Advances and Applications in Mathematical Sciences ISSN 0974-6803
Volume 22, Issue 5, March 2023, Pages 997-1011
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MODAL OPERATORS ON PICTURE FUZZY MATRICES

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Abstract

In this paper, we introduce Modal Operators on Picture Fuzzy Matrix (PicFM) and derive some results.

1. Introduction

In 1965, Zadeh introduced the concept of fuzzy sets (FS), a powerful tool for dealing with fuzzy. As a result, Atanassov introduced a new concept in 1983 called the Intuitionistic Fuzzy Set (IFS), an extension of FS. After the introduction of FS, Hashimoto [3] introduced the concept of fuzzy matrix (FM).

For motivations to deal with completely different kinds of uncertainties, there are many generalizations and modifications of FS theory, such as vague sets, rough sets, soft sets, IFS theory, and FS theory.

Due to some limitations on true and false membership values, FS and its extensions only handle uncertain data and vague and inconsistent data that

2020 Mathematics Subject Classification: Primary 03E72, Secondary 15B15.

Keywords: Picture Fuzzy Set (PicFS), Picture Fuzzy Matrices (PicFMs), Modal Operator (MO).

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Received April 22, 2022; Accepted December 12, 2022

may actually exist. For example, in some areas of the science discipline, it has been pointed out that the two elements do not seem to be free to represent a particular type of data. In such cases, neutrality is needed to fully represent the data. For example, in Medicine, a disease can have three types of effects (positive, neutral, negative) on selected symptoms. Therefore, it removes the limitations of IFS and handles additional uncertainties achievable in a reasonable state.

PicFS was started by Cuong and Kreinovich [4] as a generalization of IFS. Recently, Shovan Dogra and Pal [5] studied the concept of the PicFM and its applications, and the MO of Intuitionistic FM [8] was studied by P. Murugadas, S. Sriram, and T. Muthuraji.

This article explores MOs on PicFMs and describes some properties.

2. Preliminaries

Hereafter $\mathcal{P}_{xy}$ means PicFMs of order $x \times y$ and $\mathcal{P}_x$ denotes PicFMs of order $x \times x$.

For basic theory about PicFS and PicFMs see (4, 5).

Definition 2.1. For $a = \langle \chi^t, \chi^n, \chi^f \rangle, b = \langle \phi^t, \phi^n, \phi^f \rangle \in PicFS,$ we define joint $(\vee)$ and meet $(\wedge)$ operations as,

(1) $\langle \chi^t, \chi^n, \chi^f \rangle \vee \langle \phi^t, \phi^n, \phi^f \rangle = \langle \max(\chi^t, \phi^t), \max(\chi^n, \phi^n), \min(\chi^f, \phi^f) \rangle$ $= \langle (c^t, c^n, c^f) \rangle$ if $c^t + c^n + c^f \leq 1$, otherwise find $\max\{c^t, c^n, c^f\}$ and replace $\max\{c^t, c^n, c^f\}$ by 1- (sum of the rest of the Components)

(2) $\langle \chi^t, \chi^n, \chi^f \rangle \wedge \langle \phi^t, \phi^n, \phi^f \rangle = \langle \min(\chi^t, \phi^t), \min(\chi^n, \phi^n), \max(\chi^f, \phi^f) \rangle$

(3) $a^c = \langle \chi^t, \chi^n, \chi^f \rangle$.

Definition 2.2. For PicFMs $y1 = (\langle y1^t_{xy}, y1^n_{xy}, y1^f_{xy} \rangle)_{m \times n}$, $y2 = (\langle y2^t_{xy}, y2^n_{xy}, y2^f_{xy} \rangle)_{m \times n}$.

Define

(1) $y1 \vee y2 = (\langle y1_{xy}^t \vee y2_{xy}^t,\ y1_{xy}^n \vee y2_{xy}^n,\ y1_{xy}^f \wedge y2_{xy}^f \rangle)$

(2) $y1 \wedge y2 = (\langle y1_{xy}^t \wedge y2_{xy}^t,\ y1_{xy}^n \wedge y2_{xy}^n,\ y1_{xy}^f \vee y2_{xy}^f \rangle)$

(3) $y1 \times y2 = (\langle \vee (y1_{xy}^t \wedge y2_{xy}^t),\ \vee (y1_{xy}^n \wedge y2_{xy}^n),\ \wedge (y1_{xy}^f \vee y2_{xy}^f) \rangle)$

(4) $y1^T = (\langle y1_{ji}^t,\ y1_{ji}^n,\ y1_{ji}^f \rangle)$ ($y1^T$ is transpose of $y1$)

(5) $y1 \leq y2$ iff $y1_{xy}^t \leq y2_{xy}^t,\ y1_{xy}^n \leq y2_{xy}^n,\ y2_{xy}^f \geq y2_{xy}^f$

(6) $y1^c = (\langle y1_{xy}^t,\ y1_{xy}^n,\ y1_{xy}^f \rangle)$

(7) $y1 \oplus y2 = (\langle y1_{xy}^t \vee y2_{xy}^t,\ y1_{xy}^n \wedge y2_{xy}^n,\ y1_{xy}^f \wedge y2_{xy}^f \rangle)$

(8) $y1 \odot y2 = (\langle y1_{xy}^t \wedge y2_{xy}^t,\ y1_{xy}^n \wedge y2_{xy}^n,\ y1_{xy}^f \vee y2_{xy}^f \rangle)$.

3. Results Using Modal Operators in PFM

In this section, we define the Model operators $\square, \lozenge$ for PFM and discuss the relation between these operators.

