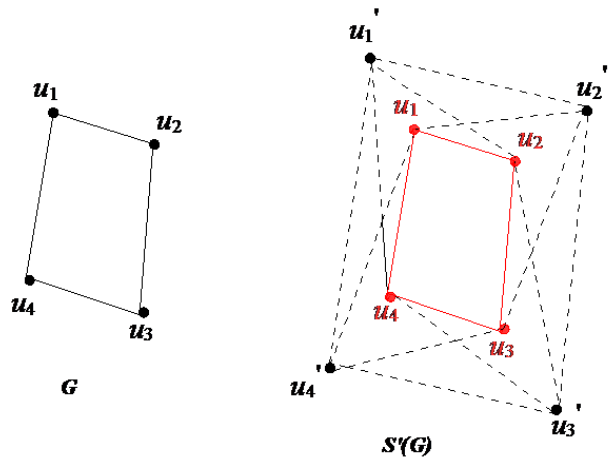

Figure 1. Splitting graph of a 2-Regular graph.

Theorem 3.1. *For any path P_n, $n \geq 4$, the dom-chromatic number of the splitting graph of P_n is given by*

$$\gamma_{dc}(S'(P_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \left\lceil \frac{n}{2} \right\rceil & \text{if } n \equiv 1, 3 \pmod 4 \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod 4). \end{cases}$$

Proof. Consider the path P_n, $n \geq 4$. Let the vertices of P_n be labeled as u_i, $i = 1, 2, \ldots, n$. Clearly P_n is 2 colorable. Therefore the vertices of P_n can be colored alternatively with colors 1 and 2. Let G be the splitting graph $S'(P_n)$ of the path P_n. Let u_i', $i = 1, 2, \ldots, n$ represent the copies of u_i, $i = 1, 2, \ldots, n$. The splitting graph G thus obtained can also be colored with $\chi(P_n)$ colors, by assigning the same color of P_n to each copy of the corresponding vertices u_i', $i = 1, 2, \ldots, n$. We now find the dom-chromatic number of G under the following cases.

Case (i). $n \equiv 0 (\text{mod } 4)$.

The $2n$ vertices of G can be partitioned into set of 8 vertices as in B_k. Figure 2. Let $B_k = \{u_i, u_i' : 4k - 3 \leq i \leq 4k\}$, for all $k = 1, 2, \ldots, \frac{n}{4}$ represent the $\frac{n}{4}$ such components. Each component is dominated by 2 adjacent vertices which is the minimum dominating set of each component. Hence $D = \{u_{4i-2}, u_{4i-1}' : 1 \leq i \leq \frac{n}{4}\}$ is the minimum dominating set which contains at least one vertex from each color class of G.

Dom-chromatic number of each component = domination number of each component $= 2$. Hence the cardinality of the dc-set $= \gamma_{dc}(G) = |D|$ $= 2 \times \frac{n}{4} = \frac{n}{2}$.

Case (ii). $n \equiv 1 (\text{mod } 4)$.

The $2n$ vertices of G can be partitioned into $\frac{n-1}{4}$ components with 8 vertices in each component and the remaining 2 vertices as shown in the Figure 3. Then $V(G) = B_k \cup \{u_n, u_n'\}$ where $B_k = \{u_n, u_n' : 4k - 3 \leq i \leq 4k\}$, for all $k = 1, 2, \ldots, \frac{n-1}{4}$ represents the $\frac{n-1}{4}$ such components. Each component B_k, $k = 1, 2, \ldots, \left\lfloor \frac{n-2}{4} \right\rfloor$ is dominated by 2 adjacent vertices $\left\{u_{4i-2}, u_{4i-1}' : 1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor\right\}$ and the component $B_{\frac{n-1}{4}}$ is dominated by 3 adjacent vertices $\{u_{n-3}, u_{n-2}', u_{n-1}\}$. Hence $D = \left\{u_{4i-2}, u_{4i-1}' : 1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor\right\}$ $\cup \{u_{n-3}, u_{n-2}', u_{n-1}\}$ is the minimum dominating set which contains at least one vertex from each color class of G.

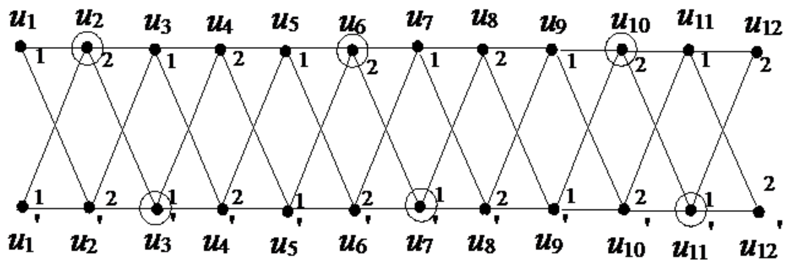

Figure 2. Splitting graph $S'(P_{12})$.

