Advances and Applications in Mathematical Sciences
Volume 21, Issue 2, December 2021, Pages 519-529

ODD GRACEFUL LABELING OF SSG WITH STAR RELATED GRAPHS

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Abstract

In 1991, Gnanajothi [4] defined a graph G with q edges to be odd graceful [3] if there is an injection g from $V(G)$ to the set $\{0, 1, 2, \ldots, (2q-1)\}$ such that, when each edge xy is assigned the label $|g(x)-g(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, (2q-1)\}$.

In this paper, we prove the odd graceful labeling of star graphs attached to the apex and all the path vertices of the Sub divided shell graph.

1. Introduction

Alexander Rosa [7] initiated the concept of graph labeling in the year 1967. Graph labeling is the assignment of labels to vertices or edges or both of a graph which are represented by integers subject to certain conditions. Graceful labeling is the first labeling method in graph theory. The graceful labeling of a graph G with q edges is an injection g from the vertices of $V(G)$

2020 Mathematics Subject Classification: 05C78.
Keywords: Graceful labeling, Odd graceful labeling, shell graph, star graph, sub divided shell graph.
Received May 17, 2021; Accepted June 7, 2021

to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge xy is assigned the label $|g(x) - g(y)|$, the resulting edge labels are distinct. Deb and Limaye [2] introduced the shell graph. Jeba Jesintha and Ezhilarasi Hilda [6] introduced subdivided shell graph. Gnanajothi [4] proved the following graphs are odd graceful; the path P_n, the cycle C_n if and only if n is even, the complete bipartite graph $K_{m,n}$ and the crowns $C_{n0}K_1$, if and only if n is even. Gnanajothi conjectured that all trees are odd graceful and proved the conjecture for all trees with order up to 10. Ibrahim Moussa [5] proved that $C_m \cup P_n$ which is the union of the cycle C_m and the path P_n admits odd graceful labeling. Sub divided shell flower graphs are odd graceful proved by Jeba Jesintha and Ezhilarasi Hilda [6]. Shadow graph $D_2(P_n)$ are odd graceful graph proved by Vaidya and Lekha [8]. Vaidya and Lekha Bijukumar [9] proved that splitting graph of a star $K_{1,4}$ admits odd graceful labeling. Badr [1] proved that the revised friendship graphs $F(kC_4)$, $F(kC_8)$, $F(kC_{12})$, $F(kC_{16})$ and $F(kC_{20})$ are odd graceful, where k is any positive integer. For a detailed study on odd graceful labeling, refer to "Dynamic survey of graph labeling" by Joseph Gallian [3].

2. Basic Definitions

In this section, we recall some definitions which taken from "Dynamic Survey of Graph Labeling" by Joseph Gallian [3].

Definition 2.1. A star graph [3] is defined as the tree which has one vertex and adjacent to all the other vertices. The star graph is denoted as S_r where r is the number of pendant vertices connected tone vertex. Refer Figure 1.

Definition 2.2. Shell graph [5] is defined as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex. Shell graphs are also known as fans which are the join of K_1 and P_n, the path with, n^t vertices. The vertices and edges $v_1, e_1, v_2, e_2, v_3, e_3, \ldots, v_{n-1}, e_{n-1}, v_n$ constitute the path of the shell graph where the edge e_i is incident with v_i and v_{i+1}. Refer Figure 1.

Definition 2.3. When each edge in the path of the shell graph is sub divided, then a Sub divided Shell graph is obtained. The common vertex v_0 is called the apex in both shell graph and sub divided shell graph. The edge e_i in the shell graph is sub divided into two edges $v_i w_i$ and $w_i v_{i+1}$.

Note that there are $(n-1)$ cycles at the apex v_0. These $(n-1)$ cycles are termed as the petals of the sub divided shell graph and it is denoted by m. Refer Figure 1.

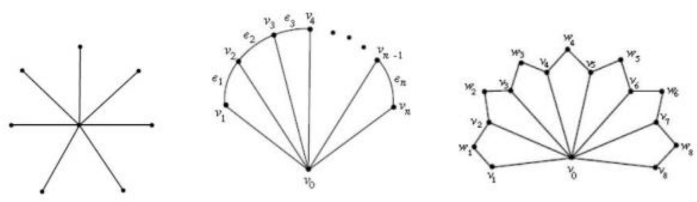

Figure 1. Star graph S_7, Shell graph with n petals, Sub divided Shell Graph with 7 petals.

