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OSCILLATORY BEHAVIOUR OF FOURTH ORDER DELAY DIFFERENCE EQUATIONS

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Abstract

We study the oscillatory and almost oscillatory behavior of fourth order delay difference equation of the form, $\Delta(p_n\Delta(q_n\Delta(r_n\Delta x_n))) + a_{n+1}f(x_{n-m+1}) - a_n f(x_{n-m+1}) = t_n$ where, $\{p_n\} > 0$, $\{q_n\} > 0$, $\{r_n\} > 0$, $\{a_{n+1}\} > 0$, $\{a_n\} > 0$ and $\{t_n\} > 0$. The necessary and sufficient conditions of oscillation and almost oscillation of the given equation are obtained. We also provide examples for illustrating our results.

1. Introduction

This paper deals with oscillatory and almost oscillatory behavior for

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solutions of fourth order delay difference equation given by the form,

$$\Delta(p_n\Delta(q_n\Delta(r_n\Delta x_n))) + a_{n+1}f(x_{n-m+1}) - a_n f(x_{n-m+1}) = t_n \tag{1}$$

where, $\{p_n\}, \{q_n\}, \{r_n\}, \{a_{n+1}\}, \{a_n\}, \{t_n\}$ are sequences if real numbers, $f : \mathbb{R} \to \mathbb{R}$ is continuous, $\{p_n\} > 0, \{q_n\} > 0, \{r_n\} > 0, \{a_{n+1}\} > 0, \{a_n\} > 0$ and $\{t_n\} > 0$ for every $n \geq n_0 \in N_0, \alpha f(\alpha) > 0$ for $\alpha \neq 0$, and m is a non-negative integer. The sequence $\{x_n\}$ is a real sequence for the solution of (1) for every $n \geq n_0 - m + 1$ and satisfies (1) for all $n > n_0$. Difference equations occur in the field of dynamical system, mathematical biology, economics, statistics, see for example [1-10].

For every large n a non-trivial solution $\{x_n\}$ of difference equation is oscillatory if the terms are neither eventually positive nor eventually negative and non-oscillatory otherwise. Thus (1) is oscillatory if all solutions are oscillatory and (1) is almost oscillatory if all solutions $\{x_n\}$ are either oscillatory or satisfy the condition $\lim_{n\to\infty} \Delta^i x_n = 0$ for $i = 0, 1, 2$. In section 3 the necessary and sufficient conditions of oscillation and almost oscillation of (1) are obtained. Examples are provided to prove the results.

2. Methodology

The oscillation and almost oscillation of fourth order delay difference equations are studied by Riccati transformation technique, comparison method and summation averaging method.

3. Oscillation Theorems

Assume $\Delta a_n \geq 0$ for every $n \geq n_0$ and the following is considered

$$\sum_{n=n_0}^{\infty} \frac{1}{p_n} = \sum_{n=n}^{\infty} \frac{1}{q_n} = \sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty \tag{2}$$

Theorem 1. *Let* $f(\alpha) = \alpha$ *and* $t_n \equiv 0$, *then there exist real valued functions t and T, where* $T : N_0 \times N_0 \to R$ *such that*

$$T(n, n) = 0$$

$$T(n, s) > 0$$

$$\Delta_2 T(n, s) \leq 0$$

$$-\Delta_2 T(n, s) = t(n, s)\sqrt{T(n, s)}$$

where $\Delta_2 T(n, S) = T(n, s+1) - T(n, s)$. *If for every* $n > s \geq n_0$,

$$\lim_{n\to\infty} \sup \frac{1}{T(n, M)} \sum_{s=M}^{n-1}\left[-\frac{T(n, s)(a_{s+1}-a_s)p_s r_{s-m} t^2(n, s)}{4(s-m-N_1)(s-m-N_2)}\right] \tag{3}$$

and

$$\sum_{i=n}^{n+m-1}(a_{i+1}-a_i)\left[\left(\sum_{j=n}^{i}\frac{1}{r_j}\right)\left(\sum_{k=j}^{i}\frac{1}{q_k}\right)\left(\sum_{k=j}^{i}\frac{1}{p_l}\right)\right] > 1 \tag{4}$$

then all solutions for (1) *are oscillatory.*

Proof. Assume $\{x_n\}$ to be non-oscillatory solution for (1). With no loss of generality, assume $\{x_n\}$ to be eventually non-negative. Then, there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-m} > 0$ for every $n \geq n_1$. From equation (1), we have

$$\Delta(p_n\Delta(q_n\Delta(r_n\Delta x_n))) + a_n f(x_{n-m+1}) - a_{n+1} f(x_{n-m+1})$$

