Advances and Applications in Mathematical Sciences
Volume 21, Issue 6, April 2022, Pages 3431-3443

SOFT R-METRIC SPACES AND SOME OF ITS FIXED POINT RESULTS

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Abstract

Here, we have presented soft R-metric space built on soft elements, by the help of R-metric space. After that, some definitions and properties regarding soft R-metric spaces are discussed with examples. Finally, in soft R-metric spaces, various significant fixed point theorems have been established.

1. Introduction

Molodtsov [3] was the one who came up with the idea for the soft set, an alternative way for handling uncertainties adhering real life situations unlocked broad areas to the researchers. As a result, researchers have started to formalized different mathematical structures viz. group, ring, field, vector space, metric space, normed linear space, topology, etc. in this setting, and it progressed very fast. Das and Samanta [8, 11] proposed the thought of soft real numbers, soft metric and also studied some of its properties. After that, several fixed point theorems have conferred in [2, 4, 6, 7].

2020 Mathematics Subject Classification: Primary 54E50; Secondary 47H10, 06D72, 54E99.
Keywords: Soft set, Soft R-metric space, Fixed soft element.
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Received: August 31, 2021; Accepted December 19, 2021

R-metric space was first introduced by S. Khalehoghli, H. Rahimi and M. Eshaghi Gordji [12] in 2020, and they also proved some fixed point results with applications.

With the help of R-metric space, we define soft R-metric space in this study. Following that, we provide various definitions and results for soft R-metric space, along with appropriate examples. Finally, we used soft mapping [1, 9] to prove a number of valuable fixed point results in this setting.

2. Preliminaries

We present certain definitions and preliminary [3, 5, 10, 11, 12] results in this area, which are important for the main discussions.

Definition 2.1. Accepts a function $F: A^* \to P(X)$, with $P(X)$, power set of $X(\neq \phi)$ and $A^*(\neq \phi) \subseteq E$, representing a set of parameters. At that time (F, A^*) is named as soft set over X.

Definition 2.2. Assume $X(\neq \phi)$ and $A^*(\neq \phi)$. Then

- $\widetilde{X}$ is named absolute soft set over X wherever $\tilde{X}(b^*) = X, \forall\, b^* \in A^*$.
- a soft real number $\widetilde{r}$ is a soft set (F, A^*) over $\mathbb{R}$, where $F(b^*) \in \mathbb{R}, \forall\, b^* \in A^*$.

In particular, if $F(b^*) = s \in \mathbb{R}, \forall\, b^* \in A^*$, then this soft real number is called constant soft real number and represented by $\bar{s}$.

Definition 2.3. A soft element on X is a function ε from A^* to X and $\varepsilon \,\widetilde{\in}\, (F, A^*)$ in the case of $\varepsilon(b^*) \in F(b^*), \forall\, b^* \in A^*$.

3. Soft R-metric Spaces

We present a soft R-metric space and a few basic properties are discussed. Also, some fixed point results are established in this space.

Definition 3.1. Assume that a soft metric space $(\widetilde{X}, d, A)$ and a relation R on $\widetilde{X}$. At that moment we called $(\widetilde{X}, d, E, R)$ is a soft R-metric space.

Example 3.2. Express $d : SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ as,

$$d(\widetilde{p^*}, \widetilde{q^*}) = \begin{cases} \overline{0}, & \text{if } \widetilde{p^*} = \widetilde{q^*} \\ \overline{1}, & \text{if } \widetilde{p^*} \neq \widetilde{q^*}, \end{cases}$$

where $\widetilde{X}$ stands an absolute soft set. $(\widetilde{X}, d, E)$ is then a discrete soft metric space, explained in Example 2.2.2 of reference [6].

Now, for each $\gamma^* \in E$, define $\widetilde{p}(\gamma^*) R \widetilde{q}(\gamma^*)$ if $\widetilde{p}(\gamma^*) \cdot \widetilde{q}(\gamma^*) \geq \widetilde{p}(\gamma^*)$ or $\widetilde{q}(\gamma^*)$, $\forall\ \widetilde{p}, \widetilde{q} \,\tilde{\in}\, SE(\widetilde{X})$. Here, $(\widetilde{X}, d, E, R)$ is a soft R-metric space.

Remark 3.3. The conception of soft R-metric space makes it obvious that,

- if $R(\neq \phi)$ is soft R-metric space, it is soft metric space also.
- if we take $R = (\widetilde{X}, E) \times (\widetilde{X}, E)$, then the soft R-metric space $(\widetilde{X}, d, E, R)$ is equivalent to the soft metric space $(\widetilde{X}, d, E)$.