3. Main Result

In this section, we prove the odd graceful labeling of star graphs attached to the apex and all the path vertices of the sub divided shell graph.

Theorem 1. *The Subdivided Shell Graph with Star graph S_r attached to its apex is odd graceful.*

Proof. Let G be the graph obtained by attaching the star graph S_r to the apex of the sub divided shell graph. Let $V(G)$ and $E(G)$ be the number of vertices and edges of the graph G respectively, where $|V(G)| = p$ and $|E(G)| = q$. The graph G is described as follows: Let $u_1, u_2, u_3, \ldots, u_r$ denote the pendant vertices of S_r in anticlockwise direction. Let the apex of the sub divided shell graph be denoted as v. Let $w_1, w_2, w_3, \ldots, w_n$ be the path vertices of SSG which are marked in clock wise direction. Let m denote the number of petals in SSG. The graph G has $p = r + 2(m+1)$ vertices and $q = r + (3m+1)$ edges.

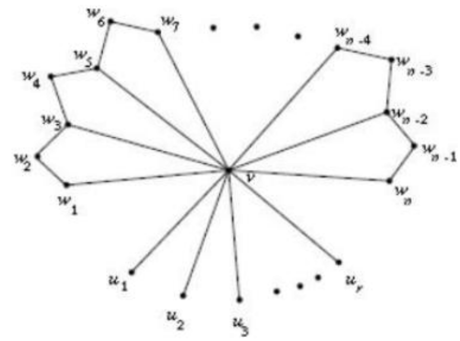

Figure 2. Sub divided Shell Graph with Star Graph attached to the apex.

The vertex labels for pendant vertices of S_r :

$$g(u_i) = 2(i-1) \text{ for } 1 \leq i \leq r. \tag{3.1}$$

The vertex label for the apex of SSG:

$$g(v) = 2q - 1. \tag{3.2}$$

The vertex labels for the path vertices of SSG:

$$g(w_i) = 2r + (i-1) \text{ for } i = 2k+1 \text{ and } 1 \leq k \leq \frac{(n+1)}{2} \tag{3.3}$$

$$g(w_i) = 2q - 2m - (i+1) \text{ for } i = 2k \text{ and } 1 \leq k \leq \frac{(n-1)}{2}. \tag{3.4}$$

From the above equations (3.1) to (3.4), it is observed that the vertex labels of G are distinct. The edge labels of G are labeled as follows:

$$|\, g(v) - g(u_i) \,| = |\, 2q - 2i + 1 \,| \text{ for } 1 \leq i \leq r \tag{3.5}$$

$$|\, g(w_{2i}) - g(w_{2i+1}) \,| = |\, 2q - 2m - 2r - 4i - 1 \,| \text{ for } 1 \leq i \leq \frac{n-1}{2} \tag{3.6}$$

$$|\, g(w_{2i-1}) - g(w_{2i}) \,| = |\, 2q - 2m - 2r - 4i + 1 \,| \text{ for } 1 \leq i \leq \frac{n+1}{2} \tag{3.7}$$

$$|\, g(v) - g(w_{2i-1}) \,| = |\, 2q - 2i - 2r + 1 \,| \text{ for } 1 \leq i \leq \frac{n+1}{2}. \tag{3.8}$$

We call these to f edge labels as E_1, E_2, E_3 and E_4 given in equations (3.5) to (3.8) respectively.

$$E_1 = (2q-1), (2q-3), \dots, (2q-2r+1) \tag{3.9}$$

$$E_2 = (2q-2m-2r-5), (2q-2m-2r-9), \dots, (2q-2m-2r-2n-3) \tag{3.10}$$

$$E_3 = (2q-2m-2r-3), (2q-2m-2r-7), \dots, (2q-2m-2r-2n+3) \tag{3.11}$$

$$E_4 = (2q-2r-1), (2q-2r-3), \dots, (2q-2r-n). \tag{3.12}$$

By denoting $E = E_2 \cup E_3 \cup E_4 \cup E_1$, we can note that the edge labels are distinct odd numbers from the set $\{1, 3, 5, \dots, (2q-1)\}$. Thus, the odd graceful labeling for S_r attached to the apex of SSG is odd graceful. Illustration for the above the ore m for a graph G with $m = 8$, $r = 6$, $n = 17$, is given in Figure 3.

