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A SECOND TYPE OF HOLLING FUNCTIONAL RESPONSE OF STABILITY ANALYSIS FOR PREY-PREDATOR AND HOST ECOSYSTEM

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Abstract

A three-species ecosystem with a prey-predator interaction and a third species that serves as a predator's host is examined for stability. Species interact in biological system for the sake of survival and to meet their dietary needs. During the inter-competition encounter, the prey uses several protective techniques to flee from predator. Predator species are both commensal to the host and competitors to the prey. The model is Holling Type-II functional responses in nonlinear differential equations. The suggested system investigates all the existing equilibrium points of the three species, bionomic equilibrium, optimal prey harvesting, in addition to the mortality rates of commensal and host species. The system subsequently analyses the stability of coexistence both locally and globally in an optimistic equilibrium condition.

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1. Introduction

The ecological system consists of all the organisms and exposes the interaction between the many species in the habitat. Species interactions within biological webs involve four major types of two-way communication such as competition, commensalism, mutualism and predation for their physical space and nutrition diet. This paper mainly discussed the two contacts namely competition and commensalism. The interspecific competition occurs when members of diverse species struggle for shared resources. Commensalism is a symbiotic relationship in which members of one species obtain food, locomotion, refuge, etc., from the other (host) species neither wounded nor affected. Commensalism plays a crucial position in evolution while the interactions and adaptation accumulate over time.

Lotka and Volterra [1, 21] defined the prey-predator dynamic system as a form of ordinary differential equations for a steady population density variable in the limit of the enormous size of the population. Several ecologists and mathematicians are part in contributing various modelling concepts of the prey-predator system in the population ecology for the treatises of H. I. Freedman, J. D. Murray and J. N. Kapur, [9, 10, 11]. Holling (1959) proposed non linear functional responses of this type based on a general argument about the allocation of a predator's time between two actions such as prey finding and prey handling. The functional response is the rate at which a predator consumes prey effectively attacked as a function of prey density. It depicts how a predator reacts to changes in its prey's density. Holling and Hassell (1978) classified functional response into three categories. Type-I has a linear relationship between prey population and the largest number of prey killed, whereas Type-II has a monotonic relationship between prey population and the fraction of prey consumed. Type-III is characterized by a sigmoid relationship, in which the significant percentage of prey consumed is positively density-dependent over certain prey population regions.

Many researchers work deals with Holling Type-II and Type-III functional responses with an effective manner in the population interaction and harvesting function. S. T. Motuma [18] described the functional response of the prey-predator system and harvesting function in the population interaction. G. M. Vijaya Lakshmi [7] focused on Holling Type-II functional responses with square root performing in the effect of herd behaviour of prey-redactor model. M. N. Srinivas, et al. [15] determined the stochastic analysis and optimal harvesting strategy of a two-species commensal system. K. Madhusudhan Reddy, et al. [12] analyzed the prey harvesting and alternative food for predators in the two-species ecological system. Debasis Mukherjee [6] explained the effect of refuge and immigrations of the three species dynamic model where such a predator consumes of two opposing species. Geremew Kenassa Edessa, et al. [8] considered three species ecosystem with a sigmoid functional response form D. Pal, et al. [5] studied the interval biological parameters for performing in the optimal harvesting of prey-predator system. M. Gunasekaran, et al. [14] inspected the optimal harvesting of all the three species with bionomic equilibrium. K. Sujatha, et al. [13] examines the optimal control of disease in Eco-Epidemiological system R. P. Gupta et al. [17] examine the bifurcation analysis and prey harvesting study made from the modified Leslie-Gower predator-prey model with Michaelis-Menten type B. Hari Prasad, et al. [3, 4] investigated the stability analysis of Syn-Eco-System and Prey-Predator, Host-commensal with the mortality rate.

Thadei Sagaamiko, et al. [20] investigate predator survival using a set of parameters, a threshold, and a death rate. Asifa Tassaddiq et al. [2] explores the ratio dependent in two dimensional model and this system implemented chaos control strategies, determined positive fixed point under Neimark sacker bifurcation and phase plane analysis. Yusrianto et al. [22] developed one prey and one predator model under constructing second type holling functional response for examines the threshold harvesting and stability analysis for the predator. Nijamuddin Ali et al. [16] investigate the prey-predator food chain model and the study concentrated on the biological feasible equilibrium, Hopf-Andronov bifurcation with suitable parameters for the competitive species. Sahabuddin Sarwardi et al. [19] research focused on developing a competitive model and to build a type II functional response in holling for the fraction of prey habitat. This system feasibility determined with different stability condition.

2. Representation of the Mathematical Model

A second type Holling functional response is incorporated in the active growth of a prey-predator model with interspecific competition. The system consists of Prey (S_1), Predator (S_2), and Host (S_3). Here predators are competing naturally with prey and commensal to the Host species. Mortality rates are inspected for commensal and host species.