Definition 3.4. A R-sequence is a sequence of soft elements $\{\widetilde{x_t}\}$ of a soft R-metric space $(\widetilde{X}, d, E, R)$ such that for each $\gamma^* \in E$,

$$\widetilde{x_t}(\gamma^*) R \widetilde{x_{t+k}}(\gamma^*), \forall t, k \in \mathbb{N}.$$

Example 3.5. Take the discrete soft metric space $(\widetilde{X}, d, E)$ described in Example 3.2. Take $E = \{\alpha_1, \alpha_2\}$. Now, for each $\alpha \in E$, define $\widetilde{x_r}(\alpha) R \widetilde{x_s}(\alpha)$ if $\widetilde{x_r}(\alpha) \geq \widetilde{x_s}(\alpha), \forall\ \tilde{x}, \tilde{y} \,\tilde{\in}\, SE(\widetilde{X}); r < s$. Therefore, if we consider $\widetilde{x_t}(\alpha_j) = 1 + \frac{j}{t}; j = 1, 2,$ then $\{\widetilde{x_t}\}$ is a soft R-sequence. But, if $\widetilde{x_t}(\alpha_j) = 1 + \frac{j}{t}; j = 1, 2,$ then $\{\widetilde{x_t}\}$ is not a soft R-sequence.

Definition 3.6. In a soft R-metric space $(\widetilde{X}, d, E, R)$, let $\{\widetilde{x_t}\}$ stand a soft R-sequence. Then we claim $\{\widetilde{x_t}\}$ converges to a soft element $\widetilde{y} \,\tilde{\in}\, SE(\widetilde{X})$ if $\forall\, \alpha \in E, [d(\widetilde{x_t}, \tilde{y})](\alpha) \to 0,$ as $t \to \infty$. This implies that for every $\tilde{\epsilon} \,\tilde{>}\, \overline{0}$ and for all $\alpha \in E, \exists$ a $N_1 (= N_{\epsilon_\alpha}) \in \mathbb{N}$, so that $[d(\widetilde{x_t}, \tilde{y})](\alpha) < \tilde{\epsilon}(\alpha)$, whenever $t > N_1$. We denote it as $\widetilde{x_t} \to \tilde{y}$, as $t \to \infty$.

Example 3.7. Take $X = \mathbb{R}$, set of real numbers. Define $d : SE(\widetilde{X}) \times SE(\widetilde{X}) \in \mathbb{R}(E)^{*}$ as $[d(\widetilde{p}, \widetilde{q})](\alpha_1) = [| \widetilde{p}, \widetilde{q} |](\alpha_1), \forall \alpha_1 \in E$. $(\widetilde{X}, d, E)$ is then a soft metric space. Define $\{\widetilde{x_t}\}$ of $(\widetilde{X}, d, E)$ by $\widetilde{x_t}(\alpha_1) = \frac{1}{t}, \forall t \in \mathbb{N}$ and $\forall \alpha_1 \in E$. Now, for each $\alpha_1 \in E$, define $\widetilde{x_r}(\alpha_1) R \widetilde{x_s}(\alpha_1)$ if $\widetilde{x_r}(\alpha_1) \cdot \widetilde{x_s}(\alpha_1) \geq \widetilde{x_r}(\alpha_1)$ or $\widetilde{x_s}(\alpha_1), \forall r, s \in \mathbb{N}$. $(\widetilde{X}, d, E, R)$, is then a soft R-metric space and $\widetilde{x_t} \xrightarrow{R} \overline{0}$. Again, if we consider $E = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\widetilde{x_t}(\alpha_j) = 2 + \frac{j}{t}; t \in \mathbb{N}, j = 1, 2, 3, 4$. Then $\widetilde{x_t} \xrightarrow{R} \overline{2}$, as $t \to \infty$.

Definition 3.8. Consider a soft R-metric space $(\widetilde{X}, d, E, R)$. If there occurs a soft R-sequence $\{\widetilde{x_t}\}$ in (X, E) such that $x_t \neq \widetilde{y}, \forall t \in \mathbb{N}$ and $\widetilde{x_t} \xrightarrow{R} \widetilde{y}$ as $t \to \infty$, then a soft element $\widetilde{y} \widetilde{\in} SE(\widetilde{X})$ is termed a "soft R-limit of (X, E)". This implies for any $\widetilde{\in} \widetilde{>} \overline{0}$ and for any $\alpha \in E$, $\exists N_1 (= N_{\in_\alpha}) \in \mathbb{N}$, for which $[d(\widetilde{x_t}, \widetilde{y})](\alpha) < \widetilde{\in}(\alpha)$, whenever $t > N_1$.

Example 3.9. Select the soft R-metric space $(\widetilde{X}, d, E, R)$ from Example 3.7. Choose $(Y, E) \widetilde{\subset} \widetilde{\mathbb{R}}$ where $y(\alpha) = (0, 1], \forall \alpha \in E$ and a R-sequence $\{\widetilde{x_t}\}$ in $(\widetilde{Y}, d, E, R)$ with $\widetilde{x_t}(\alpha) = \frac{1}{t}, \forall t \in \mathbb{N}, \alpha \in E$. Then there is no such $\widetilde{x} \widetilde{\in} SE(\widetilde{Y})$ for which $\widetilde{x_t} \xrightarrow{R} \widetilde{x}$ in $(\widetilde{Y}, d, E, R)$. But if we consider a R-sequence $\{\widetilde{y_t}\}$ in $(\widetilde{Y}, d, E, R)$ where $\widetilde{y_t}(\alpha) = \frac{1}{4}, \forall t \in \mathbb{N}, \alpha \in E$. Then $\widetilde{y_t} \xrightarrow{R} \overline{\frac{1}{4}}$ as $t \to \infty$. Again, let $E = \{\alpha_1, \alpha_2, \alpha_3\}$ and let $\{\widetilde{x_t}\}$ be a R-sequence of $(\widetilde{Y}, d, E, R)$ where $\widetilde{x_t}(\alpha_j) = \frac{1}{2} + \frac{j}{t}, \forall t \in \mathbb{N}, j = 1, 2, 3$. Then $\widetilde{x_t} \xrightarrow{R} \overline{\frac{1}{2}}$ as $t \to \infty$.

