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OBSERVATIONS ON TERNARY QUADRATIC EQUATION $3x^2 + 2y^2 = 275z^2$

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Abstract

The Ternary Quadratic Diophantine Equation $3x^2 + 2y^2 = 275z^2$ is analyzed for its infinite number of non-zero integral solutions.

Introduction

Diophantine equations have stimulated the interest of various mathematicians. Diophantine equations with higher degree greater than three can be reduced into equations of degree 2 or 3 and it can be easily solved. In [1-3], theory of numbers is discussed. In [4-5], quadratic Diophantine equations are discussed. In [6-11], cubic, biquadratic and higher order equations are considered for its integral solutions.

This communication concerns with yet another interesting ternary quadratic equation $3x^2 + 2y^2 = 275z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

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Notations

$T_{6,\,n} = n(2n-1) =$ Hexagonal number of rank n

$T_{8,\,n} = n(3n-2) =$ Octagonal number of rank n

$T_{10,\,n} = n(4n-3) =$ Decagonal number of rank n

$T_{12,\,n} = n(5n-4) =$ Dodecagonal number of rank n

$T_{14,\,n} = n(6n-5) =$ Tetradecagonal number of rank n

$T_{16,\,n} = n(7n-6) =$ Hexadecagonal number of rank n

$T_{18,\,n} = n(8n-7) =$ Octadecagonal number of rank n

$T_{20,\,n} = n(9n-8) =$ Icosagonal number of rank n

$T_{22,\,n} = n(10n-9) =$ Icosidigonal number of rank n

$T_{24,\,n} = n(11n-10) =$ Icositetragonal number of rank n

$T_{26,\,n} = n(12n-11) =$ Icosihexagonal number of rank n

$T_{28,\,n} = n(13n-12) =$ Icosioctagonal number of rank n

$T_{30,\,n} = n(14n-13) =$ Triacontagonal number of rank n

$Cs_n = n^2 + (n-1)^2 =$ Centered Square number of rank n

$GnO_n = (2n-1) =$ Gnomonic number of rank n

Method of Analysis

The ternary quadratic Diophantine equation to be solved for its non-zero integral solutions is

$$3x^2 + 2y^2 = 275z^2. \tag{1}$$

The substitution of linear transformations

$$x = X + 2T \text{ and } y = X - 3T \tag{2}$$

In (1) leads to,

$$X^2 + 6T^2 = 55Z^2. \tag{3}$$

Pattern 1.

Assume,

$$z = z(a, b) = a^2 + 6b^2 \tag{4}$$

where a and b are non-zero integers

$$55 = (1 + 3i\sqrt{6})(1 - 3i\sqrt{6}). \tag{5}$$

Using (4) and (5) in (3), and using factorization method,

$$(X + i\sqrt{6}T)(X - i\sqrt{6}T) = (1 + 3i\sqrt{6})(1 - 3i\sqrt{6})(a + i\sqrt{6}T)^2(a - i\sqrt{6}b)^2. \tag{6}$$

Equating the like terms of the equation and comparing the real and imaginary parts, we get

$$X = a^2 - 36ab - 6b^2,$$

$$T = 3a^2 + 2ab - 18b^2.$$

Substituting the values of X and T in equation (2) the corresponding integer solutions are,

$$x = x(a, b) = 7a^2 - 32ab - 18b^2,$$

$$y = y(a, b) = -8a^2 - 42ab + 48b^2,$$

$$z = z(a, b) = a^2 + 6b^2.$$

Observations

1. $2x(a, a) - 65y(a, a) + 7z(a, a) = 49a^2$ Perfect square.
2. $x(a, a) - y(a, a) - 9T_{16, a} \equiv 0 \pmod 6$
3. $2x(a, a) - 65y(a, a) + 42z(a, a) = 49a^2$ Nasty number.
4. $2x(a, a) + 65y(a, a) + z(a, a) - T_{16, a} \equiv 0 \pmod 6$
5. $y(a, a) - x(a, a) - 21T_{8, a} \equiv 0 \pmod 2$