The following model was designed for holling Type-II functional response which claimed that the standard dealing time was zero and type-II response function was effectively transformed into a Lotka-Volterra model.

$$\frac{dS_1}{dT} = a_1 S_1\left(1 - \frac{S_1}{K_1}\right) - \frac{bS_1S_2}{1 + T_h bS_1} - qE_1S_1$$

$$\frac{dS_2}{dT} = a_2 S_2\left(1 - \frac{S_2}{K_2}\right) - \frac{ebS_1S_2}{1 + T_h bS_1} + CS_2S_3 - m_1S_2$$

$$\frac{dS_3}{dT} = a_3 S_3\left(1 - \frac{S_3}{K_3}\right) - m_2S_3 \tag{1}$$

To determine the system's stability, make certain assumptions for the above model.

$$S_1 = \alpha x,\ S_2 = \beta y,\ S_3 = \gamma z,\ T = \eta t,$$

with rescaling variable

$$\eta = \frac{1}{a_1},\ \alpha = K_1,\ \beta = K_2,\ \gamma = K_3,$$

and other parameters are

$$\theta = \frac{m_1}{a_1},\ \omega = \frac{m_2}{a_1},\ \lambda = \frac{bK_2}{a_1},\ \mu = \frac{beK_1}{a_1},\ \delta = \frac{CK_3}{a_1}$$

$$\rho = \frac{a_2}{a_1},\ \tau = \frac{a_3}{a_1},\ h = T_h bK_1,\ H = \frac{qE_1}{a_1}$$

The dimensionless version of the given model is shown below.

$$\frac{dx}{dt} = x(1 - x) - \frac{\lambda xy}{1 + hx} - Hx$$

$$\frac{dy}{dt} = \rho y(1 - y) - \frac{\mu xy}{1 + hx} + \delta yz - \theta y$$

$$\frac{dz}{dt} = \tau z(1 - x) - \omega z$$

Let's pretend that the average handling time is zero. $(h = 0)$

Therefore, the above model is composed as

$$\frac{dx}{dt} = x(1 - x) - \lambda xy - Hx$$

$$\frac{dy}{dt} = \rho y(1 - y) - \mu xy + \delta yz - \theta y$$

$$\frac{dz}{dt} = \tau z(1 - z) - \omega z \qquad (2)$$

The following assumptions were made by formulating the mathematical model (1)

(i) $S_1(0) > 0$, $S_2(0) > 0$ and $S_3(0) > 0$ are all positive initial conditions. Where $S_1(T)$, $S_2(T)$ and $S_3(T)$ represent the prey, predator and host population densities respectively.

(ii) In the nonexistence of predator, the local prey population grows up logistically with an underlying biological growth rate a_1, having ecological carrying capacity K_1, optimal harvesting rate qE_1 for the catching capability coefficient, and the effort given to the population S_1.

(iii) During the inter competition, the prey population will decrease exponential while compete with predator and vice versa for average handling time of predator T_h, predator searching efficiency 'b' and consumption rate 'e' and to form the second type holling functional response.

(iv) The predator population continues to expand logistically with growth rate a_2, ecological carrying capacity K_2 and mortality rate m_1 in the presence of prey.

(v) It was assumed that the predator acts as a commensal species C (coefficient of commensalism) and gets benefit from the host species for their alternative food.

(vi) The host species is increased at the rate a_3, ecological carrying capacity K_3 and mortality rate m_2.

3. Stability Analysis of the Equilibrium State

In this segment identified all the possible existence of equilibrium state and their stability analysed by equating $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$ and $\frac{dz}{dt} = 0$ are as follows.

I. Trivial stability state

(i) $\bar{x} = 0, \bar{y} = 0, \bar{z} = 0$

II. Axial stability state

(ii) $\bar{x} = 1 - H, \bar{y} = 0, \bar{z} = 0$

(iii) $\bar{x} = 0, \bar{y} = 1 - \frac{\theta}{\rho}, \bar{z} = 0$

(iv) $\bar{x} = 0, \bar{y} = 0, \bar{z} = 1 - \frac{\omega}{\tau}$

III. The condition of one of the three species has been washed away

(v) $\bar{x} = \frac{\rho(1-H) - \lambda(\rho - \theta)}{\rho - \lambda\mu}, \bar{y} = \frac{(\rho - \theta) - \mu(1-H)}{\rho - \lambda\mu}, \bar{z} = 0$

(vi) $\bar{x} = 0, \bar{y} = \frac{\tau(\rho - \theta) + \delta(\tau - \omega)}{\rho\tau}, \bar{z} = \frac{\rho(\tau - \omega)}{\rho\tau}$

(vii) $\bar{x} = (1 - H), \bar{y} = 0, \bar{z} = \frac{(\tau - \omega)}{\tau}$

(iv) The state in which all the three species are exist

(viii) $\bar{x} = \frac{\rho\tau(1-H) - \tau\lambda(\rho - \theta) - \lambda\delta(\tau - \omega)}{\rho\tau - \lambda\mu\tau},$

$$\bar{y} = \frac{\tau(\rho - \theta) + \delta(\tau - \omega) - \mu\tau(1-H)}{\rho\tau - \lambda\mu\tau}, \bar{z} = \frac{(\tau - \omega)(\rho - \lambda\mu)}{\rho\tau - \lambda\mu\tau}$$

4. The Steady States and their Existence

To analysis the dynamic steady state of the proposed model to apply the jacobian matrix at any subjective point of view is described by

$$J(x, y, z) = \begin{bmatrix} 1 - 2x - \lambda y - H & -\lambda x & 0 \\ -\mu y & \rho - 2\rho y - \mu x + \delta z - \theta & \delta y \\ 0 & 0 & \tau - 2\tau z - \omega \end{bmatrix}$$

The resultant of the deviation matrixes to express each equilibrium state is as follows.