Theorem 3.10. *A sequence's limit in a soft R-metric space $(\widetilde{X}, d, E, R)$ is unique if it exists.*

Proof. Pick $\{\widetilde{x_t}\}$ from $(\widetilde{X}, d, E, R)$, where $\widetilde{x_t} \xrightarrow{R} \widetilde{p}$ and $\widetilde{x_t} \xrightarrow{R} \widetilde{q}$

with $\tilde{p} \neq \tilde{q}$. Then there exist at least one $\gamma^* \in E$, for which $\tilde{p}(\gamma^*) \neq \tilde{q}(\gamma^*)$. i.e., $[d(\tilde{p}, \tilde{q})](\gamma^*) \neq 0$. Take $\tilde{\in} \tilde{>} \bar{0}$ with $\tilde{\in}(\gamma^*) = \in_{\gamma^*}$. Now, since $\widetilde{x_t} \xrightarrow{R} \tilde{p}$ and $\widetilde{x_t} \xrightarrow{R} \tilde{q}$, so corresponding to every $\tilde{\in} \tilde{>} \bar{0}$, $\exists$ natural numbers $K_1 = K_1(\tilde{\in})$ and $K_2 = K_2(\tilde{\in})$ such that,

$$[d(\widetilde{x_t}, \tilde{p})](\gamma^*) < \tilde{\in}(\gamma^*), \text{ whenever } t > K_1$$

$$\text{and } [d(\widetilde{x_t}, \tilde{q})](\gamma^*) < \tilde{\in}(\gamma^*), \text{ whenever } t > K_2.$$

Take $K = \max\{K_1, K_2\}$. Then $[d(\widetilde{x_t}, \tilde{p})](\gamma^*) < \tilde{\in}(\gamma^*)$, whenever $t > K$ and $[d(\widetilde{x_t}, \tilde{q})](\gamma^*) < \tilde{\in}(\gamma^*)$, whenever $t > K$. i.e., $[d(\widetilde{x_t}, \tilde{p})](\gamma^*) < \tilde{\in}_{\gamma^*}$ whenever $t > K$ and $[d(\widetilde{x_t}, \tilde{q})](\gamma^*) < \tilde{\in}_{\gamma^*}$ whenever $t > K$.

Now, $[d(\tilde{p}, \tilde{q})](\gamma^*) \leq [d(\widetilde{x_t}, \tilde{p})](\gamma^*) + [d(\widetilde{x_t}, \tilde{q})](\gamma^*)$

$\Rightarrow [d(\tilde{p}, \tilde{q})](\gamma^*) < \in_{\gamma^*} + \in_{\gamma^*}, \forall t > K = 2 \in_{\gamma^*}$

Since, $\in_{\gamma^*} > 0$ is arbitrary, we infer that $[d(\tilde{p}, \tilde{q})](\gamma^*) = 0$, which is a contradiction. Since, $\gamma^* \in E$ is arbitrary, so $\tilde{p} = \tilde{q}$.

Definition 3.11. An R-sequence $\{\widetilde{x_t}\}$ of a soft R-metric space $(\tilde{X}, d, E, R)$ is termed "bounded", if for each $\alpha \in E, x_r(\alpha) R \widetilde{x_s}(\alpha)$ or $\widetilde{x_s}(\alpha) R \widetilde{x_r}(\alpha)$ and $\exists M > 0$ for which $[d(\widetilde{x_r}, \widetilde{x_s})](\alpha) \leq M, \forall r, s \in \mathbb{N}$.

Example 3.12. The sequence $\{\widetilde{x_t}\}$ in $(\tilde{X}, d, E, R)$ created in example 3.7 is bounded.

Definition 3.13. If each soft R-sequence in $\tilde{X}$ has a subsequence that is R-converges to a soft element in $\tilde{X}$, the soft R-metric space $(\tilde{X}, d, E, R)$ is termed "R-compact".

Example 3.14. Select $(\tilde{X}, d, E, R)$ from example 3.7.

Take $E = \{\alpha_1, \alpha_2, \alpha_3\}$. Let $(Y, E) \tilde{\subset} \tilde{\mathbb{R}}$, where $Y(\alpha_j) = [0, 1], \forall \alpha_j \in E$; $j = 1, 2, 3$. Now, choose a soft R-sequence $\{\widetilde{y_t}\}$ in (Y, E) such that $\widetilde{y_t}(\alpha_j) = \frac{1}{n}, t \in \mathbb{N}, j = 1, 2, 3$. Then $(\tilde{Y}, d, E, R)$ is compact soft R-metric space.