Definition 3.1. For PicFM $y1 = (\langle y1_{xy}^t,\ y1_{xy}^n,\ y1_{xy}^f \rangle)$. We define $\square y1 = (\langle y1_{xy}^t,\ y1_{xy}^n,\ 1 - y1_{xy}^f \rangle)$ and $\lozenge y1 = (\langle 1 - y1_{xy}^t,\ y1_{xy}^n,\ 1 - y1_{xy}^f \rangle)$.

The $\lozenge y1$ and $\square y1$ need not be a PicFM.

Proposition 3.2. *For PicFM* $y1 = (\langle y1_{xy}^t,\ y1_{xy}^n,\ y1_{xy}^f \rangle) \in \mathcal{P}_{x \times y}$ *we have*

(i) $\square(\lozenge y1) = \lozenge y1$

(ii) $\lozenge(\square y1) = \square y1$

(iii) $\square\square y1 = \square y1$

(iv) $\lozenge\lozenge y1 = \lozenge y1$.

Proof. (i) $\lozenge y1 = (\langle 1 - y1_{xy}^f,\ y1_{xy}^n,\ y1_{xy}^f \rangle)$

$\Box(\Diamond y1) = (\langle 1 - y1^{f}_{xy},\ y1^{n}_{xy},\ 1 - (1 - y1^{f}_{xy})\rangle)$

$= (\langle 1 - y1^{f}_{xy},\ y1^{n}_{xy},\ y1^{f}_{xy}\rangle)$

$= \Diamond y1$

$\therefore \Box(\Diamond y1) = \Diamond y1.$

(ii) $\Box y1 = (\langle y1^{t}_{xy},\ y1^{n}_{xy},\ 1 - y1^{t}_{xy}\rangle)$

$\Diamond(\Box y1) = (\langle 1 - (1 - y1^{t}_{xy}),\ y1^{n}_{xy},\ 1 - y1^{t}_{xy}\rangle)$

$= (\langle y1^{t}_{xy},\ y1^{n}_{xy},\ 1 - y1^{t}_{xy}\rangle)$

$= \Box y1$

$\therefore \Diamond(\Box y1) = \Box y1.$

(iii) $\Box\Box y1 = \Box(\Box y1)$

$= \Box(\langle y1^{t}_{xy},\ y1^{n}_{xy},\ 1 - y1^{t}_{xy}\rangle)$

$= (\langle y1^{t}_{xy},\ y1^{n}_{xy},\ 1 - y1^{t}_{xy}\rangle)$

$= \Box y1$

$\therefore \Box\Box y1 = \Box y1.$

(iv) $\Diamond\Diamond y1 = \Diamond(y1)$

$= \Diamond(\langle 1 - y1^{f}_{xy},\ y1^{n}_{xy},\ y1^{f}_{xy}\rangle)$

$= (\langle 1 - y1^{f}_{xy},\ y1^{n}_{xy},\ y1^{f}_{xy}\rangle)$

$= \Diamond y1$

$\therefore \Diamond\Diamond y1 = \Diamond y1.$

Proposition 3.3. *For PicFM* $y1 = (\langle y1^{t}_{xy},\ y1^{n}_{xy},\ y1^{f}_{xy}\rangle) \in \mathcal{P}_{m\times n}$ *we have*

(i) $(\Box y1^{c})^{c} = \Diamond y1$, (ii) $(\Diamond y1^{c})^{c} = \Box y1$.

Proof. $y1 = (\langle y1_{xy}^t, y1_{xy}^n, y1_{xy}^f\rangle)$ then $y1^c = (\langle y1_{xy}^f, y1_{xy}^n, y1_{xy}^t\rangle)$

(i) Consider $(\Box y1^c) = (\langle y1_{xy}^f, y1_{xy}^n, 1 - y1_{xy}^f\rangle)$

$(\Box y1^c)^c = (\langle 1 - y1_{xy}^f, y1_{xy}^n, y1_{xy}^f\rangle) = \Diamond y1$

$\therefore (\Box y1^c)^c = \Diamond y1$.

(ii) Consider $(\Diamond y1^c) = (\langle 1 - y1_{xy}^t, y1_{xy}^n, y1_{xy}^t\rangle)$

$(\Diamond y1^c)^c = (\langle y1_{xy}^t, y1_{xy}^n, 1 - y1_{xy}^t\rangle) = \Box y1$

$\therefore (\Diamond y1^c)^c = \Box y1$.

Proposition 3.4. *For PFMts* $y1 = (\langle y1_{xy}^t, y1_{xy}^n, y1_{xy}^f\rangle)$, $y2 = (\langle y2_{xy}^t, y2_{xy}^n, y2_{xy}^f\rangle) \in \mathcal{P}_{m\times n}$, *then* $\Box y1 \odot \Box y2 = \Box(y1^c \oplus y2^c)^c$.

Proof. $\Box y1 \odot \Box y2 = (\langle y1_{xy}^t \wedge y2_{xy}^t, y1_{xy}^n \wedge y2_{xy}^n, (1 - y1_{xy}^t) \vee (1 - y1_{xy}^t)\rangle)$ (3.1)

Now, $(y1^c \oplus y2^c) = (\langle y1_{xy}^f \vee y2_{xy}^f, y1_{xy}^n \wedge y2_{xy}^n, y1_{xy}^t \wedge y2_{xy}^t\rangle)$

$(y1^c \oplus y2^c)^c = (\langle y1_{xy}^t \wedge y2_{xy}^t, y1_{xy}^n \wedge y2_{xy}^n, y1_{xy}^f \vee y2_{xy}^f\rangle)$

$\Box(y1^c \oplus y2^c)^c = (\langle y1_{xy}^t \wedge y2_{xy}^t, y1_{xy}^n \wedge y2_{xy}^n, 1 - (y1_{xy}^t \wedge y2_{xy}^t)\rangle)$ (3.2)

Claim. $(1 - y1_{xy}^t) \vee (1 - y2_{xy}^t) = 1 - (y1_{xy}^t \wedge y2_{xy}^t)$. (3.3)

