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Graph linkedness with prescribed lengths

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Abstract Given a multigraph H, a graph G is H-*linked* if every injective map $f : V(H) \to V(G)$ can be extended to an H-subdivision (f, g) in G for some g. Given a multigraph H and an integer sequence $w = \{w_e \mid e \in E(H),\ w_e \geq 2\}$, a graph G is (H, w, m)-*linked* if every injective map $f : V(H) \to V(G)$ can be extended to an H-subdivision (f, g) in G such that each path $g(e)$ has length $w_e, \ldots,$ or $w_e + m$. If $m = 0$, then we say G is (H, w)-*linked*. We show that the sharp minimum degree condition for a graph to be H-linked is the same as the sharp minimum degree condition for a large graph to be (H, w, m)-linked for $m \geq 1$ and all sets w with each value $w_e \in w$ at least 14. Additionally, we establish a sharp minimum degree condition for a large graph to be (H, w)-linked.

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1 Introduction

Unless otherwise noted, H refers to a multigraph with at least one edge and possibly with loops, and G refers to a simple graph. Let "$\hookrightarrow$" denote that a map is injective, and let $\mathcal{P}(G)$ be the set of all paths in G. By an *embedding* of H into G, we mean a pair (f,g) of maps $f : V(H) \hookrightarrow V(G)$ and $g : E(H) \hookrightarrow \mathcal{P}(G)$ that maps all edges $v_iv_j \in E(H)$ to edge-disjoint $f(v_i), f(v_j)$-paths in G. If the embedding (f,g) maps edges in $E(H)$ to internally vertex-disjoint paths in G, then we say the embedding, or corresponding subgraph of G, is called an *H-subdivision* in G. We call $f(V(H))$ the set of *ground vertices* and $g(E(H))$ the set of *edge-paths*. Given H, we say G is *H-linked* if every map $f : V(H) \hookrightarrow V(G)$ can be extended to an H-subdivision (f,g) in G. For brevity, let $|H| = |V(H)|$ and $e(H) = |E(H)|$.

Although initially discovered in 1970 by Jung in [10], the concept of H-linkedness did not develop until Gould's and Whalen's contributions [9] and [17] in the early 2000's. Since then, there has been progress on the minimum degree criteria for a graph to be H-linked. Kostochka and Yu proved sharp minimum degree conditions for *loopless* multigraphs, while Ferrara, Gould, Tansey, and Whalen independently showed similar results in [6] for multigraphs possibly *with loops*. Kostochka and Yu determined in [13] that a graph G of order $n \geq 5e(H)+6$ with $\delta(G) \geq \frac{n+e(H)-2}{2}$ is H-linked, and that the lower bound on $\delta(G)$ is sharp for bipartite H. Let $B(H)$ be the number of edges in a maximum edge-cut of H. In [14], the same authors showed that if $\delta(H) \geq 2$, then a graph G of order $n \geq 7.5e(H)$ with the sharp condition $\delta(G) \geq \frac{n+B(H)-2}{2}$ is H-linked. For the rest of our paper, assume that H may contain loops. Ferrara, Gould, Tansey, and Whalen discovered the sharp condition $\delta(G) \geq \frac{n+B(H)-2}{2}$ for a sufficiently large graph G of order n to be H-linked. The authors of [14] and [6] then united their results in [8] by first defining $b(H)$, a generalized version of $B(H)$, and developed the generalized sharp condition $\delta(G) \geq \frac{n+b(H)}{2}$ for G to be H-linked for connected *and disconnected* (and hence, for all) multigraphs H as well. Ferrara et al. later analogously defined $b(H)$ as

$$b(H) = \max_{\substack{R \cup N \cup U = V(H) \\ e(R,U) \geq 1}} \{|N| + e(R,U)\} \tag{1}$$

in [5] and then generalized the result in [8] by proving a sharp condition on $\sigma_2(G)$, the minimum degree sum (of non-adjacent vertices) of G, for H-linkedness in G. Let h_0 denote the number of isolated vertices in H. As shown in [5], a graph G of order n with $\delta(G) \geq 4e(H) + h_0$ and $\sigma_2(G) \geq n + b(H) - 2$ is H-linked. Sharp connectivity criteria for H-linkedness remain unknown.