Proposition 1. *The trivial stability state of the point* $E_1(0, 0, 0)$ *which is always unstable if* $H < 1, \rho > \theta$ *and* $\tau > \omega$, *under positive growth rate of the each species.*

Proof. The system's jacobian matrix at the trivial stability point $E_1(0, 0, 0)$ is known by

$$J(E_1) = \begin{bmatrix} 1 - H & 0 & 0 \\ 0 & \rho - \theta & 0 \\ 0 & 0 & \tau - \omega \end{bmatrix}$$

The characteristic roots of $J(E_1)$ is $\lambda_1 = 1 - H, \lambda_2 = \rho - \theta$ and $\lambda_3 = \tau - \omega$. Since growth rate of each species is higher than the mortality rate and the harvesting rate. Hence all the three characteristic roots are positive then the system $J(E_1)$ is unstable.

Proposition 2. *The axial state in which predator and host free equilibrium point* $E_2(1 - H, 0, 0)$ *is unstable if* $\rho > \mu(1 - H) + \theta$ *and* $\tau > \omega$ *even the prey population is at high density level.*

Proof. The system's jacobian matrix at the prey exist axial stability point E_2 is identified by

$$J(E_2) = \begin{bmatrix} -1 + H & -\lambda(1 - H) & 0 \\ 0 & \rho - \mu(1 - H) - \theta & 0 \\ 0 & 0 & \tau - \omega \end{bmatrix}$$

The characteristic roots of $J(E_2)$ is

$\lambda_1 = -1 + H, \lambda_2 = \rho - \mu(1 - H) - \theta$ and $\lambda_3 = \tau - \omega$.

Since $\lambda_1 < 0, \lambda_2 > 0$ and $\lambda_3 > 0$. Hence the steady state of the system $J(E_2)$ is unstable and saddle points exist.

Proposition 3. *The axial state in which prey and host free balance point* $E_3 = \left(0, 1 - \frac{\theta}{\rho}, 0\right)$ *is always unstable if* $\left[\lambda\left(1 - \frac{\theta}{\rho}\right) + H\right] > 1$ *and* $\tau > \omega$. *Whereas the predator at high density level.*

Proof. The system's jacobian matrix at the predator exist axial stability point E_3 is identified by

$$J(E_3) = \begin{bmatrix} 1 - \lambda\left(1 - \frac{\theta}{\rho}\right) - H & 0 & 0 \\ -\mu\left(1 - \frac{\theta}{\rho}\right) & -\rho + \theta & \delta\left(1 - \frac{\theta}{\rho}\right) \\ 0 & 0 & \tau - \omega \end{bmatrix}$$

The characteristic roots of $J(E_3)$ is $\lambda_1 = 1 - \lambda\left(1 - \frac{\theta}{\rho}\right) - H$, $\lambda_2 = -\rho + \theta$ and $\lambda_3 = \tau - \omega$. Here one of the root λ_2 is negative and another two's are positive. So, saddle point exists in this state of the equilibrium point. i.e., $\lambda_1 > 0, \lambda_2 < 0$ and $\lambda_3 > 0$ Hence the axial state of the system $J(E_3)$ is unstable.

Proposition 4. *The axial state in which prey and predator free balance point* $E_4\left(0, 0, 1 - \frac{\omega}{\tau}\right)$ *is always unstable if* $H < 1$ *and* $\theta < \left[\rho + \delta\left(1 - \frac{\omega}{\tau}\right)\right]$. *Whereas the host species at high density level.*

Proof. The system's jacobian matrix at the host exist axial equilibrium point E_4 is recognized by

$$J(E_4) = \begin{bmatrix} 1 - H & 0 & 0 \\ 0 & \rho + \delta\left(1 - \frac{\omega}{\tau}\right) - \theta & 0 \\ 0 & 0 & -\tau + \omega \end{bmatrix}$$

The characteristic roots of $J(E_4)$ is $\lambda_1 = 1 - H$, $\lambda_2 = \rho + \delta\left(1 - \frac{\omega}{\tau}\right) - \theta$ and $\lambda_3 = -\tau + \omega$. Here one of the roots λ_3 is negative and another two roots λ_1, λ_2 are positive. So, saddle point exists in this state of the equilibrium point. i.e., $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 < 0$. Hence the axial state of the system $J(E_4)$ is unstable.

Proposition 5. *The boundary state in which the host washed out the balance point* $E_5 = \left(\frac{(1-H)\rho - \lambda(\rho-\theta)}{\rho-\lambda\mu}, \frac{(\rho-\theta)-(1-H)\mu}{\rho-\lambda\mu}, 0\right)$ *is unstable if* $\tau > \omega$.