Case (i) If $y1 \geq y2$ then $y1_{xy}^t \geq y2_{xy}^t$, $y1_{xy}^n \geq y2_{xy}^n$ and $y1_{xy}^n \leq y2_{xy}^n$

LHS of equation (3.3), $y1_{xy}^t \geq y2_{xy}^t$ then $(1 - y1_{xy}^t) \leq (1 - y2_{xy}^t)$

$(1 - y1_{xy}^t) \vee (1 - y2_{xy}^t) = 1 - y2_{xy}^t$ (3.4)

RHS of equation (3.3), $1 - (y1_{xy}^t \wedge y2_{xy}^t) = 1 - y2_{xy}^t$ (3.5)

From equations (3.4) and (3.5), we get, LHS = RHS

Case (ii) If $y1 < y2$ then $y1^t_{xy} < y2^t_{xy}$, $y1^n_{xy} < y2^n_{xy}$ and $y1^f_{xy} > y2^f_{xy}$

LHS of equation (3.3), $y1^t_{xy} < y2^t_{xy}$ then $(1 - y1^t_{xy}) > (1 - y2^t_{xy})$

$$(1 - y1^t_{xy}) \vee (1 - y2^t_{xy}) = 1 - y1^t_{xy} \tag{3.6}$$

RHS of equation (3.3), $1 - (y1^t_{xy} \wedge y2^t_{xy}) = 1 - y1^t_{xy}$ (3.7)

From equations (3.6) and (3.7), we get, LHS = RHS

From Case (i) and Case (ii) we get, $\Box y1 \odot \Box y2 = \Box(y1^c \oplus y2^c)^c$.

Proposition 3.5. *For PicFMts* $y1 = (\langle y1^t_{xy}, y1^n_{xy}, y1^f_{xy}\rangle)$, $y2 = (\langle y2^t_{xy}, y2^n_{xy}, y2^f_{xy}\rangle) \in \mathcal{P}_{x \times y}$, *then* $\Box y1 \oplus \Box y2 = (y1^c \odot y2^c)^c$.

Proof. $\Box y1 \odot \Box y2 = (\langle y1^t_{xy} \vee y2^t_{xy}, y1^n_{xy} \wedge y2^n_{xy}, (1 - y1^t_{xy}) \wedge (1 - y1^t_{xy})\rangle)$ (3.8)

Now, $(y1^c \odot y2^c) = (\langle y1^f_{xy} \wedge y2^f_{xy}, y1^n_{xy} \wedge y2^n_{xy}, y1^t_{xy} \vee y2^t_{xy}\rangle)$

$(y1^c \odot y2^c)^c = (\langle y1^t_{xy} \vee y2^t_{xy}, y1^n_{xy} \wedge y2^n_{xy}, y1^f_{xy} \wedge y2^f_{xy}\rangle)$

$$\Box(y1^c \odot y2^c)^c = (\langle y1^t_{xy} \vee y2^t_{xy}, y1^n_{xy} \wedge y2^n_{xy}, 1 - (y1^t_{xy} \vee y2^t_{xy})\rangle) \tag{3.9}$$

Claim: $(1 - y1^t_{xy}) \wedge (1 - y2^t_{xy}) = 1 - (y1^t_{xy} \vee y2^t_{xy})$ (3.10)

Case (i) If $y1 \geq y2$ then $y1^t_{xy} \geq y2^t_{xy}$, $y1^n_{xy} \geq y2^n_{xy}$ and $y1^f_{xy} \leq y2^f_{xy}$

LHS of equation (3.3), $y1^t_{xy} \geq y2^t_{xy}$ then $(1 - y1^t_{xy}) \leq (1 - y2^t_{xy})$

$$(1 - y1^t_{xy}) \wedge (1 - y2^t_{xy}) = 1 - y2^t_{xy} \tag{3.11}$$

RHS of equation (3.10), $1 - (y1^t_{xy} \vee y2^t_{xy}) = 1 - y2^t_{xy}$ (3.12)

From equations (3.11) and (3.12), we get, LHS = RHS.

Case (ii). If $y1 < y2$ then $y1^t_{xy} < y2^t_{xy}$, $y1^n_{xy} < y2^n_{xy}$ and $y1^f_{xy} > y2^f_{xy}$

LHS of equation (3.10), $y1^t_{xy} < y2^t_{xy}$ then $(1 - y1^t_{xy}) > (1 - y2^t_{xy})$

$$(1 - y1^t_{xy}) \wedge (1 - y2^t_{xy}) = 1 - y1^t_{xy} \tag{3.13}$$

RHS of equation (3.10), $1 - (y1^t_{xy} \vee y2^t_{xy}) = 1 - y1^t_{xy}$ (3.14)

From equations (3.13) and (3.14), we get, LHS = RHS.

From Case (i) and Case (ii) we get, $\Box y1 \oplus \Box y2 = \Box (y1^c \odot y2^c)^c$.

Proposition 3.6. *For PicFMs* $y1 = (\langle y1^t_{xy}, y1^n_{xy}, y1^f_{xy} \rangle)$, $y2 = (\langle y2^t_{xy}, y2^n_{xy}, y2^f_{xy} \rangle) \in \mathcal{P}_{x \times y}$, *then* $\lozenge y1 \odot \lozenge y2 = (y1^c \oplus y2^c)^c$.