Definition 3.15. An R-sequence $\{\widetilde{x_t}\}$ in $(\tilde{X}, d, E, R)$ is R-Cauchy, if for each $\alpha \in E$ and $\tilde{\in} > \bar{0}$, $\exists$ a number $n(\in \mathbb{N})$, for which $[d(\widetilde{x_r}, \widetilde{x_s})](\alpha) < \tilde{\in}(\alpha), \forall r, s \geq n$ and $\widetilde{x_r}(\alpha) R \widetilde{x_s}(\alpha)$ or $\widetilde{x_s}(\alpha) R \widetilde{x_r}(\alpha)$.

i.e., for every $\alpha \in E, [d(\widetilde{x_r}, \widetilde{x_s})](\alpha) \to 0$ as $r, s \to \infty$, and $\widetilde{x_r}(\alpha) R \widetilde{x_s}(\alpha)$ or $\widetilde{x_s}(\alpha) R \widetilde{x_r}(\alpha)$.

Example 3.16. In a soft R-metric space $(\tilde{Y}, d, E, R)$ defined in example 3.14, for any $\alpha \in E, [d(\widetilde{y_r}, \widetilde{y_s})](\alpha) \to 0$, as $r, s \to \infty$.

So, the soft R-sequence $\{\widetilde{y_t}\}$ is a R-Cauchy sequence.

Theorem 3.17. *Each convergent R-sequence in a soft R-metric space is an R-Cauchy sequence.*

Proof. In $(\tilde{X}, d, E, R)$ assume $\{\widetilde{x_t}\}$, a convergent R-sequence with a limit $\tilde{p} \tilde{\in} SE(\tilde{X})$. Then for every $\tilde{\in} > \bar{0}$ and $\alpha \in E, \exists$ a number $N = N_{\in_\alpha}(\in \mathbb{N})$ with, $[d(\widetilde{x_t}, \tilde{p})](\alpha) < \frac{\tilde{\in}}{2}(\alpha)$, whenever $t > N$. Then for $r^*, s^* \geq N$ with $\widetilde{x_{r^*}}(\alpha) R \widetilde{x_{s^*}}(\alpha)$ we have,

$$[d(\widetilde{x_{r^*}}, \widetilde{x_{s^*}})](\alpha) \leq [d(\widetilde{x_{r^*}}, \tilde{p})](\alpha) + [d(\tilde{p}, \widetilde{x_{s^*}})](\alpha) < \frac{\tilde{\in}}{2}(\alpha) + \frac{\tilde{\in}}{2}(\alpha) = \tilde{\in}(\alpha).$$

Therefore, $\{\widetilde{x_t}\}$ is an R-Cauchy sequence, as $\alpha \in E$ is arbitrary.

Theorem 3.18. L*et* $\{\widetilde{x_t}\}$ *be an R-Cauchy sequence in* $(\tilde{X}, d, E, R)$ *with a subsequence that R-converges to* $\tilde{p} \tilde{\in} SE(\tilde{X})$. *Then* $\{\widetilde{x_t}\}$ *is R-converges to* $\tilde{p}$.

Proof. Let $\{\widetilde{x_{kt}} : k_1 < k_2 < k_3 < \ldots\}$ be a subsequence of an R-Cauchy sequence $\{\widetilde{x_t}\}$ in $(\widetilde{X}, d, E, R)$ which R-converges to $\widetilde{p} \,\widetilde{\in}\, SE(\widetilde{X})$. Then corresponding to any $\widetilde{\in} \,\widetilde{>}\, \overline{0}$, $\exists n \in \mathbb{N}$ so that for every $\gamma^* \in E$,

$$[d(\widetilde{x_{k_1}}, \tilde{p})](\gamma^*) < \frac{\tilde{\in}}{2}(\gamma^*), \forall\, i \geq n \tag{1}$$

Again, since $\{\widetilde{x_t}\}$ is a R-Cauchy sequence, so $\exists$ a natural number r so that $\widetilde{x_1}(\gamma^*) R \widetilde{x_J}(\gamma^*)$ and

$$[d(\widetilde{x_1}, \widetilde{x_J})](\gamma^*) < \frac{\tilde{\in}}{2}(\gamma^*), \forall\, i, j \geq r \tag{2}$$

Now, let $s = \max\{n, r\}$ and $\widetilde{x_1}(\gamma^*) R \widetilde{x_{k_1}}(\gamma^*)$. Then $\forall\, i \geq s$, we have

$$[d(\widetilde{x_1}, \tilde{p})](\gamma^*) \leq [d(\widetilde{x_1}, \widetilde{x_{k_1}})](\gamma^*) + [d(\widetilde{x_{k_1}}, \tilde{p})](\gamma^*) < \frac{\tilde{\in}}{2}(\gamma^*) + \frac{\tilde{\in}}{2}(\gamma^*) = \tilde{\in}(\gamma^*).$$

Since, $\gamma^* \in E$ is arbitrary, so the R-sequence $\{\widetilde{x_t}\}$ is converges to $\widetilde{p}(\widetilde{\in}\, SE(\widetilde{X}))$.

Definition 3.19. If each R-Cauchy sequence in $(\widetilde{X}, d, E, R)$ is R-converges to a soft element in $(\widetilde{X}, d, E, R)$, the soft R-metric space $(\widetilde{X}, d, E, R)$ is termed "R-complete".