Proof. The system's jacobian matrix at the prey and predator exist boundary balance point E_5 is recognized by

$$J(E_5) = \frac{1}{(\rho-\lambda\mu)}$$

$$\begin{bmatrix} -(1-H)\rho + \lambda(\rho-\theta) & -\lambda[(1-H)\rho - \lambda(\rho-\theta)] & 0 \\ -\mu[(\rho-\theta)-(1-H)\mu] & \rho[(1-H)\mu - (\rho-\theta)] & \delta[(\rho-\theta)-(1-H)\mu] \\ 0 & 0 & \tau-\omega \end{bmatrix}$$

The Characteristic equation of $J(E_5)$ is

$$\lambda^2 - (trace\, J(x, y))\lambda + (\det J(x, y)) = 0 \text{ and } \lambda_3 = \tau - \omega$$

The characteristic roots of the above equation is

$$\lambda_1 = \frac{1}{2}[-(trace\, J(x, y)) + \sqrt{(trace\, J(x, y))^2 - 4(\det J(x, y))}] < 0$$

$$\lambda_2 = \frac{1}{2}[-(trace\, J(x, y)) - \sqrt{(trace\, J(x, y))^2 - 4(\det J(x, y))}] < 0$$ and $\lambda_3 = \tau - \omega$.

Where $[trace\, J(x, y)^2 - 4(\det J(x, y))] > 0$, $-(trace\, J(x, y)) < 0$ and $\sqrt{trace\, J(x, y)^2 - 4(\det J(x, y))} < trace\, J(x, y)$.

Where

$$(trace\, J(x, y)) = [(-(1-H)\rho + \lambda(\rho-\theta)) + \rho[(1-H)\mu - (\rho-\theta)]/(\rho-\lambda\mu)] < 0$$

$$(\det J(x, y)) = \frac{[(-(1-H)\rho + \lambda(\rho-\theta)) \times \rho[(1-H)\mu - (\rho-\theta)]] - [(-\lambda[(1-H)\rho - \lambda(\rho-\theta)]) \times (-\mu[(\rho-\theta)-(1-H)\mu])]}{(\rho-\lambda\mu)^2} > 0$$

Since $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_3 > 0$ (always). Hence the system of boundary state $J(E_5)$ is unstable and saddle points exist.

Proposition 6. *The boundary state in which the prey washed out balance point* $E_6 = \left(0, \frac{\tau(\rho-\theta)+\delta(\tau-\omega)}{\rho\tau}, \frac{\rho(\tau-\omega)}{\rho\tau}\right)$ *is asymptotically stable if* $[(1-H)\rho\tau < \lambda[\tau((\rho-\theta)+\delta(\tau-\omega))]]$ *else the state is unstable if* $[(1-H)\rho\tau > \lambda[\tau((\rho-\theta)+\delta(\tau-\omega))]]$.

Proof. The system's jacobian matrix at the predator and host exist boundary stability point E_6 is predictable by

$$J(E_6) = \frac{1}{\rho\tau}$$

$$\begin{bmatrix} (1-H)\rho\tau - \lambda[\tau(\rho-\theta)+\delta(\tau-\omega)] & 0 & 0 \\ -\mu[\tau(\rho-\theta)+\delta(\tau-\omega)] & -\rho[\tau(\rho-\theta)+\delta(\tau-\omega)] & \delta[\tau(\rho-\theta)+\delta(\tau-\omega)] \\ 0 & 0 & -\tau+\omega \end{bmatrix}$$

The characteristic equation of $J(E_6)$ is

$$\left[\frac{(1-H)\rho\tau - \lambda[\tau(\rho-\theta)+\delta(\tau-\omega)]}{\rho\tau} - \lambda\right]\left[\frac{-\rho[\tau(\rho-\theta)+\delta(\tau-\omega)]}{\rho\tau} - \lambda\right]$$

$$\left[\frac{-(\tau-\omega)}{\rho\tau} - \lambda\right] = 0$$

The characteristic roots of the above equation is

$\lambda_1 = \frac{(1-H)\rho\tau - \lambda[\tau(\rho-\theta+\delta(\tau-\omega))]}{\rho\tau}$, $\lambda_2 = \frac{-\rho[\tau(\rho-\theta)+\delta(\tau-\omega)]}{\rho\tau}$ and $\lambda_3 = \frac{-(\tau-\omega)}{\rho\tau}$.

Case (i). Since $\lambda_1 < 0$ if $(1-H)\rho\tau < \lambda[\tau((\rho-\theta)+\delta(\tau-\omega))]$, $\lambda_2 < 0$ (always) and $\lambda_3 < 0$ (always). Hence the system of state $J(E_6)$ is asymptotically stable.

Case (ii). Since $\lambda_1 > 0$ if $(1-H)\rho\tau > \lambda[\tau((\rho-\theta)+\delta(\tau-\omega))]$, $\lambda_2 < 0$ (always) and $\lambda_3 < 0$ (always). Hence the system of boundary state $J(E_6)$ is unstable and saddle points exist.