Proof. $\lozenge y1 = (\langle 1 - y1^f_{xy}, y1^n_{xy}, y1^f_{xy} \rangle)$ and $\lozenge y2 = (\langle 1 - y2^f_{xy}, y2^n_{xy}, y2^f_{xy} \rangle)$

$$\lozenge y1 \odot \lozenge y2 = (\langle (1 - y1^f_{xy}) \wedge (1 - y2^f_{xy}), y1^n_{xy} \wedge y2^n_{xy}, y1^f_{xy} \vee y2^f_{xy} \rangle) \tag{3.15}$$

Now, $(y1^c \oplus y2^c) = (\langle y1^f_{xy} \vee y2^f_{xy}, y1^n_{xy} \wedge y2^n_{xy}, y1^t_{xy} \wedge y2^t_{xy} \rangle)$

$$(y1^c \oplus y2^c)^c = (\langle y1^t_{xy} \wedge y2^t_{xy}, y1^n_{xy} \wedge y2^n_{xy}, y1^f_{xy} \vee y2^f_{xy} \rangle)$$

$$\lozenge (y1^c \oplus y2^c)^c = (\langle 1 - (y1^f_{xy} \vee y2^f_{xy}), y1^n_{xy} \wedge y2^n_{xy}, y1^f_{xy} \vee y2^f_{xy} \rangle) \tag{3.16}$$

Claim: $(1 - y1^f_{xy}) \wedge (1 - y2^f_{xy}) = 1 - (y1^f_{xy} \vee y2^f_{xy})$ (3.17)

Case (i) If $y1 \geq y2$ then $y1^t_{xy} \geq y2^t_{xy}$, $y1^n_{xy} \geq y2^n_{xy}$ and $y1^f_{xy} \leq y2^f_{xy}$

LHS of equation (3.17), $y1^f_{xy} \leq y2^f_{xy}$ then $(1 - y1^f_{xy}) \geq (1 - y2^f_{xy})$

$$(1 - y1^f_{xy}) \wedge (1 - y2^f_{xy}) = 1 - y2^f_{xy} \tag{3.18}$$

RHS of equation (3.17), $1 - (y1^f_{xy} \vee y2^f_{xy}) = 1 - y2^n_{xy}$. (3.19)

From equations (3.18) and (3.19), we get, LHS = RHS.

Case (ii) If $y1 < y2$ then $y1^t_{xy} < y2^t_{xy}$, $y1^n_{xy} < y2^n_{xy}$ and $y1^f_{xy} > y2^f_{xy}$

LHS of equation (3.17), $y1^f_{xy} > y2^f_{xy}$ then $(1 - y1^f_{xy}) < (1 - y2^f_{xy})$

$$(1 - y1^f_{xy}) \wedge (1 - y2^f_{xy}) = 1 - y1^f_{xy} \tag{3.20}$$

RHS of equation (3.17), $1-(y1_{xy}^{f} \vee y2_{xy}^{f}) = 1 - y1_{xy}^{f}$ (3.21)

From equations (3.20) and (3.21), we get, LHS = RHS.

From Case (i) and Case (ii) we get, $\Diamond y1 \odot \Diamond y2 = \Diamond(y1^{c} \oplus y2^{c})^{c}$.

Proposition 3.7. *For PicFMts* $y1 = (\langle y1_{xy}^{t}, y1_{xy}^{n}, y1_{xy}^{f}\rangle)$, $y2 = (\langle y2_{xy}^{t}, y2_{xy}^{n}, y2_{xy}^{f}\rangle) \in \mathcal{P}_{x\times y}$, *then* $\Diamond y1 \oplus \Diamond y2 = (y1^{c} \odot y2^{c})^{c}$.

Proof. $\Diamond y1 = (\langle 1 - y1_{xy}^{f}, y1_{xy}^{n}, y1_{xy}^{f}\rangle)$ and $\Diamond y2 = (\langle 1 - y2_{xy}^{f}, y2_{xy}^{n}, y2_{xy}^{f}\rangle)$

$$\Diamond y1 \oplus \Diamond y2 = (\langle (1 - y1_{xy}^{f}) \vee (1 - y2_{xy}^{f}), y1_{xy}^{n} \wedge y2_{xy}^{n}, y1_{xy}^{f} \vee y2_{xy}^{f}\rangle) \quad (3.21)$$

Now, $(y1^{c} \odot y2^{c}) = (\langle y1_{xy}^{f} \wedge y2_{xy}^{f}, y1_{xy}^{n} \wedge y2_{xy}^{n}, y1_{xy}^{t} \vee y2_{xy}^{t}\rangle)$

$$(y1^{c} \oplus y2^{c})^{c} = (\langle y1_{xy}^{t} \vee y2_{xy}^{t}, y1_{xy}^{n} \wedge y2_{xy}^{n}, y1_{xy}^{f} \wedge y2_{xy}^{f}\rangle)$$

$$\Diamond(y1^{c} \odot y2^{c})^{c} = (\langle 1 - (y1_{xy}^{f} \wedge y2_{xy}^{f}), y1_{xy}^{n} \wedge y2_{xy}^{n}, y1_{xy}^{f} \wedge y2_{xy}^{f}\rangle) \quad (3.22)$$

Claim. $(1 - y1_{xy}^{f}) \vee (1 - y2_{xy}^{f}) = 1 - (y1_{xy}^{f} \wedge y2_{xy}^{f})$ (3.23)

Case (i) If $y1 \geq y2$ then $y1_{xy}^{t} \geq y2_{xy}^{t}$, $y1_{xy}^{n} \geq y2_{xy}^{n}$ and $y1_{xy}^{f} \leq y2_{xy}^{f}$

LHS of equation (3.23), $y1_{xy}^{f} \leq y2_{xy}^{f}$ then $(1 - y1_{xy}^{f}) \geq (1 - y2_{xy}^{f})$

$$(1 - y1_{xy}^{f}) \vee (1 - y2_{xy}^{f}) = 1 - y2_{xy}^{f} \quad (3.24)$$

RHS of equation (3.23), $1 - (y1_{xy}^{f} \wedge y2_{xy}^{f}) = 1 - y2_{xy}^{f}$ (3.25)

From equations (3.24) and (3.25), we get, LHS = RHS.

