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S-TOPOLOGICAL BE-ALGEBRAS

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Abstract

In this paper, we introduce the concept of S-topological BE-algebras (STBE-algebras) as a generalization of the concept of topological BE-algebra by using the concepts of semi-open sets and prove some of its properties. We also introduce the topological concepts, open sets, closed sets, interior and closure in STBE-algebras and arrive at the topological properties of STBE-algebras.

1. Introduction

In [4] H. S. Kim and Y. H. Kim introduced the notion of BE-algebras, which is a generalization of BCK-algebras. They also introduced the notion of commutative BE-algebras and studied their properties and characterization. In [6], S. Mehrshad and J. Golzarpoor studied the topological BE-algebras and discussed their properties. In [7], M. Jansi and V. Thiruveni introduced the notion of ideals in TSBF-algebras. In this paper, we introduce the notion of S-topological BE-algebras. In this way, we study the connection between BE-algebras and topology and discuss some of its properties. The concept of S-topological BE-algebras is a generalization of the concept of topological BE-algebras. An S-topological BE-algebra is a BE-algebra X together with a special type of topology that makes the operation defined on X, S-topological continuous.

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2. Preliminaries

Definition 2.1 [4]**.** A BE-algebra $(X, *, 1)$ of type (2, 0) (A non-empty set together with a binary operation * and a constant 1) satisfying the following conditions.

1. $x * x = 1$
2. $x * 1 = 1$
3. $1 * x = x$
4. $x * (y * z) = y * (x * z), \forall\ x,\ y,\ z \in X.$

Definition 2.2 [11]**.** A non-empty subset A of a BE-algebra X is called a sub-algebra of X if $x * y \in A, \forall\ x,\ y \in A.$

Definition 2.3 [11]**.** A non-empty subset I of a BE-algebra X is called an ideal of X if

1. $\forall\ x \in X$ and $\forall\ a \in I \Rightarrow x * a \in I$
2. $\forall\ x \in X$ and $\forall\ a, b \in I \Rightarrow (a * (b * x)) * x \in I$

Definition 2.4 [5]**.**

(i) A subset A of a topological space is said to be semi-open if $A \subseteq \overline{Int\ A}.$

(ii) The complement of semi-open set is called semi-closed.

(iii) The semi-closure of a subset A of a topological space is the intersection of all semi-closed set containing A. It is denoted by $\overline{A}^s$.

(iv) A subset A of a topological space is said to be regular-open if $A = \overline{Int\ A}.$

Definition 2.5 [10]**.** A topological space (X, τ) is called semi-T_1 if for each two distinct points $x, y \in X$ there exists two semi-open sets U and V such that U contains x but not y and V contains y but not x.

Definition 2.6 [10]**.** A topological space (X, τ) is called semi-T_2 if for each two distinct points $x, y \in X$ there exists two disjoint semi-open sets U and V such that $x \in U$ and $y \in V$.

Definition 2.7 [6]**.** Let $(X, *, 1)$ be a BE-algebra and $F \subseteq X$. The F is a filter when it satisfies the following conditions.

1. $1 \in X$.

2. If $1 \neq x \in F$ and $x * y \in F$, then $y \in F$.

3. S-Topological BE-algebras

Definition 3.1. A BE-algebra $(X, *)$ together with a topology τ is called a S-topological BE-algebra (STBE-algebra) if the function $f : X \times X \to X$ given by $f(x, y) = x * y$, which satisfies the condition that for each open set M containing $x * y$, there exists an open set U containing x and a semi-open set V containing y such that $U * V \subseteq M$, $\forall\ x, y \in X$.

Example 3.2. Consider a BE-algebra $X = \{1, a, b, c, d\}$ with the following Cayley table.

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Consider $\tau_s = \{X, \phi, \{a, c, d\}, \{b, d\}\}$. Clearly, * is S-topologically continuous. Then $(X, *, \tau_s)$ is a STBE-algebra.