Hence the cardinality of the dc-set $= \gamma_{dc}(G) = |D| = 2 \times \left\lfloor \frac{n-2}{4} \right\rfloor + 3$

$= 2 \times \left(\frac{n-1}{4} - 1 \right) + 3 = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$.

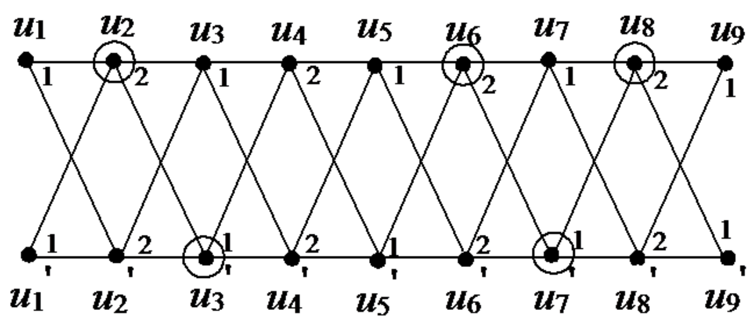

Figure 3. Splitting graph $S'(P_9)$.

Case (iii). $n \equiv 2 (\text{mod } 4)$.

The $2n$ vertices of G can be partitioned into $\frac{n-2}{4}$ components with 8 vertices in each component and the remaining 4 vertices as shown in the Figure 4. Then $V(G) = B_k \cup \{u_{n-1}, u'_{n-1}, u_n, u'_n\}$ where $B_k = \{u_i, u'_i : 4k - 3 \leq i \leq 4k\}$, for all $k = 1, 2, \ldots, \frac{n-2}{4}$ represents the $\frac{n-2}{4}$ such components. Each component B_k, $k = 1, 2, \ldots, \frac{n-2}{4}$ is dominated by 2 adjacent vertices $\{u_{4i-2}, u'_{4i-1} : 1 \leq i \leq \frac{n-2}{4}\}$ and the remaining 4 vertices are dominated by 2 adjacent vertices $\{u_{n-1}, u'_n\}$. Hence $D = \{u_{4i-2}, u'_{4i-1} : 1 \le i \le \frac{n-2}{4}\} \cup \{u_{n-1}, u'_n\}$ is the minimum dominating set which contains at least one vertex from each color class of G.

Hence the cardinality of the dc-set $= \gamma_{dc}(G) = |D| = 2 \times \left(\frac{n-2}{4}\right) + 2 = \left(\frac{n-2}{2}\right) + 2 = \frac{n+2}{2} = \frac{n}{2} + 1.$

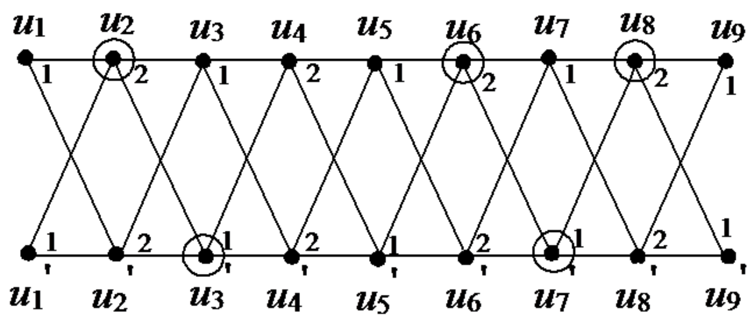

Figure 4. Splitting graph $S'(P_{10})$.

Case (iv). $n \equiv 3 (\bmod 4)$.

The $2n$ vertices of G can be partitioned into $\frac{n-3}{4}$ components with 8 vertices in each component and the remaining 6 vertices as shown in the Figure 5. Then $V(G) = B_k \cup \{u_i, u'_i : n-2 \le i \le n\}$ where $B_k = \{u_i, u'_i : 4k-3 \le i \le 4k\}$, for all $k = 1, 2, \ldots, \frac{n-3}{4}$ represents the $\frac{n-3}{4}$ such components. Each component B_k, $k = 1, 2, \ldots, \frac{n-3}{4}$ is dominated by 2 adjacent vertices $\{u_{4i-2}, u'_{4i-1} : 1 \le i \le \frac{n-3}{4}\}$ and the remaining 6 vertices are dominated by 2 adjacent vertices $\{u_{n-1}, u'_n\}$. Hence $D = \{u_{4i-2}, u'_{4i-1} : 1 \le i \le \frac{n-3}{4}\} \cup \{u_{n-1}, u'_n\}$ is the minimum dominating set which contains at least one vertex from each color class of G.