Case (ii) If $y1 < y2$ then $y1_{xy}^{t} < y2_{xy}^{t}$, $y1_{xy}^{n} < y2_{xy}^{n}$ and $y1_{xy}^{f} > y2_{xy}^{f}$

LHS of equation (3.23), $y1_{xy}^{f} > y2_{xy}^{f}$ then $(1 - y1_{xy}^{f}) < (1 - y2_{xy}^{f})$

$$(1 - y1_{xy}^{f}) \vee (1 - y2_{xy}^{f}) = 1 - y2_{xy}^{f} \quad (3.26)$$

RHS of equation (3.23), $1 - (y1_{xy}^{f} \wedge y2_{xy}^{f}) = 1 - y2_{xy}^{f}$ (3.27)

From equations (3.26) and (3.27), we get, LHS = RHS.

From Case (i) and Case (ii) we get, $\Diamond y1 \oplus \Diamond y2 = \Diamond(A^c \odot y2^c)^c$.

Proposition 3.8. *For PicFMs* $y1 = (\langle y1^t_{xy}, y1^n_{xy}, y1^f_{xy}\rangle)$, $y2 = (\langle y2^t_{xy}, y2^n_{xy}, y2^f_{xy}\rangle) \in \mathcal{P}_{x\times y}$, *then* $\Box y1 \oplus \Box y2 = \Box(y1 \oplus y2)$.

Proof. $\Box y1 \oplus \Box y2 = (\langle y1^t_{xy} \vee y2^t_{xy}, y1^n_{xy} \wedge y2^n_{xy}, (1 - y1^t_{xy}) \wedge (1 - y1^t_{xy})\rangle)$

$$\tag{3.28}$$

Now, $(y1 \oplus y2) = (\langle y1^t_{xy} \vee y2^t_{xy}, y1^n_{xy} \wedge y2^n_{xy}, y1^f_{xy} \wedge y2^f_{xy}\rangle)$

$$\Box(y1 \oplus y2) = (\langle y1^t_{xy} \vee y2^t_{xy}, y1^n_{xy} \wedge y2^n_{xy}, 1 - (y1^t_{xy} \vee y2^t_{xy})\rangle) \tag{3.29}$$

Claim. $(1 - y1^t_{xy}) \wedge (1 - y2^t_{xy}) = 1 - (y1^t_{xy} \vee y2^t_{xy})$ (3.30)

From equation (3.10), we get, $\Box y1 \oplus \Box y2 = \Box(y1 \oplus y2)$.

Proposition 3.9. *For PicFMts* $y1 = (\langle y1^t_{xy}, y1^n_{xy}, y1^f_{xy}\rangle)$, $y2 = (\langle y2^t_{xy}, y2^n_{xy}, y2^f_{xy}\rangle) \in \mathcal{P}_{x\times y}$, *then* $\Diamond y1 \oplus \Diamond y2 = \Diamond(y1 \oplus y2)$.

Proof. $\Diamond y1 = (\langle 1 - y1^t_{xy}, y1^n_{xy}, y1^f_{xy}\rangle)$ and $\Diamond y2 = (\langle 1 - y2^f_{xy}, y2^n_{xy}, y2^f_{xy}\rangle)$

$$\Diamond y1 \oplus \Diamond y2 = (\langle (1 - y1^f_{xy}) \vee (1 - y1^f_{xy}), y1^n_{xy} \wedge y2^n_{xy}, y1^f_{xy} \wedge y2^f_{xy}\rangle) \tag{3.31}$$

Now, $(y1 \oplus y2) = (\langle y1^t_{xy} \vee y2^t_{xy}, y1^n_{xy} \wedge y2^n_{xy}, y1^f_{xy} \wedge y2^f_{xy}\rangle)$

$$\Diamond(y1 \oplus y2) = (\langle 1 - (y1^f_{xy} \wedge y2^f_{xy}), y1^n_{xy} \wedge y2^n_{xy}, y1^f_{xy} \wedge y2^f_{xy}\rangle) \tag{3.32}$$

Claim. $(1 - y1^f_{xy}) \vee (1 - y2^f_{xy}) = 1 - (y1^f_{xy} \wedge y2^f_{xy})$ (3.33)

From equation (3.23), we get, $\Diamond y1 \oplus \Diamond y2 = \Diamond(y1 \oplus y2)$.

Proposition 3.10. *For PicFMs* $y1 = (\langle y1^t_{xy}, y1^n_{xy}, y1^f_{xy}\rangle)$, $y2 = (\langle y2^t_{xy}, y2^n_{xy}, y2^f_{xy}\rangle) \in \mathcal{P}_{x\times y}$, *then* $\Box y1 \odot \Box y2 = \Box(y1 \odot y2)$.

Proof. $\Box y1 \odot \Box y2 = (\langle y1^t_{xy} \wedge y2^t_{xy}, y1^n_{xy} \wedge y2^n_{xy}, (1 - y1^t_{xy}) \vee (1 - y1^t_{xy})\rangle)$

$$\tag{3.34}$$

Now, $(y1 \odot y2) = (\langle y1_{xy}^{t} \wedge y2_{xy}^{t},\ y1_{xy}^{n} \wedge y2_{xy}^{n},\ y1_{xy}^{f} \vee y2_{xy}^{f}\rangle)$

$$\Box(y1 \odot y2) = (\langle y1_{xy}^{t} \wedge y2_{xy}^{t},\ y1_{xy}^{n} \wedge y2_{xy}^{n},\ 1-(y1_{xy}^{t} \wedge y2_{xy}^{t})\rangle) \quad (3.35)$$

Claim: $(1-y1_{xy}^{t}) \vee (1-y2_{xy}^{t}) = 1-(y1_{xy}^{t} \wedge y2_{xy}^{t})$ (3.36)

From equation (3.3), we get, $\Box y1 \oplus \Box y2 = \Box(y1 \oplus y2)$.