So that $\Delta(p_n\Delta(q_n\Delta(r_n\Delta x_n))) < 0$ (5)

Thus, $\{q_n\Delta(r_n\Delta x_n)\}$ and $\{\Delta x_n\}$ are monotonic and of eventually one sign. Let's claim there exist an integer $n_2 \geq n_1$ such that

$$\Delta(q_n\Delta(r_n\Delta x_n)) > 0 \tag{6}$$

Also lets claim that there exist an integer $n_3 \geq n_2$ such that

$$\Delta(r_n\Delta x_n) > 0 \tag{7}$$

In order to prove this we assume the contrary that $\Delta(q_n\Delta(r_n\Delta x_n)) \leq 0$. So there is another integer $n_4 \geq n_3$ such that $p_{n4}\Delta(q_{n4}\Delta(r_{n4}\Delta x_{n4})) < 0$. Then for every $n \geq n_4$ we get,

$$p_n \Delta(q_n \Delta(r_n \Delta x_n)) \leq p_{n4} \Delta(q_{n4} \Delta(r_{n4} \Delta x_{n4})) < 0 \tag{8}$$

By taking summation for above inequality from n_4 to $n - 1$ we get,

$$r_n \Delta x_n - r_{n4} \Delta x_{n4} < p_{n4} \Delta(q_{n4} \Delta(r_{n4} \Delta x_{n4})) \sum_{s=n_4}^{n-1} \frac{1}{q_s} \sum_{s=n_3}^{n-1} \frac{1}{p_s}$$

Here $r_n \Delta x_n \to -\infty$ as $n \to \infty$. Summing again, we get a contradiction to $x_n > 0$. We now consider two cases:

Case I. Let $\Delta x_n \geq 0$ for $n \geq n_3$ and define,

$$z_n = \frac{p_n \Delta(q_n \Delta(r_n \Delta x_n))}{y_{n-m+1}}$$

where $z_n > 0$ for $n \geq n_3$

$$\Delta z_n \leq a_n - a_{n+1} - \frac{\Delta x_{n-m+1}}{x_{n-m+1}} z_{n+1} \tag{9}$$

By (5) and (7) and with $\Delta p_n \geq 0$ we obtain,

$$\Delta^3 (r_n \Delta x_n) \leq 0 \tag{10}$$

Here (10) implies $\Delta(r_n \Delta x_n)$ is non-increasing. Now consider the equality,

$$r_n \Delta x_n = c_N \Delta x_N + \sum_{s=N_1}^{n-1} \Delta(q_s \Delta(r_s \Delta x_s)) + \sum_{s=N_2}^{n-1} \Delta(r_s \Delta x_s)$$

for $n \geq N_1 \geq n_2$ and $n \geq N_2 \geq n_3$ such that

$$r_n \Delta x_n \geq (n - N_1)(n - N_2) \Delta(r_n \Delta x_n) \tag{11}$$

From (11) we obtain,

$$\Delta x_{n-m} \geq \frac{(n - m + N_1)(n - m - N_2) p_n \Delta(q_n \Delta(r_n \Delta x_n))}{p_n r_{n-m}} \tag{12}$$

for $n > N_1 + m + 1 > N_2 + m + 1 = M$. From (9) and (12) we get,

$$\Delta z_n \leq a_n - a_{n+1} - \frac{(n-m+N_1)(n-m-N_2)p_n\Delta(q_n\Delta(r_n\Delta x_n))}{x_{n-m}p_n r_{n-m}} z_{n+1} \tag{13}$$

With (5) and using the fact $\Delta y_n \geq 0$, (13) yields

$$\Delta z_n \leq a_n - a_{n+1} - \frac{(n-m+N_1)(n-m-N_2)}{p_n r_{n-m}} z_{n+1}^2$$

For every $n \geq M$ we have,

$$\sum_{n=M}^{n-1} T(n, s)(a_{s+1} - a_s) \leq T(n, M)z_M$$

$$- \sum_{s=M}^{n-1}\left[Z_{s+1}(-\Delta_2 T(n, s)) + \frac{(s-m-N_1)(s-m-N_2)T(n, s)}{p_s r_{s-m}} z_{s+1}^2 \right]$$

$$= T(n, M)z_M$$

$$- \sum_{s=M}^{n-1} + \left[\frac{\sqrt{T(n, s)}t(n, s)z_{s+1}(s-m-N_1)(s-m-N_2)T(n, s)}{p_s r_{s-m}} z_{s+1}^2 \right]$$

$$\leq T(n, M)z_M + \sum_{s=M}^{n-1} \frac{p_s r_{s-m} t^2(n, s)}{4(s-m-N_1)(s-m-N_2)}$$