Example 3.20. In example 3.14, $(\widetilde{X}, d, E, R)$ is a R-complete metric space.

Theorem 3.21. *Each soft R-metric space that is R-compact is R-complete.*

Proof. Assume a compact soft R-metric space $(\widetilde{X}, d, E, R)$. So in $(\widetilde{X}, d, E, R)$, every R-sequence has a convergent subsequence, which is an R-Cauchy (as of Theorem 3.17). Hence every R-sequence in $(\widetilde{X}, d, E, R)$ has a R-Cauchy subsequence.

Now, to prove $(\widetilde{X}, d, E, R)$ is R-complete, let $\{\widetilde{x_t}\}$ be a R-Cauchy sequence.

Then from R-compactness, $\{\widetilde{x_t}\}$ has a R-convergent subsequence. i.e., The given R-Cauchy sequence is converges to some soft element in $\widetilde{X}$ (from Theorem 3.18). Therefore, $(\widetilde{X}, d, E, R)$ is soft R-complete.

Corollary 3.22. *It's possible that the converse of the previous theorem is incorrect. e.g., Select* $(\widetilde{\mathbb{R}}, d, E, R)$, *where* $\mathbb{R}$ *is endowed with the usual metric* d, *and* $(\mathbb{R}, d, E)$ *is generated by the crisp metric space* $(\mathbb{R}, d)$. *Then* $(\widetilde{\mathbb{R}}, d, E, R)$ *is R-complete, as every R-Cauchy sequence in* $(\widetilde{\mathbb{R}}, d, E, R)$ *is R-converges to some soft element in* $\widetilde{\mathbb{R}}$. *But* $(\widetilde{\mathbb{R}}, d, E, R)$ *is not compact, as the R-sequence* $\{\widetilde{x_t}\}$ *where* $\widetilde{x_t}(\alpha_i) = ti, \forall\, \alpha_i \in E$, *has no R-convergent subsequence.*

Definition 3.23. A function $f : (\widetilde{X}, d, E) \to (\widetilde{X}, d, E)$ is termed "soft R-preserving" if $\forall\, \widetilde{p}, \widetilde{q} \mathrel{\widetilde{\in}} SE(\widetilde{X})$, and for each $\gamma^* \in E, \widetilde{p}(\gamma^*)\, R\, \widetilde{q}(\gamma^*)$ $\Rightarrow f(\widetilde{p}(\gamma^*))\, Rf\, (\widetilde{q}(\gamma^*))$.

Example 3.24. Take $X = \mathbb{R}$, set of all real numbers and $E = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Let $\widetilde{p}(\alpha_i) = i, \forall\, \widetilde{p} \mathrel{\widetilde{\in}} SE(\widetilde{X}), i = 1, 2, 3, 4$.

Now, for each $\gamma^* \in E$, define a soft relation R as $\widetilde{p}(\gamma^*)\, R\, \widetilde{q}(\gamma^*)$ if $\widetilde{p}(\gamma^*) < \widetilde{q}(\gamma^*), \forall\, \widetilde{p}, \widetilde{q} \mathrel{\widetilde{\in}} SE(\widetilde{X})$. Then $f : (\widetilde{X}, d, E) \to (\widetilde{X}, d, E)$ where $f(\tilde{p})$ $= \tilde{p} + \overline{1}$, is a soft R-preserving as $\widetilde{p}(\alpha_i) < \widetilde{q}(\alpha_j)$ for $i < j, i = 1, 2, 3$ and $j = 2, 3, 4 \Rightarrow f(\widetilde{p}(\alpha_i)) < f(\widetilde{q}(\alpha_j))$. i.e., $\widetilde{p} R \widetilde{q}, \forall\, \widetilde{p}, \widetilde{q} \mathrel{\widetilde{\in}} SE(\widetilde{X}) \Rightarrow f(\widetilde{p})\, Rf\, (\widetilde{q})$.

Definition 3.25. A function f is defined on $(\widetilde{X}, d, E, R)$ is termed soft R-continuous at $\widetilde{p} \mathrel{\widetilde{\in}} SE(\widetilde{X})$, if for every R-sequence $\{\widetilde{x_t}\}$ in $(\widetilde{X}, d, E, R)$ with $\widetilde{x_t} \xrightarrow{R} \tilde{p}$, implies $f(\widetilde{x_t}) \xrightarrow{R} f(\tilde{p})$. f is termed soft R-continuous on $(\widetilde{X}, d, E, R)$ if f is soft R-continuous at every point of $(\widetilde{X}, d, E, R)$.

Example 3.26. In Example 3.14, the function f defined on $(\widetilde{Y}, d, E, R)$ by $f(\widetilde{p}) = \widetilde{p} + \overline{1}, \forall\, \widetilde{p} \mathrel{\widetilde{\in}} SE(\widetilde{Y})$, is soft R-continuous at $\overline{0}$.