Theorem 3.3. *Let* $(X, *, \tau_s)$ *be a STBE-algebra*, $A \subseteq X$ *and* $x \in X$. *Then the following conditions hold*.

1. $\overline{A} * x \subseteq \overline{A * x}$

2. If $\overline{A} * x$ is closed, then $\overline{A} * x = \overline{A * x}$.

Proof. 1. Let $y \in \overline{A} * x$ and M be any open set containing y. Then $y = a * x$, $a \in \overline{A}$.

Since X is a STBE-algebra, there exists an open set U of a and a semi-open set V of x such $U * V \subseteq M$.

Since $a \in \overline{A}$, every open set of a intersects A. So, there is an element b such that $b \in A \cap U$.

Therefore, $b * x \in A * x$ and $b * x \in U * x \subseteq U * V \subseteq M$.

That is every open set M of y contains an element $b * x$ from $A * x$.

Hence $y \in \overline{A * x}$. So, $\overline{A} * x \subseteq \overline{A * x}$.

2. Let $\overline{A} * x$ is closed. Let $y \in \overline{A * x}$.

Suppose $y \notin \overline{A} * x$. Then $y \in (\overline{A} * x)^c$.

Since $\overline{A} * x$ is closed, $(\overline{A} * x)^c$ is open. Clearly, $A * x \subseteq (\overline{A} * x)^c$. So, $A * x \cap (\overline{A} * x)^c = \phi$.

Now, since $y \in \overline{A * x}$, every open set of y intersects $A * x$. So $A * x \cap (\overline{A} * x)^c \neq \phi$, which is a contradiction. Therefore, $y \in \overline{A} * x$, which implies $\overline{A * x} \subseteq \overline{A} * x$.

From 1 it follows that $\overline{A} * x = \overline{A * x}$.

Theorem 3.4. *Let $(X, *, \tau_s)$ be a STBE-algebra, $A \subseteq X$ and $x \in X$. Then the following conditions hold.*

1. $x * \overline{A}^s \subseteq \overline{x * A}$

2. *If $x * \overline{A}^s$ is closed, then $* \overline{A}^s = \overline{x * A}$.*

Proof. Let $y \in x * \overline{A}^s$ and M be any open set containing y. Then $y = x * a$, $a \in \overline{A}^s$.

Since X is a STBE-algebra, there exists an open set U of x and a semi-open set V of a such that $U * V \subseteq M$.

Since $a \in \overline{A}^s$ every semi-open set of a intersects A. So, there is an element b such that $b \in A \cap V$.

Therefore, $x * b \in x * A$ and $x * b \in x * V \subseteq U * V \subseteq M$.

That is every open set M of y contains an element $x * b$ from $x * A$.

Hence $y \in \overline{x * A}$. So, $x * \overline{A}^s \subseteq \overline{x * A}$.

2. Let $x * \overline{A}^s$ is closed. Let $y \in \overline{x * A}$.

Suppose $y \notin x * \overline{A}^s$. Then $y \in (x * \overline{A}^s)^c$.

Since $x * \overline{A}^s$ is closed, $(x * \overline{A}^s)^c$ is open. Clearly, $x * A \subseteq (x * \overline{A}^s)^c$. So, $x * A \bigcap (x * \overline{A}^s)^c = \phi$.

Now, since $y \in \overline{x * A}$, every open set of y intersects $x * A$. So $x * A \bigcap (x * \overline{A}^s)^c \neq \phi$, which is a contradiction. Therefore, $y \notin x * \overline{A}^s$, which implies $\overline{x * A} \subseteq x * \overline{A}^s$.

From 1 it follows that $\overline{x * A} = x * \overline{A}^s$

With similar argument, we can prove the following theorem.

Theorem 3.5. *Let* $(X, * \tau_s)$ *be a STBE-algebra and* $A, B \subseteq X$. *Then the following conditions hold.*

1. $\overline{A} * \overline{B} \subseteq \overline{A * B}$

2. *If* $A * \overline{B}^s \subseteq \overline{A * B}$ *is closed, then* $\overline{A} * \overline{B} = \overline{A * B}$.