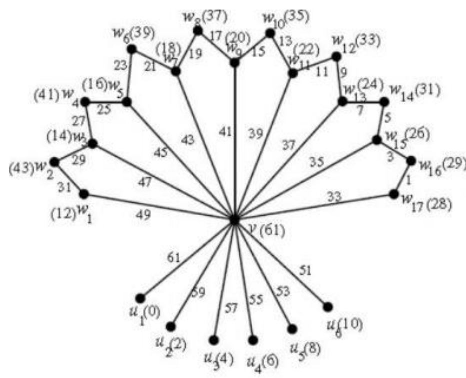

Figure 3. Odd grace full a belling of Sub divided Shell Graph with star graph when $m = 8$, $r = 6$, and $n = 17$.

Theorem 2. *Sub divided shell Graph with star graphs S_r attached to the apex and odd path vertices is odd grace full.*

Proof. Let G be the graph obtained by attaching star graphs S_r to the apex and odd path vertices of the subdivided shell graph. The graph G is described as follows:

Let $w_1, w_2, w_3, \dots, w_n$ be the path vertices of SSG which are marked in clock wise direction. Let w_1, w_3, w_5, w_n denote the odd path vertices of SSG. Let m denote the number of petals in SSG. Let $w_1^1, w_1^2, w_1^3, \dots, w_1^j,$

$(j = 1, 2, \ldots, r)$ be the pendant vertices of S_r attached to w_1. Let $w_3^1, w_3^2, w_3^3, \ldots, w_3^j (j = 1, 2, \ldots, r)$ be the pendant vertices of S_r attached to w_3. Similarly, let $w_n^1, w_n^2, w_n^3, \ldots, w^{nj} (j = 1, 2, \ldots, r)$ be attached to w_n. The graph G has $p = 2(r + 1) + m(r + 2)$ vertices and $q = r(m + 2) + (3m + 1)$ edges. The vertex label for the apex of SSG:

$$g(v) = 2q - 1 \tag{3.13}$$

$$g(u_i) = 2(i - 1), \text{ for } 1 \leq i \leq r. \tag{3.14}$$

The vertex labels for the path vertices of SSG:

$$g(w_i) = 2r + (i - 1) \text{ for } i = 2k + 1 \text{ and } 1 \leq k \leq \frac{(n + 1)}{2} \tag{3.15}$$

$$g(w_i) = 2r + 2n - (i + 1) \text{ for } i = 2k \text{ and } 1 \leq k \leq \frac{(n + 1)}{2}$$

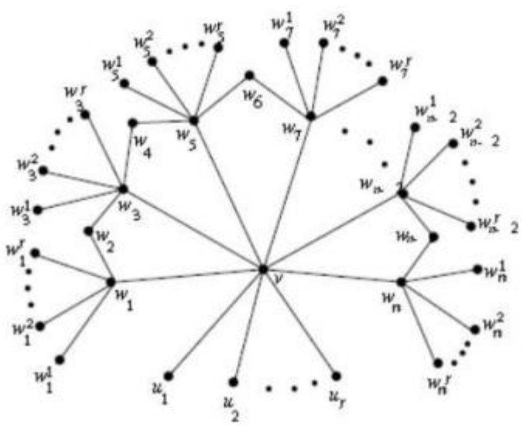

Figure 4. Sub divided Shell Graph with Star Graph attached to the apex and odd path vertices.

The vertex labels for pendant vertices of S_r attached to w_i (i is odd):

$$g(w_i^j) = r(i + 1) + 2j + 2n + (i - 4) \text{ for } i = 1, 3, \ldots, n \text{ and } 1 \leq j \leq r. \tag{3.17}$$

From the equations (3.13) to (3.17), it is observed that the vertex labels of G are distinct. The edge labels of G are labeled as follows:

$$|\,g(v)-g(u_i)\,| = |\,2q-2i+1\,| \text{ for } 1 \le i \le r \tag{3.18}$$

$$|\,g(w_{2i})-g(w_{2i+1})\,| = |\,2n-4i-1\,| \text{ for } 1 \le i \le \frac{n-1}{2} \tag{3.19}$$

$$|\,g(w_{2i-1})-g(w_{2i})\,| = |\,2n-4i+1\,| \text{ for } 1 \le i \le \frac{n+1}{2} \tag{3.20}$$

$$|\,g(v)-g(w_{2i-1})\,| = |\,2q-2r-2i+1\,| \text{ for } 1 \le i \le \frac{n+1}{2} \tag{3.21}$$