With above inequality we get,

$$\lim_{n\to\infty} \sup \frac{1}{T(n, M)} \sum_{s=M}^{n-1}\left[T(n, s)(a_{s+1} - a_s) - \frac{p_s r_{s-m} t^2(n, s)}{4(s-m-N_1)(s-m-N_2)} \right] \leq z_M$$

which is a contradiction to (3).

Case II. Let $\Delta x_n < 0$ for every $n \geq n_3$. Summing (1) from s to n we get,

$$p_{n+1}\Delta(q_{n+1}\Delta(r_{n+1}x_{n+1})) - p_s\Delta(q_s\Delta(r_s\Delta x_s)) + \sum_{i=s}^{n} a_{i+1}x_{i-m+1} - \sum_{i=s}^{n} a_i x_{i-m+1} = 0$$

So

$$-\Delta(q_n\Delta(r_n\Delta x_n))+\frac{1}{p_n}\left(\sum_{i=s}^{n}a_{i+1}x_{i-m+1}-\sum_{i=s}^{n}a_ix_{i-m+1}\right)\leq 0$$

Thus we have,

$$q_n\Delta(r_n\Delta x_n)+\sum_{i=n}^{\infty}\left(\sum_{j=n}^{i}\frac{1}{p_j}\right)(a_{i+1}x_{i-m+1}-a_ix_{i-m+1})\leq 0 \tag{14}$$

Summing again from s to n and with the fact that $c_n\Delta y_n\leq 0$ we get,

$$-\Delta(r_n\Delta x_n)+\frac{1}{q_n}\sum_{i=n}^{\infty}\left(\sum_{j=n}^{i}\frac{1}{p_j}\right)(a_{i+1}x_{i-m+1}-a_ix_{i-m+1})\leq 0$$

In a similar way we have,

$$r_n\Delta x_n+\sum_{i=n}^{\infty}\left(\sum_{j=n}^{i}\frac{1}{q_j}\right)\left(\sum_{k=j}^{i}\frac{1}{p_k}\right)(a_{i+1}x_{i-m+1}-a_ix_{i-m+1})\leq 0 \tag{15}$$

A final summation of (15) yield,

$$\sum_{i=n}^{\infty}\left[\left(\sum_{j=n}^{i}\frac{1}{r_j}\right)\left(\sum_{k=j}^{i}\frac{1}{q_k}\right)\left(\sum_{l=k}^{i}\frac{1}{p_l}\right)\right](a_{i+1}-a_i)x_{i-m+1}\leq x_n$$

Or

$$\sum_{i=n}^{n+m-1}\left[\left(\sum_{j=n}^{i}\frac{1}{r_j}\right)\left(\sum_{k=j}^{i}\frac{1}{q_k}\right)\left(\sum_{l=k}^{i}\frac{1}{p_l}\right)\right](a_{i+1}-a_i)x_{i-m+1}\leq x_n \tag{16}$$

Here $\{x_n\}$ is decreasing. Hence (16) yield

$$\sum_{i=n}^{n+m-1}(a_{i+1}-a_i)\left[\left(\sum_{j=n}^{i}\frac{1}{r_j}\right)\left(\sum_{k=j}^{i}\frac{1}{q_k}\right)\left(\sum_{l=k}^{i}\frac{1}{p_l}\right)\right]\leq 1$$

This gives a contradiction to (4). Thus the proof is completed.

Example 1. Consider the difference equation

$$\Delta^4 x_n - \frac{16n + 32}{n - m + 1} x_{n-m+1} = 0 \tag{A1}$$

where m is a odd positive integer, satisfies all conditions of Theorem 1 for $T(n, s) = (n - s)$ and $t(n, s) = \frac{1}{\sqrt{n - s}}$. Hence every solution of (1) becomes oscillatory. Here $\{x_n\} = \{(-1)^n n\}$ is such a solution of (A1).