Theorem 3.6. *In a commutative STBE-algebra,* $(X, *, \tau_s)$ *if* $\{1\}$ *is open, then* τ_s *is discrete.*

Proof. Let $x \in X$. Assume that, $\{1\}$ is open.

Since, $x * x = 1, \forall x \in X$ and X is a STBE-algebra, for every open set M of 1, there exists an open set U of x and a semi-open set V of x such $U * V \subseteq M$. In particular, $U * V \subseteq \{1\}$.

Let $G = U \bigcap V$. Now, $x \in G$ and G is semi-open.

Let $y \in G$ and $x \neq y$. Then $x * y = y * x = 1$, this implies that $x = y$ (since X is commutative).

Hence $G = \{x\}$ is open also.

Since x is arbitrary, $\{x\}$ is open for all $x \in X$. Hence τ_s is discrete.

Theorem 3.7. *Let* $(X, *, \tau_s)$ *be a STBE-algebra and* Y *be an open BE-subalgebra of* X, *then* Y *is also an STBE-algebra with respect to a subspace topology* τ_{Y_s}.

Proof. Let $x * y \in Y$. Since Y is a subalgebra of X, $x, y \in Y$.

Let M be an open set of $x * y$ in Y. Then $M = Y \cap U$, where U is an open set of $x * y$ in X.

Since X is a STBE-algebra, there exists open set W_1 of x and semi-open set W_2 of y such that

$W_1 * W_2 \subseteq U$.

Then $W_1 \cap Y$ is an open set of x in Y and $W_2 \cap Y$ is a semi-open set of y in Y.

Now, $(W_1 \cap Y) * (W_2 \cap Y) = (W_1 * W_2) \cap Y \subseteq U \cap Y = M$.

Hence, $(Y, *, \tau_{Y_s})$ is an STBE-algebra.

Definition 3.8. A non-empty subset A of a STBE-algebra $(X, *, \tau_s)$ is called an ideal of X if the following conditions are satisfied.

1. $1 \in A$.

2. For all $x \neq 1$ in X and for all $y \in A$, if $x * y \in A$ then $x \in A$.

Theorem 3.9. *Let* $(X, *, \tau_s)$ *be a STBE-algebra and* A *be an open ideal in* X, *then* A *is semi-closed and regular-open.*

Proof. Suppose $x \notin A$.

Since $x * x = 1$, for every open set A of 1, there exists open set M of x and a semi-open set N if x such that $M * N \subseteq A$.

Let $G = M \cap N$. Then $x \in G$ and G is semi-open. Also, $G * G \subseteq M * N \subseteq A$.

Since A is an ideal, we have $G \subseteq A$, which is a contradiction.

Thus, $G \subseteq X \setminus A$. Then A is semi-closed.

Since A is open, we have $A \subseteq Int(\overline{A})$. Therefore $A = Int(\overline{A})$. Hence A is regular open.

Theorem 3.10. *If A is an ideal of a STBE-algebra X and $1 \in Int(A)$, then A is open.*

Proof. Let $x \in A$. Since $1 \in Int(A)$, there is an open set U of 1 such that $U \subseteq A$.

Since, X is a STBE-algebra, for every open set U of 1, there exists an open set W_1 of x and a semiopen set W_2 of x such that $W_1 * W_2 \subseteq U$.

Therefore, $W_1 * W_2 \subseteq A$ and $W_1 * x \subseteq U$.

Let $y \in W_1 \cap (X \setminus A)$. Then $y * x \in A$. This implies $x \in A$ for $x \in A$ and A is an ideal.

This is not possible. Therefore, $W_1 \subseteq A$. Thus $x \in Int(A), \forall x \in A$. Hence A is open.