Given a multigraph H and an integer sequence $w = \{w_e \mid e \in E(H), w_e \geq 2\}$, a graph G is *(H,w,m)-linked* if every map $f : V(H) \hookrightarrow V(G)$ can be extended to an H-subdivision (f,g) in G such that each path $g(e)$ has length $w_e, \ldots,$ or $w_e + m$. If $m = 0$, then we say

G is (H,w)-*linked*. For all sets w with each value $w_e \in w$ at least 14, we establish a sharp minimum degree condition for a large graph G to be (H,w,m)-linked for $m \geq 1$. The value 14 is used in Theorems 1.1 and 1.2 solely for technical reasons (see Lemmas 3.4 and 3.5, along with the proof of Lemma 2.3) and most likely is not sharp. We hope to improve on this lower bound in the future.

If G is (H,w,m)-linked for specific H and w, then we can choose the specific length within m of each edge-path for an H-subdivision with given ground vertices. While this is similar to the idea of *pan-H-linked* graphs, as defined by Ferrara, Magnant and Powell in [7], there are significant differences as well. Namely, pan-H-linkedness does not specify lengths of individual edge-paths in G. On the other hand, pan-H-linkedness implies an H-subdivision in G can be extended to span G. As a result, Theorems 1.1 and 1.2 are neither stronger nor weaker than Theorem 6 in [7].

In order to state our main results, we recall the term $b(H)$ from (1). Theorem 1.1 states that given a sequence w with values all at least 14, a sufficiently large graph G with $\delta(G) \geq \frac{n+b(H)-2}{2}$ is also (H,w,m)-linked for $m \geq 1$. Note that this value of $\delta(G)$ is the same as that in [5] and [8]. Hence, the necessary minimum degree for G to be (H,w,m)-linked for $m \geq 1$ equals the necessary minimum for G to be H-linked – we only require a larger lower bound on the order of G. If we also consider $e(H)$ and the number of isolated vertices in H, then we get a similar sharp lower bound for $\delta(G)$ when establishing (H,w)-linkedness in G.

We now state our main results.

Theorem 1.1 *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \geq 14\}$ be a sequence of integers. If G is a graph of order $n \geq n(H,w)$ with $\delta(G) \geq \frac{n+b(H)-2}{2}$, then G is (H,w,m)-linked for $m \geq 1$. Furthermore, the lower bound for $\delta(G)$ is sharp.*

The sharpness of Theorem 1.1 is established in [8], where Gould et al. first show in their Lemma 4 that a connected graph H has $|H|-1$ edges in a maximum edge-cut if and only if H contains no even cycles. The sharp example is then split into two cases depending on whether H contains a component with no even cycles. If no component of H contains an even cycle, then the vertices of H are partitioned into partite sets X and $Y = V(H) \setminus X$ of a maximum edge cut of H, and $b(H) = e(X,Y)$. Otherwise, the vertices of a subgraph H_1 of H whose components all contain an even cycle are partitioned analogously into a maximum edge cut of H_1. Letting $H_0 = H - H_1$ and $c(H_0)$ be the number of components of H_0, their Lemma 4 ensures that a maximum edge cut of H_0 contains $|H_0| - c(H_0)$ edges. Considering partite sets X and Y of a maximum edge cut of H_1, it follows that $b(H) = e(X,Y) + |H_0|$.

In either case, a graph G of sufficiently large order is constructed so that G consists of two complete graphs G_1 and G_2 that intersect at $b(H)-1$ vertices. It follows that G cannot contain an (X,Y)-subdivision whose ground vertices for X and Y reside in $G_1 - G_2$ and

$G_2 - G_1$, respectively.