Definition 3.27. A function f is defined on $(\widetilde{X}, d, E, R)$, is termed soft R-contraction mapping in $(\widetilde{X}, d, E, R)$ if f is soft R-preserving and if $\exists\, \overline{\lambda_1}$, with $\overline{0} \,\widetilde{<}\, \overline{\lambda_1} \,\widetilde{<}\, \overline{1}$ so that $\widetilde{p} R \widetilde{q}$, $\forall\, \widetilde{p}, \widetilde{q} \,\widetilde{\in}\, SE(\widetilde{X})$ and

$$d(f(\widetilde{p}), f(\widetilde{q})) \,\widetilde{\leq}\, \overline{\lambda_1}\; d(\widetilde{p}, \widetilde{q}), \forall\, \widetilde{p}, \widetilde{q} \,\widetilde{\in}\, SE(\widetilde{X}).$$

i.e., for any $\gamma^* \in E$, $\widetilde{p}(\gamma^*) R \widetilde{q}(\gamma^*) \Rightarrow f(\widetilde{p}(\gamma^*)) R f(\widetilde{q}(\gamma^*))$, $\forall\, \widetilde{p}, \widetilde{q} \,\widetilde{\in}\, SE(\widetilde{X})$ and $[d(f(\widetilde{p}), f(\widetilde{q}))](\gamma^*) \leq \overline{\lambda_1}\,[d(\widetilde{p}, \widetilde{q})](\gamma^*)$, $\forall\, \widetilde{p}, \widetilde{q} \,\widetilde{\in}\, SE(\widetilde{X})$.

Example 3.28. Select $(\widetilde{\mathbb{R}}, d, E)$ from Example 2.3.5 of [6]. For each $\beta^* \in E$, define $\widetilde{p}(\beta^*) R \widetilde{q}(\beta^*)$ if $\widetilde{p}(\beta^*) \leq \widetilde{q}(\beta^*)$, $\forall\, \widetilde{p}, \widetilde{q} \,\widetilde{\in}\, SE(\widetilde{\mathbb{R}})$. Choose a soft R-preserving mapping f on $(\widetilde{X}, d, E, R)$ so that $f(\beta^*) = f_{\beta^*} \cdot f$ is then a soft R-contraction mapping which is already shown in Example 4.2 of [6].

Example 3.29. Let $\mathbb{R}$ is endowed with usual metric d and $(\widetilde{\mathbb{R}}, d, E)$ is a soft metric generated by the crisp metric space $(\mathbb{R}, d)$.

Choose $(X, E) \,\widetilde{\subset}\, \mathbb{R}$, where $X(\gamma^*) = [1, \infty); \gamma^* \in E$. Now, for each $\gamma^* \in E$, define $\widetilde{p}(\beta^*) R \widetilde{q}(\beta^*)$ if $\tilde{p}(\beta^*) \leq \tilde{q}(\gamma^*)$, $\forall\, \tilde{p}, \tilde{q} \,\widetilde{\in}\, SE(\tilde{X})$.

$\therefore (\widetilde{X}, d, E, R)$ is a soft R-metric space. Again, define $f : (\widetilde{\mathbb{R}}, d, E, R) \to (\widetilde{\mathbb{R}}, d, E, R)$ by $f(\widetilde{p}) = \frac{\widetilde{p}}{3}$, $\forall\, \widetilde{p}, \widetilde{q} \,\widetilde{\in}\, SE(\widetilde{X})$, where $\tilde{p}(\gamma^*) = p_{\gamma^*}$, $\forall\, \gamma^* \in E$. Then f is soft R-preserving.

Now, $[d(f(\widetilde{p}), f(\widetilde{q}))](\gamma^*) = |\, f(\widetilde{p})(\gamma^*) - f(\widetilde{q})(\gamma^*) \,|$

$$= \left| \frac{\widetilde{p}}{3}(\gamma^*) - \frac{\widetilde{q}}{3}(\gamma^*) \right|$$

$$= \left| \frac{p_{\gamma^*}}{3} - \frac{q_{\gamma^*}}{3} \right|$$

$$= \frac{1}{3} |\, \widetilde{p}(\gamma^*) - \widetilde{q}(\gamma^*) \,|$$

$$= \frac{1}{3}[d(\tilde{p}, \tilde{q})](\gamma^*)$$

Since this holds for all $\gamma^* \in E$, $d(f(\tilde{p}), f(\tilde{q})) \tilde{\leq} \overline{\gamma^*} d(\tilde{p}, \tilde{q})$, $\overline{0} \tilde{<} \overline{\gamma^*} \tilde{<} \overline{1}$.

Therefore, f is Soft R-contraction mapping.

Theorem 3.30. *Assume* $(\tilde{X}, d, E, R)$, *a complete soft R-metric space, and a soft R-contraction map ping f define on* $(\tilde{X}, d, E, R)$. *Also, suppose there exist* $\widetilde{y_0} \tilde{\in} SE(\tilde{X})$ *such that* $\widetilde{y_0} R\tilde{q}, \forall \tilde{q} \tilde{\in} f(\tilde{X})$ *and R be an equivalent relation on* $(\tilde{X}, E)$. *Then in* $(\tilde{X}, d, E, R)$, *has a unique fixed soft element.*

Proof. Choose $\widetilde{y_0} \tilde{\in} SE(\tilde{X})$ and $\widetilde{y_1} = f(\widetilde{y_0}), \widetilde{y_2} = f(\widetilde{y_1}) = f^2(\widetilde{y_0}), \ldots, \widetilde{y_t} = f(\widetilde{y_{t-1}}) = f^t(\widetilde{y_0}), \ldots, \forall t \in \mathbb{N}$.