Theorem 2. *We assume* $t_{n+1} - t_n \equiv 0$. *Let* $f(\alpha) - f(\beta) = g(\alpha, \beta)(\alpha - \beta)$ *and* $g(\alpha, \beta) \geq \nu$ *for* $\nu > 0$. *Then there exist a positive sequence* $\{\phi_n\}$ *for all* $n_1 > n_0 + m$ *such that,*

$$\sum_{n=n_1}^{\infty}[(a_{n+1} - a_n)\phi_n + \frac{p_n r_{n-m}(\Delta\phi_n)^2}{4\nu\phi_n(n - m - n_0)}] = \infty \tag{17}$$

and

$$\lim_{n\to\infty} \sup (a_{i+1} - a_i) \sum_{i=n}^{n+m-1} \left[\left(\sum_{j=n}^{i} \frac{1}{r_j}\right)\left(\sum_{k=j}^{i} \frac{1}{q_k}\right)\left(\sum_{l=k}^{i} \frac{1}{p_l}\right)\right] = \infty \tag{18}$$

Then all solutions of (1) are oscillatory.

Proof. We proceed exactly as Theorem 1. Here (7) holds and $\{\Delta x_n\}$ is eventually positive then set

$$z_n = \frac{p_n \Delta(q_n \Delta(r_n \Delta x_n))\phi_n}{f(x_{n-m})}$$

For $n \geq n_3$, $z_n > 0$,

$$\Delta z_n \leq (a_n - a_{n+1})\rho_n + \frac{\Delta\phi_n}{\phi_{n+1}} \frac{\Delta f(x_{n-m})}{f(x_{n-m})} z_{n+1}$$

We get

$$\Delta z_n \leq (r_n - r_{n+1})\phi_n + \frac{\Delta\phi_n}{\phi_{n+1}} \frac{g(x_{n-m+1}, x_{n-m})}{f(x_{n-m})} z_{n+1}$$

With (12) and considering the facts that $\{p_n \Delta(q_n \Delta(r_n \Delta x_n))\}$ is decreasing and $\{x_n\}$ is increasing, we have

$$\Delta z_n \leq \phi_n(a_{n+1} - a_n) + \frac{\Delta\phi_n}{\phi_{n+1}} \frac{\mu(n - m - N_1)(n - m - N_2)\phi_n}{p_n r_{n-m} f(x_{n-m})} z_{n+1}^2$$

For $n \geq N_1 + m + 1 \geq N_2 + m + 1 > n_3$. By squaring we get,

$$\Delta z_n \leq (a_{n+1} - a_n)\phi_n + \frac{p_n r_{n-m}(\Delta\phi_n)^2}{4\nu\phi_n(n - m - N_1)(n - m - N_2)}.$$

Taking summation for above inequality from M to n and letting $n \to \infty$ with (17), we get $\lim_{n\to\infty} z_n = -\infty$. This is a contradiction to $\{z_n\}$ which is eventually positive. Now assume $\{\Delta x_n\}$ is eventually negative. Taking summation thrice for (1) exactly as Theorem 1, we can understand that,

$$\sum_{i=n}^{\infty}\left[\left(\sum_{j=n}^{i}\frac{1}{r_j}\right)\left(\sum_{k=j}^{i}\frac{1}{q_k}\right)\left(\sum_{l=k}^{i}\frac{1}{p_l}\right)\right](a_{i+1} - a_i)f(x_{i-m+1}) \leq x_n \tag{19}$$

We know that $\{x_n\}$ decreases and $f(\alpha)$ increases, then (19) follows that,

$$\sum_{i=n}^{n+m-1}\left[\left(\sum_{j=n}^{i}\frac{1}{r_j}\right)\left(\sum_{k=j}^{i}\frac{1}{q_k}\right)\left(\sum_{l=k}^{i}\frac{1}{p_l}\right)\right](a_{i+1} - a_i) \leq \frac{x_n}{f(x_n)} \tag{20}$$

It is clear that $\lim_{n\to\infty} y_n = d \geq 0$. From (18) and (20) we see that $d > 0$ is not possible. If $d = 0$ we have,

$$\lim \frac{x_n}{f(x_n)} = \lim \frac{1}{g(x_{n+1}, x_n)} \leq \frac{1}{\nu}$$

This contradicts (18). Thus the proof is completed.

Example 2. Consider the difference equation

$$\Delta(n\Delta(n\Delta^2 x_n)) + 8n^3(x_{n-4}^{\frac{1}{3}} + x_{n-4}) = 0,\ n \geq 1 \tag{A2}$$

that satisfies all the conditions of Theorem 2 for $\phi_n \equiv 1$. Hence all solutions of equation (5) are oscillatory for $\{x_n\} = \{(-1)^n n\}$.