One can easily see that $\delta(G) = \frac{n+b(H)-3}{2}$, and hence, that the condition $\delta(G) \geq \frac{n+b(H)-2}{2}$ is sharp.

Theorem 1.2 *Let H be a multigraph with h_0 isolated vertices, and let $w = \{w_e \mid e \in E(H), w_e \geq 14\}$ be a sequence of integers. If G is a graph of order $n \geq n(H,w)$ with $\delta(G) \geq \max\left\{\frac{n+b(H)-2}{2}, \frac{n+e(H)+h_0}{2}\right\}$, then G is (H,w)-linked. Furthermore, the lower bound for $\delta(G)$ is sharp.*

For Theorem 1.2, the example in [8] also establishes the sharpness of the bound for $\delta(G)$ when $b(H) - 2 \geq e(H) + h_0$. Example 1.3 demonstrates the sharpness of $\delta(G) \geq \frac{n+e(H)+h_0}{2}$ if instead $b(H) - 2 < e(H) + h_0$.

Example 1.3 Let H be a multigraph with h_0 isolated vertices. Let H_0 denote the set of isolated vertices in H. Let G be a complete tripartite graph on n vertices with independent sets A, B, and A' satisfying

$$\begin{aligned}
|A| &= \left\lceil \frac{n-(e(H)+h_0-1)}{2} \right\rceil \\
|B| &= \left\lfloor \frac{n-(e(H)+h_0-1)}{2} \right\rfloor \\
|A'| &= e(H)+h_0-1.
\end{aligned}$$

As a result, we have $\delta(G) = \left\lceil \frac{n+e(H)+h_0-2}{2} \right\rceil$. Although G is tripartite, note that $A \cup B$ is bipartite, and that A' is tiny compared to A and B. Hence, we may view G as being "almost" bipartite.

Consider a map $f : V(H) \hookrightarrow V(G)$ defined as follows: if $v \in H_0$, then let $f(v) \in A'$; otherwise, let $f(v) \in B$. Let w be a sequence of order $e(H)$ consisting of only odd integers. In order to create an H-subdivision (f,g) in G where each edge-path has odd length, we need to create paths of odd length between the ground vertices in B. It follows that each path $g(e)$ must use at least one vertex of $A' \setminus f(H_0)$. However, since $|A' \setminus f(H_0)| = e(H) - 1$, we cannot construct all of the edge-paths in our H-subdivision with the correct parity. Therefore, G is $(H,w,1)$-linked but not (H,w)-linked. This shows that the condition $\delta(G) \geq \frac{n+e(H)+h_0}{2}$ in Theorem 1.2 is sharp.

In Section 2 we state the Degree Form [3] of Szemerédi's [16] Regularity Lemma and break Theorems 1.1 and 1.2 into four lemmas, the first two of which are proved in their own subsequent sections, and the last two of which are proved in the final section.

2 Regularity and Other Preliminaries

Our approach to proving Theorems 1.1 and 1.2 combines the Regularity Lemma with the Blow-Up Lemma [11].

We define ε-regularity, which uses small vertex sets to measure how uniformly distributed edges are between two vertex subsets of a graph. Let A and B be disjoint vertex sets. The *density* of the pair (A,B) is the value

$$d(A,B) = \frac{e(A,B)}{|A||B|}.$$

Note that $0 \leq d(A,B) \leq 1$. Fix $\varepsilon > 0$. A pair (A,B) is ε*-regular* if for all subsets $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have $|d(X,Y) - d(A,B)| < \varepsilon$. Some sources, such as [12], write $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, but for our purposes this difference is insignificant. We say (A,B) is (ε,δ)*-regular* to mean (A,B) is ε-regular with density greater than δ.

Although the Regularity Lemma is true for any values of ε and δ, we assume the relationships

$$0 < \varepsilon \ll \delta \ll 1, \tag{2}$$

and $n \geq N(\varepsilon,\delta)$ for some function N, which potentially increases after each assumption of small ε and δ values.