6. $y(a, a) - x(a, a) - T_{8, a} - 10T_{14, a} - 26Gno_a \equiv 0 \pmod{26}$

7. $y(a, a) - x(a, a) - 9T_{16, a} \equiv 0 \pmod{6}$

8. $y(a, a) - 6z(a, a) - 8T_{12, a} - 16Gno_a \equiv 0 \pmod{16}$

9. $z(a, a) - x(a, a) - 9T_{18, a} \equiv 0 \pmod{63}$

10. $4z(a, a) - y(a, a) - 15T_{6, n} \equiv 0 \pmod{15}$.

Pattern 2.

Equation (3) can be written as,

$$55 = (7 + i\sqrt{6})(7 - i\sqrt{6}). \qquad (7)$$

Using (4) and (7) in (3), and using factorization method

$$(X + i\sqrt{6}T)(X - i\sqrt{6}T) = (7 + i\sqrt{6})(7 - i\sqrt{6})(a + i\sqrt{6}b)^2(a - i\sqrt{6}b)^2. \qquad (8)$$

Equating the like terms and comparing the real and imaginary parts, we get

$$X = 7a^2 - 12ab - 42b^2,$$

$$T = a^2 + 14ab - 6b^2.$$

Substituting the values of X and T in equation (2) the corresponding integer solutions are,

$$x = x(a, a) = 9a^2 + 16ab - 26b^2,$$

$$y = y(a, a) = 10a^2 + 30ab - 32b^2,$$

$$z = z(a, a) = a^2 + 6b^2.$$

Observations

1. $8x(a, a) + y(a, a) + 7z(a, a)$ Perfect square.

2. $8x(a, a) + y(a, a) + 7z(a, a)$ is a nasty number.

3. $y(a, a) - x(a, a) - 3T_{8, a} \equiv 0 \pmod{6}$.

4. $y(a, a) - x(a, a) - T_{12, a} - T_{10, a} \equiv 0 (\text{mod } 7)$.

5. $y(a, a) - x(a, a) - T_{20, a} \equiv 0 (\text{mod } 8)$.

6. $y(a, a) - x(a, a) - T_{8, a} - T_{14, a} \equiv 0 (\text{mod } 7)$.

7. $8y(a, a) + 6z(a, a) - 53T_{6, a} \equiv 0 (\text{mod } 53)$.

8. $9y(a, a) - 6x(a, a) - 22T_{8, a} - 22Gno_a \equiv 0 (\text{mod } 22)$.

9. $4y(a, a) + 4z(a, a) - 20T_{8, a} - 20Gno_a \equiv 0 (\text{mod } 20)$.

10. $5x(a, a) + 5y(a, a) + 5z(a, a) - 10T_{16, a} - 30Gno_a \equiv 0 (\text{mod } 30)$.

Pattern 3.

55 can be written as,

$$55 = \frac{(37 + i\sqrt{6})(37 - i\sqrt{6})}{5^2}. \tag{9}$$

Using (4) and (9) in (3),

$$(X + i\sqrt{6}T)(X - i\sqrt{6}T) = \frac{1}{5}(37 + i\sqrt{6})(37 - i\sqrt{6})(a + i\sqrt{6}b)^2(a - i\sqrt{6}b)^2. \tag{10}$$

Equating the like terms and comparing the real and imaginary parts, we get

$$U = \frac{1}{5}(37a^2 - 12ab - 222b^2),$$

$$V = \frac{1}{5}(a^2 - 74ab - 6b^2).$$

Since our interest is on finding integer solutions, we choose a and b suitably. So that X and T are integers. Let us take $a = 5A, b = 5B$ then the values of

$$X = X(A, B) = 185A^2 - 60AB - 1110B^2,$$

$$T = T(A, B) = 5A^2 + 370AB - 30B^2.$$

In view of (2), the integer solutions of (1) are given by,

$$x = x(A, B) = 185A^2 - 60AB - 1110B^2,$$

$$y = y(A, B) = 5A^2 + 370AB - 30B^2,$$

$$z = z(A, B) = 25A^2 + 150B^2.$$

Observations

1. $69x(A, A) + 197y(A, A) + 7z(A, A)$ is a perfect square.