Proposition 7. *The boundary state in which the predator washed out* *balance point* $E_7 = \left((1-H), 0, \frac{(\tau-\omega)}{\tau}\right)$ *is asymptotically stable if* $[\tau\mu(1-H) > (\tau(\rho-\theta)+\delta(\tau-\omega))]$ *and unstable if* $\tau\mu(1-H) < (\tau(\rho-\theta)+\delta(\tau-\omega))$.

Proof. The system's jacobian matrix at the prey and host exist boundary stability point E_7 is knowable by

$$J(E_7) = \begin{bmatrix} -(1-H) & -\lambda(1-H) & 0 \\ 0 & \frac{[\tau(\rho-\theta)-\tau\mu(1-H)+\delta(\tau-\omega)]}{\tau} & 0 \\ 0 & 0 & -\tau+\omega \end{bmatrix}$$

The Characteristic equation of $J(E_7)$ is

$$[-(1-H)-\lambda]\left[\frac{[\tau(\rho-\theta)-\tau\mu(1-H)+\delta(\tau-\omega)]}{\tau}-\lambda\right][(-\tau+\omega)-\lambda] = 0$$

The characteristic roots of above equation becomes

$$\lambda_1 = -(1-H),\ \lambda_2 = \frac{[\tau(\rho-\theta)-\tau\mu(1-H)+\delta(\tau-\omega)]}{\tau} - \lambda \text{ and } \lambda_3 = -\tau+\omega$$

Case (i). Since $\lambda_1 < 0$ (always), $\lambda_3 < 0$ (always) and $\lambda_2 < 0$ (if $\tau\mu(1-H) > \tau(\rho-\theta)+\delta(\tau-\omega)$). Hence the state of equilibrium point $J(E_7)$ is asymptotically stable.

Case (ii). Since $\lambda_1 < 0$, (always) $\lambda_3 < 0$ (always) and $\lambda_2 > 0$ (if $\tau\mu(1-H) < \tau(\rho-\theta)+\delta(\tau-\omega)$). Hence the state of equilibrium point $J(E_7)$ is unstable and saddle points exist.

Proposition 8. *The Positive interior equilibrium point* $E_8 = \left(\frac{\rho\tau(1-H)-(\rho-\theta)\tau\lambda-\lambda\delta(\tau-\omega)}{\rho\tau-\lambda\mu\tau}, \frac{\tau(\rho-\theta)+\delta(\tau-\omega)-\mu\tau(1-H)}{\rho\tau-\lambda\mu\tau}, \frac{(\tau-\omega)(\rho-\lambda\mu)}{\rho\tau-\lambda\mu\tau}\right)$ *is locally asymptotically stable, if* $h_1 > 0$, $h_3 > 0$ and $(h_1 \cdot h_2) > h_3$.

Proof. Let us assume that prey, predator and host species are exists equilibrium point E_8 is standard by Let $J(E_8) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

$$a_{11} = \frac{-\rho\tau(1-H) + \lambda(\delta(\tau-\omega) + \tau(\rho-\theta))}{\rho\tau - \lambda\mu\tau},$$

$$a_{12} = -\lambda\left[\frac{\rho\tau(1-H) - \lambda\tau(\rho-\theta) - \lambda\delta(\tau-\omega)}{\rho\tau - \lambda\mu\tau}\right],$$

$$a_{21} = -\mu\left[\frac{\tau(\rho-\theta) + \delta(\tau-\omega) - \mu\tau(1-H)}{\rho\tau - \lambda\mu\tau}\right],$$

$$a_{22} = \left[\frac{\rho(-\tau(\rho-\theta) - \delta(\tau-\omega) + \mu\tau(1-H))}{\rho\tau - \lambda\mu\tau}\right],$$

$$a_{23} = \delta\left[\frac{\tau(\rho-\theta) + \delta(\tau-\omega) - \mu\tau(1-H)}{\rho\tau - \lambda\mu\tau}\right],$$

$$a_{33} = -\tau - \omega.$$

The characteristic equation of $J(E_8)$ is

$$\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = 0$$

where $h_1 = a_{11} + a_{22} + a_{33},$

$$h_2 = a_{22}a_{33} + a_{11}a_{33} + a_{11}a_{22} - a_{12}a_{21},\ h_3 = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33}.$$

Now the result of Routh-Hurwitz criterion analyse the coexistence steady state under the positive interior equilibrium state which has negative real part iff $h_1 > 0, h_3 > 0$ and $h_1h_2 > h_3$. Hence all the three characteristic roots are negative then the steady state of the equilibrium point $J(E_8)$ is always asymptotically stable.

5. Global Steadiness Analysis

In this part, to prove the Global stability of all the possible equilibrium states of the three species eco system by suitable Lyapunov function of the given model.