Lemma 2.1 (Regularity Lemma (Degree Form) - Bollobás [2], Diestel [3]) *For every $\varepsilon > 0$, there is an $M = M(\varepsilon)$ such that if G is any graph and $\delta \in (0,1)$ is any real number, then there is a partition of $V(G)$ into $r+1$ clusters $V_0, V_1, \ldots, V_r$, and there is a subgraph $G' \subseteq G$ with the following properties:*

(1) $r \leq M$,

(2) $|V_0| \leq \varepsilon|V(G)|$,

(3) $|V_1| = \cdots = |V_r| \leq \varepsilon|V(G)|$,

(4) $\deg_{G'}(v) > \deg_G(v) - (\delta+\varepsilon)|V(G)|$ *for all* $v \in V(G)$,

(5) $e(G'[V_i]) = 0$ *for all* $i \geq 1$,

(6) for all $1 \leq i < j \leq r$ *the graph* $G'[V_i,V_j]$ *is* ε*-regular and has density either* 0 *or greater than* δ.

The sets V_i in Lemma 2.1 are called *clusters*, with V_0 being the *garbage cluster*. Typically, we are concerned with graphs G with $|V(G)| \gg M$, since the result is trivially true for G of order M. Given a graph G and appropriate choices of ε and δ, let G' be the spanning subgraph of G obtained from Lemma 2.1, and let $G'' = G' \setminus V_0$. The *reduced graph* $R = R(G,\varepsilon,\delta)$ contains a vertex v_i for each cluster V_i in G'' and has an edge between v_i and v_j if and only if $d(V_i,V_j) > \delta$. Hence, $V(R) = \{v_i \mid 1 \le i \le r\}$ and $E(R) = \{v_iv_j \mid 1 \le i,j \le r, d(V_i,V_j) > \delta\}$.

For the remainder of this work, when convenient, we will assume appropriate choices of $\varepsilon,\delta > 0$ and n such that εn is an integer. As a result, we refer to R as $R(G)$ instead of $R(G,\varepsilon,\delta)$.

The next proposition provides a minimum degree condition on the reduced graph.

Proposition 2.2 (Kühn, Osthus and Taraz [15]) *If a graph G satisfies $\delta(G) \ge \frac{n}{2}$, then for fixed $\varepsilon > 0$ and $\delta > 0$, the reduced graph $R = R(G,\varepsilon,\delta)$ satisfies*

$$\delta(R) \ge \Big(\frac{1}{2} - (\delta + 2\varepsilon)\Big) r.$$

We now state the four lemmas that combine to give Theorems 1.1 and 1.2. Note that to prove Theorem 1.1, it suffices to show that G is $(H,w,1)$-linked.

Lemma 2.3 *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \ge 14\}$ be a sequence of integers. Let $\varepsilon \le \varepsilon(H,w)$ and $\delta \le \delta(H,w)$. Let $G = G(\varepsilon,\delta)$ be a graph of order $n \ge n(\varepsilon,\delta)$ with $\delta(G) \ge \frac{n}{2}$, parameters and reduced graph $R = R(G)$. If R is connected and not bipartite, then G is (H,w)-linked.*

Lemma 2.4 *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \ge 8\}$ be a sequence of integers. Let $\varepsilon \le \varepsilon(H,w)$ and $\delta \le \delta(H,w)$. Let $G = G(\varepsilon,\delta)$ be a graph of order $n \ge n(\varepsilon,\delta)$ with $\delta(G) \ge \frac{n+b(H)-2}{2}$ and reduced graph $R = R(G)$. If R is disconnected, then G is (H,w)-linked.*