$$|\,g(w_{2i-1})-g(w_i^j)\,| = |\,2r-2j-2n-r(i+1)+3\,| \text{ for}$$

$$1 \le i \le \frac{n+1}{2},\ 1 \le j \le r \tag{3.22}$$

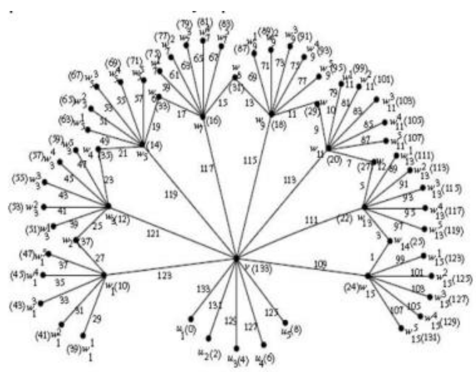

Figure 5. Odd grace full a being of Sub divided Shell Graph with star graph when $m = 7$, $r = 5$, an $n = 15$.

Let E_1, E_2, E_3, E_4 and E_5 be the sets of the edge labels given in equations (3.18) to (3.22) respectively.

$$E_1 = (2q-1), (2q-3), \ldots, (2q-2r+1) \tag{3.23}$$

$$E_2 = (2n-5), (2n-9), \ldots, 1 \tag{3.24}$$

$$E_3 = (2n-3), (2n-7), \ldots, 3 \tag{3.25}$$

$$E_4 = (2q-2r-1), (2q-2r-3), \ldots, (2q-2r-n) \tag{3.26}$$

$$E_5 = (3-2j-2n), (3-2j-2n-r), \ldots, (2r-2j-2n-r(n+1)+3). \tag{3.27}$$

By denoting $E = E_2 \cup E_3 \cup E_5 \cup E_4 \cup E_1$, we can note that the edge labels are distinct odd numbers from the set $\{1, 3, 5, \ldots, (2q-1)\}$.

Thus, the graph G obtained by joining S_r to the apex and odd path vertices of SSG admits the odd graceful labeling. Illustration for the above theorem for a graph G with $m = 7, r = 5, n = 15, p = 61, q = 67$ and $2q - 1 = 133$, is given in Figure 5.

Theorem 3. *The Star graph S_r attached to apex and all the path vertices of the sub divided shell graph is odd graceful.*

Proof. Let G be the graph obtained by attaching star graphs S_r to the apex and all the path vertices of the Sub divided shell graph. $w_1^1, w_1^2, w_1^3, \ldots, w_1^j, (j = 1, 2, \ldots, r)$ be the pendant vertices of S_r attached to w_1. Let $w_2^1, w_2^2, w_2^3, \ldots, w_2^j (j = 1, 2, \ldots, r)$ be the pendant vertices of S_r attached to w_2.

Similarly, let $w_n^1, w_n^2, w_n^3, \ldots, w_n^j (j = 1, 2, \ldots, r)$ be attached to w_n.

The graph G has $p = r(n+1) + 2(m+1)$ vertices and $q = r(n+1) + (3m+1)$ edges. The vertex label forth apex of SSG:

$$g(v) = 2q - 1 \tag{3.28}$$

$$g(u_i) = 2(i-1) \text{ for } 1 \leq i \leq r. \tag{3.29}$$

The vertex labels for the path vertices of SSG:

$$g(wi) = 2r + (i-1) \text{ for } i = 2k+1 \text{ and } 1 \leq k \leq \frac{n+1}{2} \tag{3.30}$$

$$g(wi) = 2r + 2n - (2i+1) \text{ for } i = 2k \text{ and } 1 \leq k \leq \frac{n-1}{2}. \tag{3.31}$$

The vertex labels for pendant vertices of S_r attached to w_i (i is odd):

$$g(w_i^j) = r + 4i + 2j + 4n \text{ for } i = 1, 2, 5, \ldots, n \text{ and } 1 \leq j \leq r. \tag{3.32}$$

The vertex labels for pendant vertices of S_r attached to w_i (i is even):

$$g(w_i^j) = 5r - 4i - 2j + 5n \text{ for } i = 2, 4, \ldots, n-1 \text{ and } 1 \leq j \leq r \tag{3.33}$$

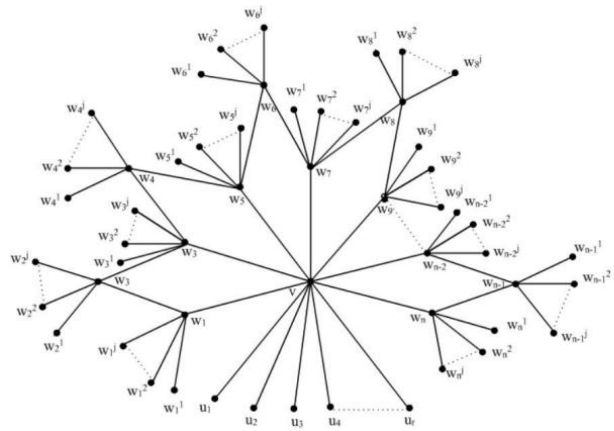

Figure 6. Sub divided Shell Graph with Star Graph attached to the apex and all the path vertices.