5.1 Global stability of the steadiness state $E_5(\bar{x}, \bar{y})$

Theorem. *The boundary equilibrium state* $E_5(\bar{x}, \bar{y})$ *is globally asymptotically stable.*

Proof. Let we apply the Lyapunov function for interior balance points E_5 as follow

$$L(\bar{x},\ \bar{y}) = \left\{x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right\} + \left\{y - \bar{y} - \bar{y}\ln\left(\frac{y}{\bar{y}}\right)\right\}$$

The time derivate of L along the solution of equation (2) is

$$\frac{dL}{dt} = \frac{dx}{dt}\left[1 - \frac{\bar{x}}{x}\right] + \frac{dy}{dt}\left[1 - \frac{\bar{y}}{y}\right]$$

Substitute $1 = \bar{x} + \lambda\bar{y} + H$ and $1 = \bar{x} + \lambda\bar{y} + H$ and $\rho = \rho\bar{y} + \mu\bar{x} + \theta$

$$\frac{dL}{dt} \leq -\left[[x - \bar{x}]^2\left(1 + \frac{\lambda + \mu}{2}\right) + [y - \bar{y}]^2\left(\rho + \frac{\lambda + \mu}{2}\right)\right]$$

$$\frac{dL}{dt} < 0$$

Thus the steadiness state E_5 is globally asymptotically stable.

5.2 Global stability of the steadiness state $E_6(\bar{y},\ \bar{z})$

Theorem. *The boundary equilibrium state $E_6(\bar{y},\ \bar{z})$ is globally asymptotically stable.*

Proof. Consider the Lyapunov function for interior balance points E_6 as described by

$$L(\bar{y},\ \bar{z}) = \left\{y - \bar{y} - \bar{y}\ln\left(\frac{y}{\bar{y}}\right)\right\} + \left\{z - \bar{z} - \bar{z}\ln\left(\frac{z}{\bar{z}}\right)\right\}$$

Differentiate ‘L’ w. r. to ‘t’ then the form reduced as

$$\frac{dL}{dt} = \frac{dy}{dt}\left[1 - \frac{\bar{y}}{y}\right] + \frac{dz}{dt}\left[1 - \frac{\bar{z}}{z}\right]$$

$$= [y - \bar{y}][\rho - \rho y + \delta z - \theta] + [z - \bar{z}][\tau - \tau z - \omega]$$

Substitute $\rho = \rho\bar{y} - \delta\bar{z} + \theta$ and $\tau = \tau\bar{z} + \omega$

$$\frac{dL}{dt} \leq -\left[[y - \bar{y}]^2\left(\rho - \frac{\delta}{2}\right) + [z - \bar{z}]^2\left(\tau - \frac{\delta}{2}\right)\right]$$

$$\frac{dL}{dt} < 0$$

Thus the steadiness state E_6 is globally asymptotically stable.

5.3 Global stability of the steadiness state $E_7(\bar{x}, \bar{z})$

Theorem. *The boundary equilibrium state $E_7(\bar{x}, \bar{z})$ is globally asymptotically stable.*

Proof. Let us consider the following Lyapunov function for the interior balance points E_7 as

$$L(\bar{x}, \bar{z}) = \left\{x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right\} + \left\{z - \bar{z} - \bar{z}\ln\left(\frac{z}{\bar{z}}\right)\right\}$$

Differentiate 'L' w. r. to 't' then the form reduced as

$$\frac{dL}{dt} = \frac{dx}{dt}\left[1 - \frac{\bar{x}}{x}\right] + \frac{dz}{dt}\left[1 - \frac{\bar{z}}{z}\right]$$

$$= [x - \bar{x}][1 - x - H] + [z - \bar{z}][\tau - \tau z - \omega]$$

Substitute $1 = \bar{x} + H$ and $\tau = \tau\bar{z} + \omega$

$$\frac{dL}{dt} \leq -[[x - \bar{x}]^2 + \tau[z - \bar{z}]^2]$$

$$\frac{dL}{dt} < 0$$

Thus the steadiness state E_7 is globally asymptotically stable.

5.4 Global stability of the steadiness state $E_8(\bar{x}, \bar{y}, \bar{z})$

Theorem. *The Positive interior equilibrium state $E_8(\bar{x}, \bar{y}, \bar{z})$ is globally asymptotically stable.*

Proof. Let us define the Lyapunov function for the positive definite equilibrium points E_8 as follows

$$L(\bar{x}, \bar{y}, \bar{z}) = \left\{x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right\} + \left\{y - \bar{y} - \bar{y}\ln\left(\frac{y}{\bar{y}}\right)\right\} + \left\{z - \bar{z} - \bar{z}\ln\left(\frac{z}{\bar{z}}\right)\right\}$$

The differential of 'L' w.r.to time 't' then reduce to the following form

$$\frac{dL}{dt} = \frac{dx}{dt}\left[1 - \frac{\bar{x}}{x}\right] + \frac{dy}{dt}\left[1 - \frac{\bar{y}}{y}\right] + \frac{dz}{dt}\left[1 - \frac{\bar{z}}{z}\right]$$

$$= [x - \bar{x}][1 - x - \lambda y - H] + [y - \bar{y}][\rho - \rho y - \mu x + \delta z - \theta]$$

$$+ [z - \bar{z}][\tau - \tau z - \omega]$$

Substitute $1 = \bar{x} + \lambda\bar{y} + H$, $\rho = \rho\bar{y} + \mu\bar{x} - \delta\bar{z} + \theta$ and $\tau = \tau\bar{z} + \omega$

$$\frac{dL}{dt} \leq -\left[[x - \bar{x}]^2\left(1 + \frac{\lambda + \mu}{2}\right) + [y - \bar{y}]^2\left(\rho + \frac{\lambda + \mu}{2}\right) + [z - \bar{z}]^2\left(\tau - \frac{\delta}{2}\right)\right]$$

$$\frac{dL}{dt} < 0$$

Therefore, L is positive definite of the system and also $L(\bar{x}, \bar{y}, \bar{z}) = 0$. Hence the steadiness state E_8 is globally asymptotically stable.