Lemma 2.5 *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \ge 3\}$ be a sequence of integers. Let $\varepsilon \le \varepsilon(H,w)$ and $\delta \le \delta(H,w)$. Consider a graph $G = G(\varepsilon,\delta)$ of order $n \ge n(\varepsilon,\delta)$ with $\delta(G) \ge \frac{n}{2}$ and reduced graph $R = R(G)$. If R is bipartite, then G is $(H,w,1)$-linked.*

Lemma 2.6 *Let H be a multigraph with h_0 isolated vertices, and let $w = \{w_e \mid e \in E(H), w_e \ge 3\}$ be a sequence of integers. Let $\varepsilon \le \varepsilon(H,w)$ and $\delta \le \delta(H,w)$. Consider a graph G of order $n \ge n(\varepsilon,\delta)$ with $\delta(G) \ge \frac{n+e(H)+h_0}{2}$, and reduced graph $R = R(G)$. If R is bipartite, then G is (H,w)-linked.*

After proving Lemmas 2.3–2.6, we may prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Since $\delta(G) \geq \frac{n+b(H)-2}{2} \geq \frac{n}{2}$, if R is connected and not bipartite, then G is (H,w)-linked (and so $(H,w,1)$-linked). Similarly, we see by Lemma 2.4 that if R is disconnected, then G is (H,w)-linked. Lastly, by Lemma 2.5, we see that G is $(H,w,1)$-linked if R is bipartite. Hence, G is (H,w,m)-linked for $m \geq 1$. □

Proof of Theorem 1.2. Since $\delta(G) \geq \frac{n}{2}$, if R is connected and not bipartite, then G is (H,w)-linked. If $e(H)+h_0 \geq b(H)-2$, then it is clear that G satisfies the conditions for Lemma 2.4 and hence is (H,w)-linked if R is disconnected. If instead $e(H)+h_0 < b(H)-2$, then $\delta(G) \geq \frac{n+b(H)-2}{2}$, and Lemma 2.4 still holds. Lastly, by Lemma 2.6, we see that G is (H,w)-linked if R is bipartite. Hence, G is (H,w)-linked. □

For all subsequent propositions and lemmas, we assume a given subgraph H and the appropriate choices of $\varepsilon = \varepsilon(H)$, $\delta = \delta(H) > 0$ for a graph G to determine the reduced graph $R = R(G,\varepsilon,\delta)$.

3 R is Connected and not Bipartite

In this section, we prove Lemma 2.3.

Our approach to the proof of Lemma 2.3 is as follows. After choosing suitably small $\varepsilon \leq \varepsilon(H,w)$ and $\delta \leq \delta(H,w)$, we choose a suitably large graph $G = G(\varepsilon,\delta)$ of order $n \geq n(H,w)$ with $\delta(G) \geq \frac{n}{2}$ and a connected, non-bipartite reduced graph $R = R(G)$. We establish the existence of a triangle in R, choose an appropriately large Δ, and apply the Blow-Up Lemma to the triangle in R. For any pair of ground vertices in G requiring an edge-path with a prescribed length, we create short paths from these vertices to vertices within our blown-up triangle. We then wind around the triangle in an appropriate fashion to yield a path of the precise length between our ground vertices.

First, we define a strong form of regularity that allows us to state the famous Blow-Up Lemma. For fixed $\varepsilon,\delta > 0$, a pair (A,B) is (ε,δ)*-super-regular* if for all subsets $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have $d(X,Y) > \delta$, along with $\deg_B(a) > \delta|B|$ for all a in A and $\deg_A(b) > \delta|A|$ for all b in B. The following lemma says that we can remove a small number of vertices from an (ε,δ)-regular pair to form an $(\varepsilon,\delta-\varepsilon)$-super-regular pair.

Fact 3.1 (Diestel [3]) *Let (A,B) be an (ε,δ)-regular pair and B' be a subset of B of size at least $\varepsilon|B|$. Then there are at most $\varepsilon|A|$ vertices $v \in A$ with $|N(v) \cap B'| < (\delta-\varepsilon)|B'|$.*

For our purposes, when Fact 3.1 is applied, it is followed with an application of Lemma 3.2 (the Blow-Up Lemma) with the substitution $\delta^* = \frac{\delta}{2} < \delta - 2\varepsilon$.