4. Properties of S-Topological BE-algebras

Definition 4.1. A S-topological BE-algebra $(X, *, \tau_s)$ is called semi-T_1 STBE-algebra if for each pair of distinct points $x, y \in X$ there exist two semi-open set U and V such that U contains x but does not contain y and V contains y but does not contain x.

Definition 4.2. A topological BE-algebra $(X, *, \tau_s)$ is called semi-T_2 STBE-algebra if for each pair of distinct points $x, y \in X$ there exist two disjoint semi-open set U and V such that $x \in U$; $y \notin U$ and $y \in V$; $x \notin V$.

Theorem 4.3. *Let $(X, *, \tau_s)$ be a BE-algebra.* If $\{1\} \in \tau$ *is closed, then X is semi-T_2 STBE-algebra.*

Proof. Let $\{1\}$ be closed and let $x, y \in X$ such that $x \neq y$. Then $x * y \neq 1$ or $y * x \neq 1$.

Without loss of generality, we assume that $y * x \neq 1$.

Then there exists an open set U containing y and a semi-open set V containing x such that $U * V \subseteq X \setminus \{1\}$. Hence X is semi-T_2 STBE-algebra.

Theorem 4.4. *Let $(X, *, \tau_s)$ be a T_0 STBE-algebra. Then X is semi T_1 STBE-algebra.*

Proof. Let $x, y \in X$ such that $x \neq y$. Then either $x * y \neq 1$ or $y * x \neq 1$ Assume that $x * y \neq 1$. Since X is T_0, there exists an open set U containing either one of $x * y \neq 1$ or 1 but not both.

Let us assume that $x * y \in U$ and $1 \notin U$.

Now, as $(X, *, \tau_s)$ is a semi S-topological BE-algebra, there exists an open set V containing x and a semi-open set W containing y such that $V * W \subseteq U$. These V and W are the required semi open sets containing x and y respectively.

If $1 \in U$ and $x * y \notin U$ then $x * x = 1 \in U$. Then there exists an open set V containing x and a semiopen set W containing x such that $V * W \subseteq U$ and $y * y = 1 \in U$.

Then there exists an open set V_1 containing y and a semi-open set W_1 containing y such that $V_1 * W_1 \subseteq U$.

Take $G = V \cap W$ and $H = V_1 \cap W_1$. Then G and H are two semi-open sets such that $x \in G$ and $y \in H$. Also, $y \notin G$ and $x \notin H$.

Hence $(X, *, \tau)$ is a semi-T_1 STBE-algebra.

Definition 4.5. Let $(X, *, \tau_s)$ be a STBE-algebra and $F \subseteq X$. Then F is a filter if it satisfies the following conditions.

1. $1 \in F$.

2. If $1 \neq x \in F$ and $x * y \in F$, then $y \in F$.

Example 4.6. Consider Example 3.2. Here * is S-topological continuous. Also, $\{1, b\}$ is a filter in X.

Theorem 4.7. *Let* $(X, *, \tau_s)$ *be a STBE-algebra and F be a filter in X. If* 1 *is an interior point of F, then F is semi-open.*

Proof. Suppose that 1 is an interior point of F. Then there exists M containing 1 such that $M \subseteq F$.

Let $x \in F$. Since $x * x = 1$, there exists an open set U containing x and a semi-open set V containing x such that $U * V \subseteq M \subseteq F$.

Now, for each y in the semi-open set V, we have $x * y \in F$.

Since F is a filter and $x \in F$, we have $y \in F$. Therefore, $x \in W \subseteq F$. Hence F is semi-open.

Theorem 4.8. *Let* $(X, *, \tau_s)$ *be a STBE-algebra and F be a filter in X. If F is open, then it is closed.*

Proof. Let F be a filter in X which is open. We show that $X \setminus F$ is also open.

Let $x \in X \setminus F$. Since F is open, 1 is an interior point of F.

Since $x * x = 1$, there exists open set V containing x and a semi-open set W containing x such that

$V * W \subseteq F$.

We claim that $V \subseteq X \setminus F$.