6. Bionomic Equilibrium

The system examine on the subject of the bionomic equilibrium which is grouping of biological and economic equilibrium. When the total profit earned from selling collected biomass equals the whole cost of harvesting effort, an economic equilibrium is considered to have been reached.

The biological balance is defined as follows:

$$\frac{dx}{dt} = 0$$

Assume that C is the cost of harvesting each unit effort of prey species (x) and P is the price of each unit biomass of prey. $R = R_1$ specifies the net revenue or economic rent at any moment t.

Let $R_1 = (Pqx - C)E_1$, where R_1 is the net revenue for prey.

The equations for the bionomic equilibrium $((x)_\infty(y)_\infty(z)_\infty(E_1)_\infty(E_2)_\infty(E_3)_\infty)$ are given below.

$$x(1 - x) - \lambda xy - Hx = 0 \tag{3}$$

Let us assume that the harvesting effect $H = qE_1$ then rewrite the above equation becomes

$$x(1-x) - \lambda xy - qE_1 x = 0 \tag{4}$$

$$\rho y(1-y) - \mu xy + \delta yz - \theta y = 0 \tag{5}$$

$$\tau z(1-z) - \omega z = 0 \tag{6}$$

$$R_1 = (Pqx - C)E_1 = 0 \tag{7}$$

In terms of achieving bionomic equilibrium, we observe the following occurrences.

Instance (i). The entire system would be stopped if $C > Pqx$, the cost is more than profits for the three species.

Instance (ii). The entire system would be in function if $C < Pqx$, the cost is less than profits for all the three species and if it is a positive value.

$$(x)_\infty = \frac{C}{Pq},\ (y)_\infty = \frac{\rho - \mu x + \delta z - \theta}{\rho},\ (z)_\infty = \frac{\tau - \omega}{\tau}$$

Now substitute $(x)_\infty$ in equation (4)-(6), then we get

$$(E_1)_\infty = \frac{1}{q}[1 - (x)_\infty - \lambda(y)_\infty] = \frac{1}{q}\left[1 - \frac{C}{Pq} - \lambda\left(\frac{\rho - \mu x + \delta z - \theta}{\rho}\right)\right]$$

Now $(E)_\infty$ to be positive i.e.,

$$(E_1)_\infty > 0 \text{ if } 1 > \frac{C}{Pq} + \lambda\left(\frac{\rho - \mu x + \delta z - \theta}{\rho}\right) \tag{8}$$

Hence the nontrivial bionomic equilibrium point $((x)_\infty (y)_\infty (z)_\infty (E_1)_\infty)$ exist if the criterion (8) holds.

7. Optimal Harvesting Policy

The present value J is a revenue stream that runs continuously by.

$$J = \int_0^\infty R_1(x,\ y,\ z,\ E_1)e^{-\delta t}dt$$

Where R_1 the net revenue or economic rent is specified as

$$R_1(x,\ y,\ z,\ E_1) = (Pqx - C)E_1$$

Where δ represent the instantaneous annual rate of discount rate. The focus of this work maximize J subject to the model of the state $0 \leq E_1 \leq (E_1)_{\max}$ by constructing the Hamiltonian function is described by

$$H = e^{-\delta t}(Pqx - C)E_1 + \lambda_1(x(1-x) - \lambda xy - qE_1 x)$$
$$+ \lambda_2(\rho y(1-y) - \mu xy + \delta yz - \theta y) + \lambda_3(\tau z(1-z) - \omega z)$$

Where λ_1, λ_2 and λ_3 are the adjoint variable

By using Pontryagin's maximum principle

$$\frac{\partial H}{\partial E_1} = 0; \frac{d\lambda_1}{dt} = -\left(\frac{\partial H}{\partial x}\right); \frac{d\lambda_2}{dt} = -\left(\frac{\partial H}{\partial y}\right); \frac{d\lambda_3}{dt} = -\left(\frac{\partial H}{\partial z}\right)$$

The control variable E_1 satisfying the constraints

$$\frac{\partial H}{\partial E_1} = \phi(t) = e^{-\delta t}(Pqx - C) - \lambda_1 qx$$

Now, Plan to maximize the Hamiltonian H by obtaining an optimal equilibrium $((x)_\infty(y)_\infty(z)_\infty(E_1)_\infty)$. Because the Hamiltonian H in the control variable E_1 is linear. Extreme control or singular controls are two possibilities for optimal control.