Proposition 3.11. *For PicFMs* $y1 = (\langle y1_{xy}^{t},\ y1_{xy}^{n},\ y1_{xy}^{f}\rangle)$, $y2 = (\langle y2_{xy}^{t},\ y2_{xy}^{n},\ y2_{xy}^{f}\rangle) \in \mathcal{P}_{x\times y}$, *then* $\Diamond y1 \odot \Diamond y2 = \Diamond(y1 \odot y2)$.

Proof. $\Diamond y1 = (\langle 1-y1_{xy}^{f},\ y1_{xy}^{n},\ y1_{xy}^{f}\rangle)$ and $\Diamond B = (\langle 1-y2_{xy}^{f},\ y2_{xy}^{n},\ y2_{xy}^{f}\rangle)$

$$\Diamond y1 \odot \Diamond y2 = (\langle (1-y1_{xy}^{f}) \wedge (1-y1_{xy}^{f}),\ y1_{xy}^{n} \wedge y2_{xy}^{n},\ y1_{xy}^{f} \vee y2_{xy}^{f}\rangle) \quad (3.37)$$

Now, $(y1 \odot y2) = (\langle y1_{xy}^{t} \wedge y2_{xy}^{t},\ y1_{xy}^{n} \wedge y2_{xy}^{n},\ y1_{xy}^{f} \vee y2_{xy}^{f}\rangle)$

$$\Diamond(y1 \odot y2) = (\langle 1-(y1_{xy}^{f} \vee y2_{xy}^{f}),\ y1_{xy}^{n} \wedge y2_{xy}^{n},\ y1_{xy}^{f} \vee y2_{xy}^{f}\rangle) \quad (3.38)$$

Claim: $(1-y1_{xy}^{f}) \wedge (1-y2_{xy}^{f}) = 1-(y1_{xy}^{f} \vee y2_{xy}^{f})$ (3.39)

From equation (3.17), we get, $\Diamond y1 \odot \Diamond y2 = \Diamond(y1 \odot y2)$.

Similarly we can prove the following Propositions.

Proposition 3.12. *For PicFMts* $y1 = (\langle y1_{xy}^{t},\ y1_{xy}^{n},\ y1_{xy}^{f}\rangle)$, $y2 = (\langle y2_{xy}^{t},\ y2_{xy}^{n},\ y2_{xy}^{f}\rangle)$ *and* $y3 = (\langle y3_{xy}^{t},\ y3_{xy}^{n},\ y3_{xy}^{f}\rangle) \in \mathcal{P}_{x\times y}$ *then*, $\Box((y1 \oplus y2) \odot y3) = (\Box y1 \oplus \Box y2) \odot \Box y3$.

Proposition 3.13. *For PicFMs* $y1 = (\langle y1_{xy}^{t},\ y1_{xy}^{n},\ y1_{xy}^{f}\rangle)$, $y2 = (\langle y2_{xy}^{t},\ y2_{xy}^{n},\ y2_{xy}^{f}\rangle)$ *and* $y3 = (\langle y3_{xy}^{t},\ y3_{xy}^{n},\ y3_{xy}^{f}\rangle) \in \mathcal{P}_{x\times y}$ *then*, $\Box((y1 \odot y2) \oplus y3) = (\Box y1 \odot \Box y2) \oplus \Box y3$.

Proposition 3.14. *For PicFMs* $y1 = (\langle y1_{xy}^{t},\ y1_{xy}^{n},\ y1_{xy}^{f}\rangle)$, $y2 = (\langle y2_{xy}^{t},\ y2_{xy}^{n},\ y2_{xy}^{f}\rangle)$ *and* $y3 = (\langle y3_{xy}^{t},\ y3_{xy}^{n},\ y3_{xy}^{f}\rangle) \in \mathcal{P}_{x\times y}$ *then*, $\Diamond((y1 \oplus y2) \odot y3) = (\Diamond y1 \oplus \Diamond y2) \odot \Diamond y3$.

Proposition 3.15. *For PicFMs* $y1 = (\langle y1^t_{xy},\ y1^n_{xy},\ y1^f_{xy}\rangle)$, $y2 = (\langle y2^t_{xy},\ y2^n_{xy},\ y2^f_{xy}\rangle)$ *and* $y3 = (\langle y3^t_{xy},\ y3^n_{xy},\ y3^f_{xy}\rangle) \in \mathcal{P}_{x\times y}$ *then,* $\Diamond((y1 \odot y2) \oplus y3) = (\Diamond y1 \odot \Diamond y2) \oplus \Diamond y3$.

4. Results using Max-Max-Min product

Proposition 4.1. *For PicFMs* $y1 = (\langle y1^t_{xy},\ y1^n_{xy},\ y1^f_{xy}\rangle)$, $y2 = (\langle y2^t_{xy},\ y2^n_{xy},\ y2^f_{xy}\rangle) \in \mathcal{P}_{x\times y}$ *then,* $\Box(y1y2) = \Box y1 \Box y2$.

Proof. $y1y2 = (\langle \sum_k y1^t_{ik} y2^t_{kj},\ \sum_k y1^n_{xk} y2^n_{ky},\ \prod_k (y1^f_{xk} + y2^f_{ky})\rangle)$

$$\Box(y1y2) = (\langle \sum_k y1^t_{xk} y2^t_{ky},\ \sum_k y1^n_{xk} y2^n_{ky},\ 1 - (\sum_k y1^t_{xk} y2^t_{ky})\rangle) \tag{4.1}$$

$$\Box y1 \Box y2 = (\langle \sum_k y1^t_{xk} y2^t_{xk},\ \sum_k y1^n_{xk} y2^n_{ky},\ \prod_k ((1 - y1^t_{xk}) + (1 - y2^t_{ky}))\rangle) \tag{4.2}$$

Claim 1. $1 - (\sum_k y1^t_{xk} y2^t_{ky}) = \prod_k ((1 - y1^t_{xk}) + (1 - y2^t_{ky}))$ (4.3)

Set $\sum_k y1^t_{xk} y2^t_{ky} = 1 - y1^t_{xl}$ for some l, then $y1^t_{xl} < y2^t_{ly}$.