If V is not a subset of $X \setminus F$, then there exists an element $y \in V \bigcap F$.

Now, for each $z \in W$, we have $y * z \in V * W \subseteq F$. Since F is a filter and $y \in F$, we have $z \in F$.

Thus $W \subseteq F$, which implies $x \in F$, which is a contradiction.

Hence, $x \in V \subseteq X \setminus F \Rightarrow X/F$ is open. Therefore, F is closed.

Definition 4.9. Let $(X, *, \tau_s)$ be a STBE-algebra, U be a non-empty subset of X and $a \in X$.

The subsets Ua and aU are defined as follows.

$Ua = \{x \in X : x * a \in U\}$ and $aU = \{x \in X : a * x \in U\}$.

Also, if $K \subseteq X$ we get $KU = \bigcup_{a \in K} aU$ and $UK = \bigcup_{a \in K} Ua$.

Theorem 4.10. *Let* $(X, *, \tau_s)$ *be a STBE-algebra and U be two nonempty subsets of X. Then*

1. *If U is open, then* Ua *is open and* aU *is semi-open.*

2. *If U is closed, then* Ua *is closed and* aU *is semiclosed.*

Proof. 1. Let U be an open set, $a \in X$ and $x \in Ua$.

Then $x * a \in U$. Then there exists an open set W containing x and a semi-open set A containing a such that $W * A \subseteq U$, $x * a \in Wa \subseteq U$. Thus $W * a \subseteq U$. So, $x \in W \subseteq Ua$ is open.

To prove aU is semi-open, let $x \in aU \Rightarrow a * x \in U$.

Then there exists an open set A containing a and a semi-open set W_1 containing x such that

$A * W_1 \subseteq U$, $a * x \in aW_1 \subseteq U$. Thus $a * W_1 \subseteq U$. So, $x \in W_1 \in aU \Rightarrow aU$ is semiopen.

2. Let U be closed. Then U^c is open.

So, by 1, $U^c a$ is open and aU^c is semi-open.

Clearly, $(Ua)^c = U^c a$ and $(aU)^c = aU^c$. Hence $(aU)^c$ is open and $(aU)^c$ is semi-open.

$\Rightarrow Ua$ is closed and aU is semiclosed.

5. Conclusion

We have studied the properties of STBE-algebras using the topological concepts like open sets, semi-open sets, closed sets and filters. We can further study about various topologies induced by the filters.

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References

[1] Alias B. Khalaf and Fatima W. Ali, On S-topological BCK-algebra, Journal of University of Dohok 23(1) (2020), 199-208.

[2] Andrzej Walendziak, On commutative BE-algebras, Scientiae Mathematicae Japonicae (e-2008), 585-588.

[3] R. Englking, General Topology, PWN-Polish Scientific Publishers, Warsaw, 1977.

[4] H. S. Kim and Y. H. Kim, On BE-algebras, Scientiae Mathematicae Japonicae 66 (2007), 113-128.

[5] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963), 36-41.

[6] S. Mehrshad and J. Golzarpoor, On topological BE-algebras, Mathematica Moravica 21(2) (2017), 1-13.

[7] M. Jansi and V. Thiruveni, Complementry role of ideals in TSBF-algebras, Malaya Journal of Mathematik 8(3) (2020), 1037-1040.

[8] M. Jansi and V. Thiruveni, Topological structures on BF-algebras, International Journal of Innovative Research in Science Engineering and Technology 6 (2017), 22594-22600.

[9] Y. B. Jun and C. H. Park, Applications of soft sets in ideal theory of BCK/BCI algebras, Inform, Sci. 178 (2008), 2466-2475.

[10] S. N. Maheswari and R. Prasad, Some new separation axioms, Ann. Soc. Sci. Bruxelles 89 (1975), 395-402.

[11] A. Rezaei and A. Borumand Saeid, Some results in BE-algebras, Analele Univesitatii Oradea Fasc. Matematica, Tom XIX(1) (2012), 33-44.