2. $69x(A, A) + 197y(A, A) + z(A, A) - T_{16, a} \equiv 0(\text{mod } 150)$

3. $y(A, A) - x(A, A) - 665T_{6, a} \equiv 0(\text{mod } 665)$

4. $y(A, A) - x(A, A) - 133T_{22, a} \equiv 0(\text{mod } 1197)$

5. $69x(A, A) + 197y(A, A) + 2z(A, A) - 30T_{22, a} - 25T_{6, a} \equiv 0(\text{mod } 295)$

6. $y(A, A) - z(A, A) - 104T_{12, a} - 208Gno_a \equiv 0(\text{mod } 208)$

7. $y(A, A) - x(A, A) + 2z(A, A) - 100T_{22, a} - 85T_{18, a} \equiv 0(\text{mod } 85)$

8. $y(A, A) - x(A, A) + 4z(A, A) - 10T_{14, a} - 25Gno_a \equiv 0(\text{mod } 25)$

9. $y(A, A) - x(A, A) - 100T_{12, a} - 125T_{10, a} - 110T_{8, a} \equiv 0(\text{mod } 995)$

10. $y(A, A) - x(A, A) + 2z(A, A) - 140T_{26, a} - 770Gno_a \equiv 0(\text{mod } 770)$

Pattern 4.

55 can be written as,

$$55 = \frac{(47 + i\sqrt{6})(47 - i\sqrt{6})}{7}. \tag{11}$$

Using (4) and (11) in (3),

$$(X + i\sqrt{6}T)(X - i\sqrt{6}T) = \frac{1}{7}(47 + i\sqrt{6})(47 - i\sqrt{6})(a + i\sqrt{6}b)^2(a - i\sqrt{6}b)^2. \tag{12}$$

Equating the like terms and comparing the real and imaginary parts, we get

$$U = \frac{1}{7}(47a^2 - 108ab - 282b^2),$$

$$V = \frac{1}{7}(9a^2 + 94ab + 54b^2).$$

Since our interest is on finding integer solutions, we choose a and b suitably. So that X and T are integers. Let us take $a = 7A$ and $b = 7B$

$$X = X(A, B) = 329A^2 - 756AB - 1974B^2,$$

$$T = T(A, B) = 63A^2 + 658AB - 378B^2.$$

In view of (2), the integer solutions of (1) are given by,

$$x = x(A, B) = 329A^2 - 756AB - 1974B^2,$$

$$y = y(A, B) = 63A^2 + 658AB - 378B^2,$$

$$z = z(A, B) = 49A^2 + 294B^2.$$

Observations

1. $z(A, A)$ is a cubic integer, $y(A, A)$ is a cubic integer.

2. $y(A, A) - x(A, A) - 196T_{30, a} \equiv 0 \pmod{2548}$

3. $y(A, A) - x(A, A) - 1372Cs_a - 1372Gno_a \equiv 0 \pmod{1372}$

4. $y(A, A) - x(A, A) - 686T_{10, a} \equiv 0 \pmod{2058}$

5. $y(A, A) - x(A, A) - 540T_{12, a} - 22Cs_a - 22Gno_a \equiv 0 \pmod{4}$

6. $y(A, A) + z(A, A) - 98T_{16, a} \equiv 0 \bmod (558)$

7. $2y(A, A) + z(A, A) - x(A, A) - 245T_{30, a} \equiv 0 \bmod (13)$

8. $y(A, A) + z(A, A) - 98T_{16, a} - 294Gno_{a,} \equiv 0 \bmod (294)$

9. $10z(A, A) + y(A, A) - x(A, A) - 441T_{30, a} \equiv 0 \bmod (5733)$

10. $y(A, A) + 3z(A, A) - 343T_{10, a} \equiv 0 \pmod{1029}$.

Conclusion

To conclude, one may search for other patterns of solutions and their corresponding properties.

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