As a result, $E_1 = (E_1)_{\max}$. Where $\phi(t) > 0$, i.e., $\lambda_1 e^{\delta t} < (P - C/qx)$

$$E_1 = (E_1)_{\min}, \text{ where } \phi(t) < 0, \text{ i.e., } \lambda_1 e^{\delta t} > (P - C/qx),$$

$$E_1 = 0, \text{ where } \phi(t) = 0, \text{ i.e., } \lambda_1 = e^{-\delta t}(P - C/qx) \tag{9}$$

The optimal control in this situation is known as singular control, and the preceding equation is a criterion for maximum of Hamiltonian H. The maximum principle of Pontryagin's for adjoint equations is

$$\frac{d\lambda_1}{dt} = -\left(\frac{\partial H}{\partial x}\right)$$

$$\frac{d\lambda_1}{dt} = -(e^{-\delta t}PqE_1 + \lambda_1(1 - 2x - \lambda y - qE_1) + \lambda_2(-\mu y)) \tag{10}$$

$$\frac{d\lambda_2}{dt} = -\left(\frac{\partial H}{\partial y}\right)$$

$$\frac{d\lambda_2}{dt} = -[\lambda_1(-\lambda x) + \lambda_2(\rho - 2\rho y - \mu x + \delta z - \theta)] \tag{11}$$

$$\frac{d\lambda_3}{dt} = -\left(\frac{\partial H}{\partial z}\right)$$

$$\frac{d\lambda_3}{dt} = -[\lambda_2(\delta y) + \lambda_3(\tau - 2\tau z - \omega)] \tag{12}$$

Now, try to determine the problem's optimal equilibrium solution.

As a result x, y, z and E_1 can be considered constants

Equation (11) is written in the following way $\frac{d\lambda_2}{dt} + A_1\lambda_2 = A_2 e^{-\delta t}$

Where $A_1 = \rho - 2\rho y^* - \mu x^* + \delta z^* - \theta$ and $A_2 = \lambda x^*(P - C/qx^*)$ solution is known by

$$\lambda_2 = \frac{A_2 e^{-\delta t}}{(A_1 - \delta)} \tag{13}$$

The above equation (10) can also be written as $\frac{d\lambda_1}{dt} + A_3\lambda_1 = -A_4 e^{-\delta t}$

where $A_3 = 1 - 2x^* - \lambda y^* - qE_1$ and $A_4 = \left(PqE_1 - \mu y^* \frac{A_2}{(A_1 - \delta)}\right)$ whose solution is specified by

$$\lambda_1 = -\frac{A_4 e^{-\delta t}}{(A_3 - \delta)} \tag{14}$$

Equation (12) can be written as $\frac{d\lambda_3}{dt} + A_5\lambda_3 = -A_6 e^{-\delta t}$

Where $A_5 = \tau - 2\tau z^* - \omega$ and $A_6 = \delta y^* \frac{A_2}{(A_1 - \delta)}$ solution is given by

$$\lambda_3 = -\frac{A_6 e^{-\delta t}}{(A_5 - \delta)} \tag{15}$$

Equating (9) and (14) then we get the singular path

$$(P - C/qx^*) = -\frac{A_4}{(A_3 - \delta)} \tag{16}$$

Thus equation (16) can also be written as

$$F(x^*) = (P - C/qx^*) + \frac{A_4}{(A_3 - \delta)} = 0 \tag{17}$$

In the interval $0 \leq (x)_\infty \leq K,$ there is only one positive root $x^* = x_\delta$ of $F(x^*) = 0.$ The inequalities listed below are true $F(0) < 0, F(K) > 0,$ $F(x^*) > 0,$ for $(x^*) > 0.$

Where $x^* = x_\delta, y^* = y_\delta, z^* = z_\delta$ and

We get $(E)_\delta = \frac{1}{q}[1 - x_\delta - \lambda y_\delta]$ where $y_\delta = \left(\frac{\rho - \mu x + \delta z - \theta}{\rho}\right)$

Here $(E)_\delta > 0$ if $1 > x_\delta + \lambda y_\delta$

From the equation (13), (14) and (15), we examine if $\lambda_1, \lambda_2, \lambda_3$ is a time-independent optimal equilibrium. Thus they remain bounded as $t \to \infty.$

From the equation (16), also have

$$(P - C/qx^*) = -\frac{A_4}{(A_3 - \delta)} \to 0 \text{ as } \delta \to \infty.$$

Thus, the prey species net economic revenue is $R_1((x)_\infty (y)_\infty (z)_\infty (E_1)_\infty) = 0.$

This implies that an indefinite discount rate would result in net economic revenue approaching zero, and the fishery will stay closed.

8. Conclusion

The proposed model investigates the dynamic stability analysis of the syn-ecosystem consisting of one host with commensal predator and prey. It is observed that the model depicts the predator can able to survive in the habitat for getting benefit from the host species even though the prey species were not available for a long time. The system was initially characterized by Holling Type-II functional response after rescaling the model rehabilitated by nonlinear format. The stability states were analyzed the eight equilibrium states and derive local and global criteria under the positive coexistence state by suitable Routh-Hurwitz criterion and Lyapunov Function respectively. Finally, the system computes the bionomic equilibrium and uses Pontryagin's maximum principle to determine optimal harvesting technique for the prey species.

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