Hence the cardinality of the dc-set $= \gamma_{dc}(G) = |D| = 2 \times \left(\frac{n-3}{4}\right) + 2$ $= \left(\frac{n-3}{2}\right) + 2 = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$.

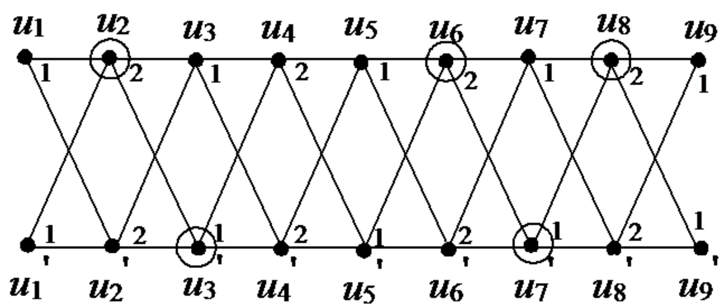

Figure 5. Splitting graph $S'(P_{10})$.

Definition 3.2 [10]**.** A graph obtained by attaching a single pendant edge to each vertex of a path P_n is called a comb graph and is denoted by P_n^+. It consists of $2n$ vertices and $2n - 1$ edges.

Theorem 3.2 [11]**.** *Let G be a comb graph* P_n^+. *Then* $\gamma_{dc}(G) = n$ *for all n.*

Theorem 3.3. *For any comb graph* P_n^+, $n > 2$ *vertices, the dom-chromatic number of the splitting graph is given by* $\gamma_{dc}(S'(P_n^+)) = \gamma_{dc}(P_n^+) = n$.

Proof. Let G be a comb graph P_n^+ with $n > 2$ vertices labeled as $u_i, v_i = 1, 2, \ldots, n$. Clearly G is 2 colorable. Let the vertices u_i, $i = 1, 2, \ldots, n$ of the graph G be colored alternately with colors 1 and 2. The pendant vertices v_i, $i = 1, 2, \ldots, n$ are colored alternately with colors 2 and 1. Let $S'(G)$ be the splitting graph of G, with vertices u_i', v_i', $i = 1, 2, \ldots, n$ which are copies of vertices u_i, v_i, $i = 1, 2, \ldots, n$. The splitting graph $S'(G)$ thus obtained can also be colored with 2 colors, by assigning the same colors of G to each copy of the corresponding vertices $u_i',. v_i'$, $i = 1, 2, \ldots, n$. Let $D = \{u_i, i = 1, 2, \ldots, n\}$ be the minimum dom-coloring set of G. In the splitting graph $S'(G)$, the same set of vertices minimum dom-coloring set D of G dominates all the vertices of $S'(G)$ which yields the minimum dom-coloring set. D is the minimum dominating set which contains at least one vertex from each color class of G. Hence D is a dom-coloring set of $S'(G)$.

Therefore the cardinality of the dc-set $= \gamma_{dc}(S'(G)) = n$.

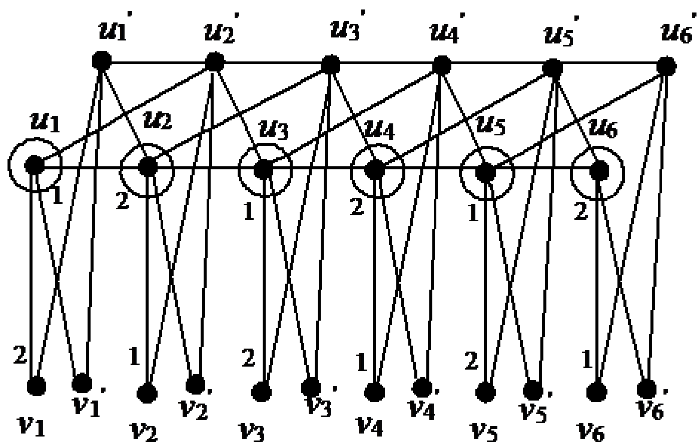

Figure 6. Splitting graph $S'(P_6^+)$.

4. Conclusion

The concepts on domination number and dom-chromatic number have been discussed in this paper. The results have been extended to the splitting graphs of path and comb graphs. Study on determining the dom-chromatic number of tree related networks are in progress.

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