From the equations (3.30) to (3.33), it is observed that the vertex labels of G are distinct. The edge labels of G are labeled as follows:

$$|\, g(v) - g(u_i) \,| = |\, 2q - 2i + 1 \,| \text{ for } 1 \le i \le r \tag{3.34}$$

$$|\, g(w_{2i}) - g(w_{2i+1}) \,| = |\, 2n - 3i - 1 \,| \text{ for } 1 \le i \le \frac{n-1}{2} \tag{3.35}$$

$$|\, g(w_{2i-1}) - g(w_{2i}) \,| = |\, 2n - 3i + 2 \,| \text{ for } 1 \le i \le \frac{n-1}{2} \tag{3.36}$$

$$|\, g(v) - g(w_{2i-1}) \,| = |\, 2q - 2r - i \,| \text{ for } 1 \le i \le \frac{n+1}{2} \tag{3.37}$$

$$|\, g(w_{2i}) - g(w_{2i-1}^{j}) \,| = |\, r - 4n - 2j - 3i - 1 \,| \text{ for } 1 \le i \le \frac{n+1}{2},\ 1 \le j \le r \tag{3.38}$$

$$|\, g(w_{2i}) - g(w_{2i}^{j}) \,| = |\, 2j + 2i - 3n - 3r - 1 \,| \text{ for }$$

$$1 \le i \le \frac{n+1}{2},\ 1 \le j \le r. \tag{3.39}$$

Let E_1, E_2, E_3, E_4, E_5 and E_6 be the sets of the edge labels given in equations (2.34) to (2.39) respectively.

$$E_1 = (2q-1), (2q-3), (2q-5), (2q-7), \ldots, (1 + 2q - 2r) \tag{3.40}$$

$$E_2 = (2n-4), (2n-7), (2n-10), (2n-13), \ldots, \left(\frac{(n+1)}{2}\right) \tag{3.41}$$

$$E_3 = (2n-1), (2n-4), (2n-7), (2n-10), \ldots, \left(\frac{(n+7)}{2}\right) \tag{3.42}$$

$$E_4 = (2q-2r-1), (2q-2r-3), \ldots, \left(2q-2r-\left(\frac{(n+1)}{2}\right)\right) \tag{3.43}$$

$$E_5 = (r-2j-4n-4), (r-2j-4n-7), \ldots, \left(r-2j-4n-\left(\frac{3(n+1)}{2}\right)\right) \tag{3.44}$$

$$E_6 = (2j-3r-3n+1), (2j-3r-3n+3), \ldots, (2j-3r-3n+n). \tag{3.45}$$

By denoting $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$, we observe that the edge labels are distinct odd numbers from the set $\{1, 3, 5, \ldots, (2q-1)\}$. Thus, the graph G is obtained by joining S_r to the apex and all the path vertices of SSG is odd graceful. The above theorem is illustrated for a graph G with $m = 5$, $r = 3$, $n = 11$, $p = 48$, $q = 52$ and $2q-1 = 103$, is given in Figure 7.

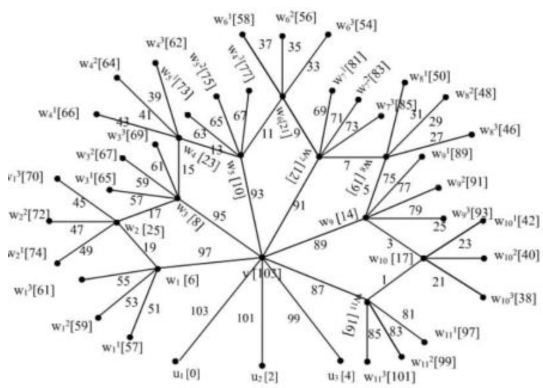

Figure 7. Sub divided Shell Graph with Star Graph when $m = 5$, $r = 3$ an $d_n = 11$.

4. Conclusion

In this paper we have proved that the star graphs attached to apex and all the path vertices of the subdivided shell graph is odd graceful. Further research can be carried out on this result on other variations in graph labeling.

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