Theorem 3. *Suppose* (17) *and* (19) *holds and if there exist a positive* *sequence* $\{\phi_n\}$ *and an oscillatory sequence* $\{\psi_n\}$ *such that* $\Delta(p_n \Delta(q_n \Delta(r_n \Delta \psi_n))) = t_n$ *for* $\lim \Delta^i \psi_n = 0$ *for* $i = 0, 1, 2$ (22) *and for some* $\gamma \in (0, 1)$ *and every* $n_2 > n_0 + m + 1$

$$\sum_{n=n_1}^{\infty} \left[(r_{n+1} - r_n)\phi_n - \frac{p_n r_{n-m} (\Delta\phi_n)^2}{4\nu\gamma(n - m - n_0)\phi_n} \right] = \infty \tag{23}$$

then (1) *is said to be almost oscillatory*.

Proof. If there exist a non-oscillatory solution $\{x_n\}$ which is positive and $\lim n \to \infty,\ y_n = 0$ then consider a function y_n defined as,

$$y_n = x_n - \psi_n \tag{24}$$

Here y_n is eventually positive, if not we will have $y_n < \psi_n$ which contradicts the nature of oscillation for $\{\psi_n\}$. So (1) implies,

$$\Delta(p_n \Delta(q_n \Delta(r_n \Delta y_n))) \le 0. \tag{25}$$

Thus $\{\Delta^2 x_n\}$ and $\{\Delta^3 x_n\}$ are eventually of one sign and monotonic. Following as before, there is an integer $n_3 \ge n_2$ such that $\Delta^2(q_n \Delta(r_n y_n)) > 0$ and $\Delta^3(r_n \Delta y_n) \le 0$.

We assume $\{y_n\}$ is eventually positive. With our assumption and using the fact that $\{y_n\}$ is increasing and $\psi_n \to 0$ as $n \to \infty$ we have,

$$x_{n-m+1} \ge \gamma y_{n-m+1} \tag{26}$$

Thus

$$f(x_{n-m+1}) \ge f(\gamma y_{n-m+1}) \tag{27}$$

Now define,

$$z_n = \frac{p_n \Delta(q_n \Delta(r_n \Delta y_n))}{f(\gamma y_{n-m})} \phi_n$$

$z_n > 0$ and $n \ge n_3$ then,

$$\Delta z_n \leq \phi_n(a_n - a_{n+1}) + \frac{\Delta\phi_n}{\phi_{n+1}} \frac{\gamma g(\gamma y_{n-m+1}, \gamma y_{n-m})\Delta y_{n-m}\phi_n}{f(\gamma y_{n-m})} z_{n+1}$$

$$\Delta z_n \leq \phi_n(a_n - a_{n+1}) + \frac{\Delta\phi_n}{\phi_{n+1}} \frac{\gamma\nu\Delta y_{n-m}\phi_n}{f(\gamma y_{n-m})} z_{n+1}$$

Proceeding as theorem 2 we obtain a contradiction to (23). Hence $\{x_n\}$ becomes eventually negative with $\{x_n\}$ decreasing to a non-negative constant d. Since $\lim_{n\to\infty} \psi_n = 0$, we have $\lim_{n\to\infty} x_n = d$. By summing (1) thrice we obtain,

$$\sum_{i=n}^{\infty}\left[\left(\sum_{j=n}^{i}\frac{1}{r_j}\right)\left(\sum_{k=j}^{i}\frac{1}{q_k}\right)\left(\sum_{l=k}^{i}\frac{1}{p_l}\right)\right](a_{i+1} - a_i)f(x_{i-m+1}) \leq y_n$$

As $\lim_{n\to\infty} \inf x_n = 0$ we get $\lim_{n\to\infty} x_n = 0$. Hence $d = 0$ and $\lim \Delta^i x_n = 0$. Thus the proof is completed.

Example 3. Consider the difference equation

$$\Delta(n\Delta((n+1)\Delta^2 x_n)) - \frac{16(x_{n-m+1})}{n-m+1} = 0 \tag{A3}$$

with m as an odd positive integer which satisfies the conditions of Theorem 3 for $\phi_n \equiv 1$. Hence all solutions of (1) are almost oscillatory. Here $\{x_n\} = \{(-1)^n\}$ is a solution of (A3).