Take $r, s \in \mathbb{N}$ with $s < r$ and $k = r - s$. Then from the given condition we have, $y_0 R f^k(y_0)$ and this implies that $f^s(y_0) R f^{s+k}(y_0)$, as f is soft R-preserving. Therefore, $\{\widetilde{y_t}\}$ is a soft R-sequence. As f is a soft R-contraction mapping, so for $\overline{\lambda_1} \tilde{\in} E$ with $\overline{0} \tilde{<} \overline{\lambda_1} \tilde{<} \overline{1}$ we have,

$$d(\widetilde{y_t}, \widetilde{y_{t+1}}) = d(f(\widetilde{y_{t-1}}), f(\widetilde{y_t})) \tilde{\leq} \overline{\lambda_1} d(\widetilde{y_{t-1}}, \widetilde{y_t}) \tilde{\leq} \overline{\lambda}_1^2 d(\widetilde{y_{t-2}}, \widetilde{y_{t-1}})$$

$$\tilde{\leq} \ldots \tilde{\leq} \overline{\lambda_1}^t d(\widetilde{y_{t-2}}, \widetilde{y_{t-1}})$$

Now, for $r^*, s^* \in \mathbb{N}$ with $r^* \geq s^*$ we have,

$$d(\widetilde{y_{s^*}}, \widetilde{y_{r^*}}) = d(\widetilde{y_{s^*}}, \widetilde{y_{s^*+1}}) + d(\widetilde{y_{s^*+1}}, \widetilde{y_{s^*+2}}) + \ldots + d(\widetilde{y_{r^*-1}}, \widetilde{y_{r^*}})$$

$$\tilde{\leq} \overline{\lambda}_1^{s^*} d(\widetilde{y_0}, \widetilde{y_1}) + \overline{\lambda_1}^{s^*+1} d(\widetilde{y_0}, \widetilde{y_1}) + \ldots + \overline{\lambda_1}^{r^*-1} d(\widetilde{y_0}, \widetilde{y_1})$$

$$= (\overline{\lambda_1}^{s^*} + \overline{\lambda_1}^{s^*+1} + \ldots + \overline{\lambda_1}^{r^*-1}) d(\widetilde{y_0}, \widetilde{y_1})$$

$$\tilde{\leq} (\overline{1} + \overline{\lambda_1} + \overline{\lambda_1}^2 + \ldots) \overline{\lambda_1}^{s^*} d(\widetilde{y_0}, \widetilde{y_1})$$

$$= \frac{\bar{\lambda}_1^{s^*}}{\bar{1} - \bar{\lambda}_1} d(\widetilde{y_0}, \widetilde{y_1})$$

Hence, $d(\widetilde{y_{s^*}}, \widetilde{y_{r^*}}) \to \bar{0}$, as $r^*, s^* \to \infty$ (since, $\bar{0} \tilde{\lesssim} \overline{\lambda_1} \tilde{\lesssim} \bar{1}$). Therefore, $\{\widetilde{y_t}\}$ is soft R-Cauchy sequence. As $(\tilde{X}, d, E, R)$, is complete, there exist $\widetilde{p_1^*} \tilde{\in} SE(\tilde{X})$ with $\widetilde{y_0} R \widetilde{p_1^*}$ so that $\widetilde{y_t} \xrightarrow{R} \widetilde{p_1^*}$. Again, as R is an equivalence relation on $(\tilde{X}, E)$, so for $\widetilde{y_0} R f(\widetilde{p_1^*})$ we have,

$$d(\widetilde{p_1^*}, f(\widetilde{p_1^*})) \tilde{\leq} d(\widetilde{p_1^*}, \widetilde{y_k}) + d(\widetilde{y_k}, f(\widetilde{p_1^*})) = d(\widetilde{p_1^*}, \widetilde{y_k}) + d(f(\widetilde{y_{k-1}}), f(\widetilde{p_1^*}))$$

$$\tilde{\leq} d(\widetilde{p_1^*}, \widetilde{y_k}) + \bar{\lambda} d(\widetilde{y_{k-1}}, \widetilde{p_1^*}) \to \bar{0}, \text{ as } k \to \infty$$

$$\Rightarrow d(\widetilde{p_1^*}, f(\widetilde{p_1^*})) = \bar{0} \Rightarrow f(\widetilde{p_1^*}) = \widetilde{p_1^*}$$

$\therefore$ f has a fixed soft element $\widetilde{p_1^*}$.

For uniqueness, let $\widetilde{p_2^*}$ be another fixed soft element of f, such that $\widetilde{y_0} R \widetilde{p_2^*}$. Then clearly $\widetilde{p_1^*} R \widetilde{p_2^*}$, as R is an equivalence relation on $(\tilde{X}, E)$.