$1 - y1^t_{xl} > 1 - y2^t_{ly}$

LHS of equation (4.3), $1 - (\sum_k y1^t_{xk} y2^t_{ky}) = 1 - y1^t_{xl}$ (4.4)

Now RHS of equation (4.3),

$$\prod_k ((1 - y1^t_{xk}) + (1 - y2^t_{xk})) = (1 - y1^t_{xk}) + (1 - y2^t_{xk}) \prod_{k \neq l} ((1 - y1^t_{xk}) + (1 - y2^t_{xk})) = (1 - y1^t_{xk}) \prod_{k \neq l} ((1 - y1^t_{xk}) + (1 - y2^t_{xk})) \tag{4.5}$$

Case (i). $y1^t_{xk} > y1^t_{xl}$ and $y2^t_{xk} < y1^t_{xl}$

$1 - y1^t_{xk} < 1 - y1^t_{xl}$ and $1 - y2^t_{xk} < 1 - y1^t_{xl}$

$1 - y2^t_{xk} > 1 - y1^t_{xl}$

From equation (4.5),

$$\prod_k ((1 - y1^t_{xk})(1 - y2^t_{ky})) = (1 - y1^t_{xl}) + (1 - y2^t_{ky}) = 1 - y1^t_{xl}. \qquad (4.6)$$

From equation (4.4) and equation (4.6), equation (4.3) holds good.

Case (ii). $y1^t_{xk} < y1^t_{xl}$ and $y2^t_{xk} > y1^t_{xl}$

$1 - y1^t_{xk} > 1 - y1^t_{xl}$ and $1 - y2^t_{xk} < 1 - y1^t_{xl}$

$1 - y2^t_{xk} > 1 - y1^t_{xl}$.

From equation (4.5),

$$\prod_k ((1 - y1^t_{xk})(1 - y2^t_{ky})) = (1 - y1^t_{xl}) + (1 - y2^t_{ky}) = 1 - y1^t_{xl} \qquad (4.7)$$

From equation (4.4) and equation (4.7), equation (4.3) holds good.

Similarly we can prove the other cases.

Proposition 4.2. *For PicFMs* $y1 = (\langle y1^t_{xy}, y1^n_{xy}, y1^f_{xy}\rangle)$, $B = (\langle y2^t_{xy}, y2^n_{xy}, y2^f_{xy}\rangle) \in \mathcal{P}_{x\times y}$ *then*, $\Diamond(y1y2) = \Diamond y1 \Diamond y2$.

Proof. $y1y2 = (\langle \sum_k y1^t_{xk}y2^t_{ky}, \sum_k y1^n_{xk}y2^n_{ky}, \prod_k (y1^f_{xk} + y2^f_{ky})\rangle)$

$$\Diamond(y1y2) = (\langle 1 - \prod_k (y1^f_{xk} + y2^f_{ky}), \sum_k y1^n_{xk}y2^n_{ky}, \prod_k (y1^f_{xk} + y2^f_{ky})\rangle) \quad (4.8)$$

$$\Diamond y1 \Diamond y2 = (\langle \sum_k (1 - y1^f_{xk})(1 - y2^f_{xk}), \sum_k y1^n_{xk}y2^n_{ky}, \prod_k (y1^f_{xk} + y2^f_{ky})\rangle) \quad (4.9)$$

Claim 1. $1 - \prod_k (y1^f_{xk} + y2f^f_{ky}) = \sum_k (1 - y1^f_{xk})(1 - y2^f_{ky}) \qquad (4.10)$

Set $\prod_k (y1^f_{xk} + y2f^f_{ky}) = y1^f_{xl}$ for some l;

LHS of equation (4.10), $1 - \prod_k (y1^f_{xk} + y2f^f_{ky}) = 1 - y1^f_{xl} \qquad (4.11)$

We have $y1_{xl}^{f} \geq y2_{ly}^{f}$ then $1 - y1_{xl}^{f} \leq 1 - y2_{ly}^{f}$

Now RHS of equation (4.10),

$$\sum_{k}(1 - y1_{xk}^{f})(1 - y2_{xk}^{f}) = (1 - y1_{xk}^{f}) \cdot (1 - y2_{xk}^{f}) + \sum_{k \neq l}(1 - y1_{xk}^{f})(1 - y2_{xk}^{f})$$

$$= (1 - y1_{xk}^{f}) + \sum_{k \neq l}(1 - y1_{xk}^{f})(1 - y2_{xk}^{f}) \tag{4.12}$$

Case (i) $y1_{xk}^{f} > y1_{xl}^{f}$ and $y2_{xk}^{f} < y1_{xl}^{f}$

$1 - y1_{xk}^{f} < 1 - y1_{xl}^{f}$ and $1 - y2_{xk}^{f} > 1 - y1_{xl}^{f}$

$1 - y2_{xk}^{f} > 1 - y1_{xl}^{f}$

From equation (4.12),

$$\sum_{k}((1 - y1_{xk}^{f})(1 - y2_{ky}^{f})) = (1 - y1_{xl}^{f}) + (1 - y2_{ky}^{f}) = 1 - y1_{xl}^{f} \tag{4.13}$$

From equation (4.11) and equation (4.13), equation (4.10) holds good.