Lemma 3.2 (Blow-Up Lemma - Komlós, Sárközy, Szemerédi [11]) *Given a graph R of order r and positive parameters δ^*, Δ, there exists an $\varepsilon_0 = \varepsilon_0(\delta^*, \Delta, r) > 0$ such that the following holds. Let $n_1, n_2, \dots, n_r$ be arbitrary positive integers, and let us replace the vertices $v_1, v_2, \dots, v_r$ of R with pairwise disjoint sets $V_1, V_2, \dots, V_r$ of orders $n_1, n_2, \dots, n_r$ (blowing up). We construct two graphs on the same vertex-set $V = \bigcup V_i$. The first graph $\mathbf{R}$ is obtained by replacing each edge $v_i v_j$ of R with the complete bipartite graph between the corresponding vertex-sets V_i and V_j . A sparser graph G is constructed by replacing each edge $v_i v_j$ with any $(\varepsilon_0, \delta^*)$-super-regular pair between V_i and V_j. If a graph H with $\Delta(H) \le \Delta$ is embeddable into $\mathbf{R}$, then it is embeddable into G.*

Essentially, when finding subgraphs of bounded minimum degree Δ, we can treat super-regular pairs like complete bipartite graphs. The use of Lemma 2.1, Fact 3.1, and Lemma 3.2 ensures that a sufficiently large and dense graph G consisting of super-regular pairs of clusters contains every subgraph H of bounded maximum degree $\Delta(H)$. This will help us greatly when establishing (H, w)-linkedness in a graph G.

Since we will apply Lemma 3.2 to a triangle, we extend the definitions of (ε, δ)-regularity and (ε, δ)-super-regularity to include triples. That is, (T_1, T_2, T_3) is an (ε, δ)-*regular triple* if the pairs (T_1, T_2), (T_2, T_3), and (T_1, T_3) are all ε-regular with density δ. Similarly, (T_1, T_2, T_3) is an (ε, δ)-*super-regular triple* if the pairs (T_1, T_2), (T_2, T_3), and (T_1, T_3) are all super-regular.

This next proposition establishes a lower bound for $\kappa(G)$ when the reduced graph is connected.

Proposition 3.3 *Let $\varepsilon, \delta \in (0, 1)$. Suppose $G = G(\varepsilon, \delta)$ is a graph of order $n \ge n(\varepsilon, \delta)$ with $\delta(G) \ge \frac{n}{2}$. If $R(G)$ has order r and is connected, then G is $\left(\frac{\varepsilon(1-\varepsilon)}{r} n\right)$-connected.*

In particular, G has connectivity a fraction of n. The proof follows from a straightforward but technical contradiction argument, which we leave to the Appendix.

Given a multigraph H, consider a graph G of order $n \ge 25(10e(H) + |H|)$ with $\delta(G) \ge \frac{n}{2}$ and $\kappa(G) \le \frac{n}{3}$, minimum cutset C and components A and B. For each vertex $c \in C$ satisfying $|N(c) \cap A| \ge 5|C|$, consider a set $A_c \subset N(c) \cap A$ of $5e(H)$ vertices so that A_c and $A_{c'}$ are disjoint for all $c' \ne c$. For all other $c \in C$, let $A_c = \emptyset$. Define A_c to be the set of *proxy* vertices of c in A. Since $\kappa(G) \le \frac{n}{3}$ and $\delta(G) \ge \frac{n}{2}$, it is apparent that no vertex $c \in C$ can have both $A_c = \emptyset$ and $B_c = \emptyset$.

The following lemma provides many short, internally disjoint paths between every pair of vertices in G.