Therefore, $d(\widetilde{p_1^*}, \widetilde{p_2^*}) = d(f(\widetilde{p_1^*}), f(\widetilde{p_2^*})) \tilde{\leq} \bar{\lambda}_1 d(\widetilde{p_1^*}, \widetilde{p_2^*})$

$\Rightarrow d(\widetilde{p_1^*}, \widetilde{p_2^*}) = \bar{0}$, as $\bar{0} \tilde{<} \bar{\lambda}_1 \tilde{<} \bar{1}. \Rightarrow \widetilde{p_1^*} = \widetilde{p_2^*}$.

Corollary 3.31. [6] *Assume* $(\tilde{X}, d, E)$, *a complete soft metric space*, *and a soft contraction mapping f defined on* $(\tilde{X}, d, E)$. *Then in* $(\tilde{X}, d, E)$, *f has a unique fixed soft element*.

Proof. Take $R = (\tilde{X}, E) \times (\tilde{X}, E)$ and fix $\widetilde{y_0} \tilde{\in} \tilde{X}$. Then clearly $\widetilde{y_0} R \tilde{q}$, $\forall\, \tilde{q} \tilde{\in} f(\tilde{X})$. Again as $(\tilde{X}, d, E)$ is complete, so it is R-complete. Also it is clear that f is soft R-contraction mapping, as it is soft contraction. Hence in $(\tilde{X}, d, E)$, f has a unique fixed soft element (from Theorem 3.30).

Theorem 3.32. *Choose a complete soft R-metric space* $(\tilde{X}, d, E, R)$ *and a* *soft mapping f on* $(\widetilde{X}, d, E, R)$ *such that* f^k *is a soft R-contraction for some* $k \in \mathbb{N}$. *Also, suppose that there exist* $\widetilde{y_0} \tilde{\in} SE(\tilde{X})$ *for which* $\widetilde{y_0} R \tilde{p}, \forall\, \tilde{p} \tilde{\in} f^k(\tilde{X})$ *and R is an equivalent relation on* $(\widetilde{X}, E)$. *Then in* $(\widetilde{X}, d, E, R)$, *f has a unique fixed soft element.*

Proof. Put $h = f^k$. Then h fulfils all the requirements of Theorem 3.30. So, there exist unique $\widetilde{q} \tilde{\in} SE(\widetilde{X})$ with $\widetilde{y_0} R \tilde{q}$, such that $h(\widetilde{q}) = \widetilde{q}$ and hence $fh(\tilde{q}) = f(\tilde{q})$. Also, $fh = hf (= f^{k+1})$. So, $hf(\widetilde{q}) = f(\widetilde{q})$.

Now, $d(\widetilde{q}, f(\widetilde{q})) = d(h(\widetilde{q}), hf(\widetilde{q})) \tilde{\leq} \overline{\lambda}_1\, d(\widetilde{q}, f(\widetilde{q}))$, $\overline{0} \tilde{\leq} \overline{\lambda}_1 \tilde{<} \overline{1}$

$\Rightarrow d(\widetilde{q}, f(\widetilde{q})) = \overline{0} \Rightarrow f(\widetilde{q}) = \widetilde{q}$.

Therefore, f has a fixed soft element $\widetilde{q}$. To verify the uniqueness, let $\widetilde{q_1^*} \tilde{\in} SE(\tilde{X})$ be another soft element different from $\widetilde{q}$ such that $\widetilde{y_0} R \widetilde{q_1^*}$ and $f(\widetilde{q_1^*}) = \widetilde{q_1^*}$. Then as f is R-preserving, so

$$f^k(\widetilde{q_1^*}) = f^{k-1}(\widetilde{q_1^*}) = f^{k-2}(\widetilde{q_1^*}) = \ldots = f(\widetilde{q_1^*}) = \widetilde{q_1^*}.$$

i.e., $\widetilde{q_1^*}$ is a fixed soft element of f^k. i.e., of h, this is inconsistent with the fact that in $(\widetilde{X}, d, E, R)$, h has a unique fixed soft element. Therefore, f has a unique fixed soft element.

Corollary 3.33. *Let* $(\widetilde{X}, d, E)$ *be a soft complete metric space and* $f : (\widetilde{X}, d, E) \to (\widetilde{X}, d, E)$ *be a soft mapping such that,* f^k *is a soft contraction. Then f has a unique fixed soft element in* $(\widetilde{X}, d, E)$.

Proof. Take $R = (\widetilde{X}, E) \times (\widetilde{X}, E)$ and fix $\widetilde{y_0} \tilde{\in} \tilde{X}$. Then clearly $\widetilde{y_0} R \tilde{q}$, $\forall\, \tilde{q} \tilde{\in} f^k(\tilde{X})$. Again, since the soft metric space $(\widetilde{X}, d, E)$ is complete, so it is R-complete. Also, it is clear that f^k is soft R-contraction mapping, as it is soft contraction. Hence from Theorem 3.32, f has a unique fixed element in $(\widetilde{X}, d, E)$.

4. Conclusion

In this study, we have presented Soft R-metric space and various definitions in this space are explored with examples. Also, we have established some properties and some significant fixed point results in this setting. Our prospect that this investigation has a great weight and to support the researchers to cultivate the new concepts in the field of fixed point results, using soft elements.

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