Case (ii) $y1_{xk}^{f} < y1_{xl}^{f}$ and $y2_{xk}^{f} > y1_{xl}^{f}$

$1 - y1_{xk}^{f} > 1 - y1_{xl}^{f}$ and $1 - y2_{xk}^{f} < 1 - y1_{xl}^{f}$

$1 - y2_{ky}^{f} > 1 - y1_{xk}^{f}$.

From equation (4.12),

$$\sum_{k}((1 - y1_{xk}^{f})(1 - y2_{ky}^{f})) = (1 - y1_{xl}^{f}) + (1 - y2_{ky}^{f}) = 1 - y1_{xl}^{f} \tag{4.14}$$

From equation (4.11) and equation (4.14), equation (4.10) holds good.

Proposition 4.3. *For PicFMs* $y1 = (\langle y1_{xy}^{t}, y1_{xy}^{n}, y1_{xy}^{f}\rangle)$, $y2 = (\langle y2_{xy}^{t}, y2_{xy}^{n}, y2_{xy}^{f}\rangle)$ *and* $y3 = (\langle y3_{xy}^{t}, y3_{xy}^{n}, y3_{xy}^{f}\rangle) \in \mathcal{P}_{x \times y}$ *then*, $\Box((y1 + y2)y3) = \Box(y1y3 + y2y3)$.

Proof. $(y1 + y2)y3 = (\langle \sum_{k}(y1_{xk}^{t} + y2_{xk}^{t}), y3_{ky}^{t}, \sum_{k}(y1_{xk}^{n} + y2_{xk}^{n}), y3_{ky}^{n},$

$\prod_k (y1^f_{xk} y2^f_{xk}) + y3^f_{ky}\rangle)$

$= (\langle \sum_k (y1_{xk} t y3_{ky} t + y2^t_{xk} y3^t_{ky}), \sum_k (y1^n_{xk} y3^n_{ky} + y2^n_{xk} y3^n_{ky}),$

$\prod_k (y1^f_{xk} + y3^f_{ky})(y2^f_{xk} + y3^f_{ky})\rangle) \;\; \square((y1 + y2)y3)$

$= (\langle \sum_k (y1^t_{xk} y3^t_{ky} + y2^t_{xk} y3^t_{ky}), \sum_k (y1^n_{xk} y3^n_{ky} + y2^n_{xk} y3^n_{ky}),$

$$1 - \sum_k (y1^t_{xk} y3^t_{ky} + y2^t_{xk} y3^t_{ky})\rangle). \tag{4.15}$$

Now, $y1y3 + y2y3$

$= (\langle \sum_k y1^t_{xk} y3^t_{ky} + \sum_k y2^t_{xk} y3^t_{ky}, \sum_k y1^n_{xk} y3^n_{ky} + \sum_k y2^n_{xk} y3^n_{ky},$

$\prod_k (y1^f_{xk} + y3^f_{ky}) \prod_k (y2^f_{xk} + y3^f_{ky})\rangle)$

$= (\langle \sum_k (y1^t_{xk} y3^t_{ky} + y2^t_{xk} y3^t_{ky}), \sum_k (y1^n_{xk} y3^n_{ky} + y2^n_{xk} y3^n_{ky}),$

$\prod_k (y1^f_{xk} + y3^f_{ky})(y2^f_{xk} + y3^f_{ky})\rangle) \;\; \square((y1 + y2)y3)$

$= (\langle \sum_k (y1^t_{xk} y3^t_{ky} + y2^t_{xk} y3^t_{ky}), \sum_k (y1^n_{xk} y3^n_{ky} + y2^n_{xk} y3^n_{ky}),$

$$1 - \sum_k (y1^t_{xk} y3^t_{ky} + y2^t_{xk} y3^t_{ky}\rangle). \tag{4.16}$$

From equation (4.15) and equation (4.16) we get, $\square((y1 + y2)y3) = \square(y1y3 + y2y3)$.

Similarly we can prove the following Propositions.

Proposition 4.4. *For PicFMts* $y1 = (\langle y1^t_{xy}, y1^n_{xy}, y1^f_{xy}\rangle)$, $y2 = (\langle y2^t_{xy}, y2^n_{xy}, y2^f_{xy}\rangle)$ *and* $y3 = (\langle y3^t_{xy}, y3^n_{xy}, y3^f_{xy}\rangle) \in \mathcal{P}_{x\times y}$ *then,* $\Diamond((y1 + y2)y3) = \Diamond(y1y3 + y2y3)$.

Proposition 4.5. *For PicFMs* $y1 = (\langle y1^t_{xy}, y1^n_{xy}, y1^f_{xy}\rangle)$,

$y2 = (\langle y2^t_{xy},\ y2^n_{xy},\ y2^f_{xy}\rangle)$ *and* $y3 = (\langle y3^t_{xy},\ y3^n_{xy},\ y3^f_{xy}\rangle) \in \mathcal{P}_{x\times y}$ *then*, $\Box(y1y2)\Box y3 = \Box y1\Box(y2y3)$.

Proposition 4.6. *For PicFMs* $y1 = (\langle y1^t_{xy},\ y1^n_{xy},\ y1^f_{xy}\rangle)$, $y2 = (\langle y2^t_{xy},\ y2^n_{xy},\ y2^f_{xy}\rangle)$ *and* $y3 = (\langle y3^t_{xy},\ y3^n_{xy},\ y3^f_{xy}\rangle) \in \mathcal{P}_{x\times y}$ *then*, $\Diamond(y1y2)\Diamond y3 = \Diamond y1\Diamond(y2y3)$.

5. Conclusion

In this paper, we have defied the Model Operators on Picture fuzzy matrices and discussed some results related to these operators. Further it is proved that necessity and possibility operator is distributive over addition.

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