Theorem 4. *Assume* (17) *holds and there exist a non-negative sequence* $\{\phi_n\}$ *and an oscillatory sequence* $\{\psi_n\}$ *such that* $\Delta^3(r_n\Delta\psi_n) = t_n$ *for* $\lim \Delta^i \psi_n = 0$ *for* $i = 0, 1, 2,$ *where* $\Delta\rho_n \leq 0$ *and* $\Delta^3\phi_n \geq 0$ *for* $n > n_0$. *Suppose*

$$\sum_{n=n_0}^{\infty}(a_{n+1} - a_n)\phi_n = \infty$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n \phi_n} \sum_{s=N}^{\infty} (s - N + 1)(a_{s+1} - a_s)\phi_{n+3} = \infty$$

are true then all solution for (1) *become almost oscillatory*.

Proof. Similarly as Theorem 3, we get

$$\Delta z_n \leq \phi_n(a_n - a_{n+1}) + \frac{\Delta\phi_n}{\phi_{n+1}} \frac{\gamma g(\gamma y_{n-m+1}, \gamma y_{n-m})\Delta y_{n-m}\phi_n}{f(\gamma y_{n-m})} z_{n+1}$$

For $n \geq N$ and $N \geq n_0$. From $\{\phi_n\}$ we obtain,

$$\Delta z_n \leq (a_n - a_{n+1})\phi_n.$$

Taking summation to above inequality from N to n and letting $n \to \infty$ implies a contradiction to (28). Hence $\{\Delta x_n\}$ should be eventually negative. Thus $\{x_n\}$ decreases to $d \geq 0$. In order to prove $d = 0$ assume $d > 0$ then there exist an integer $\mathbb{N} \geq N > 0$ then $x_{n-m+1} \geq \frac{d}{2}$. Let $\omega_n = r_n\phi_n\Delta y_n$, then

$$\Delta^3\omega_n = (a_n - a_{n+1})\phi_{n+3}f(x_{n-m+1}) - 3\Delta\phi_{n+2}\Delta(q_n\Delta(r_n\Delta x_n))$$

$$+ 3\Delta^2\phi_{n+1}\Delta(r_n\Delta x_n) + \Delta^3\phi_n(r_n\Delta x_n)$$

Here $\Delta\phi_n < 0$ and $\Delta^3\phi_n > 0$, so

$$\Delta^3\omega_n + (a_{n+1} - a_n)\phi_{n+3}f\left(\frac{d}{2}\right) \leq 0$$

for $n \geq \mathbb{N}$. Now summing this inequality from n to j with $\Delta\omega_j > 0$ then,

$$-\Delta\omega_n + f\left(\frac{d}{2}\right)\sum_{s=n}^{j}(a_{s+1} - a_s)\phi_{n+3} \leq 0$$

When $j \to \infty$ we have,

$$-\Delta\omega_n + f\left(\frac{d}{2}\right)\sum_{s=n}^{\infty}(a_{s+1} - a_s)\phi_{n+3} \leq 0$$

Again taking summation with the fact $w_j < 0$ we obtain,

$$\Delta y_n \leq \frac{f\left(\frac{d}{2}\right)}{r_n \phi_n} \sum_{s=N}^{\infty} (s - n + 1)(a_{n+1} - a_n)\phi_{n+3}$$

Taking a final summation for above inequality from $\mathcal{N}$ to $n - 1$,

$$y_n \leq y_{\mathcal{N}} \sum_{s=\mathcal{N}}^{n-1} \frac{1}{r_s \phi_s} \sum_{h=s}^{\infty} (h - s + 1)(a_{h+1} - a_h)\phi_{h+3}$$

By (29) $\lim_{n\to\infty} y_n = -\infty$ is a contradiction and the proof is complete.

Example 4. Considering the difference equation,

$$\Delta(n\Delta(n\Delta^2 x_n)) + \frac{4n^4 + 16n^3 + 18n^2 + 14n - 1}{n(n+1)(n+2)} (x_{n-2}^{1/3} + x_{n-2})$$

$$= \frac{(-1)^{n+1}}{n(n+1)(n+2)} (54n^4 + 53n^3 + 27n^2 + 10n + 2 \qquad \text{(A4)}$$

that satisfies all the conditions of Theorem 4 for $\phi_n \equiv 1$ and $\{\psi_n\} = \left\{\frac{(-1)^n}{n}\right\}$ then all solutions are almost oscillatory. Here $\{x_n\} = \{(-1)^n\}$ is an oscillatory solution of (A4).

3. Conclusion

In this paper, the oscillation and almost oscillation conditions for the solutions of (1) are established. These conditions are derived using Riccati transformation technique, comparison method and summation averaging method.

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