Lemma 3.4 *Given $\lambda > 0$, every graph G with $\kappa(G) \ge \lambda n$ and $\delta(G) \ge \frac{n}{2}$ has at least $\min\left\{\frac{n}{25}, \frac{9}{10}\lambda n\right\}$ internally disjoint paths, each of length at most 6, between every pair of vertices.*

Proof. If $\kappa(G) \geq \frac{n}{3}$, then the average path length in G is at most 4. This means that at least $\frac{n}{9}$ of these paths have length at most 6.

Now suppose $\kappa(G) \leq \frac{n}{3}$, and let C be a minimum cutset of G. By the minimum degree condition on G, there exist at most 4 components of $G \setminus C$, each having at least $\frac{n}{6}$ vertices. Let A and B be two components of $G \setminus C$. Without loss of generality, for every $u, v \in V(G)$, we have either $|N(u) \cap A| \geq \frac{n}{24}$ and $|N(v) \cap A| \geq \frac{n}{24}$, or $|N(u) \cap A| \geq \frac{n}{24}$ and $|N(v) \cap B| \geq \frac{n}{24}$, or both.

Case 1. $|N(u) \cap A| \geq \frac{n}{24}$ and $|N(v) \cap A| \geq \frac{n}{24}$.

By assumption, we have $|N(u)|, |N(v)| \geq \frac{n}{2}$. Since $|A \cup C| \leq \frac{5n}{6}$, we have $|N(u) \cap N(v)| \geq \frac{n}{6}$. Then there are at least $\frac{n}{24}$ internally disjoint u, v-paths passing through $N(u) \cap N(v)$.

Case 2. $|N(u) \cap A| \geq \frac{n}{24}$ and $|N(v) \cap B| \geq \frac{n}{24}$.

First suppose $|A| \geq |C|$ and $|B| \geq |C|$. Choosing a unique proxy vertex $a_c \in A$ for each vertex $c \in C$, we have

$$|N(a_c) \cap N(u)| \geq \frac{n}{6}.$$

A similar argument shows $|N(b_c) \cap N(v)| \geq \frac{n}{6}$ for each chosen proxy vertex $b_c \in B$ as well. For some $c \in C$, let a and b denote vertices in $N(a_c) \cap N(u)$ and $N(b_c) \cap N(v)$, respectively. It follows that we have the path $\{u, a, a_c, c, b_c, b, v\}$, and that G contains exactly $\max\left\{\lambda n - 2, \frac{n}{24} - 2\right\}$ internally disjoint u, v-paths of length 6.

Now suppose without loss of generality $|B| < |C|$. First assume $|A| \geq |C|$. Letting $A_p = \bigcup_{c \in C} a_c$ be a set of $|C|$ distinct proxy vertices in A, we have

$$\frac{n}{6} \leq |B| < |C| = |A_p| \leq \frac{n}{3} < |A| < \frac{n}{2}.$$

We also know

$$|N(u) \cap N(a_c)| \geq \frac{n}{6} \quad \text{for each } c \in C,$$
$$|N(v) \cap C| \geq \frac{n}{2} - |B| \geq \frac{n}{6}.$$

Then G contains $\max\left\{\lambda n - 2, \frac{n}{24} - 2\right\}$ internally disjoint u, v-paths of length 4 passing through $N(u) \cap N(A_p)$, A_p, and $N(v) \cap C$. Now assume $\frac{n}{6} \leq |A| < |C|$. Then we know

$$|N(u) \cap C| \geq \frac{n}{2} - |A| > \frac{n}{6},$$
$$|N(v) \cap C| \geq \frac{n}{2} - |B| > \frac{n}{6}.$$

Then we have $\max\left\{\lambda n - 2, \frac{n}{24} - 2\right\}$ internally disjoint u, v-paths of length 2 passing though C. □

The next lemma is similar to the well-known Fan Lemma, providing many short paths between a vertex and a set.

Lemma 3.5 *Let $\xi = \min\left\{\frac{1}{25}, \frac{9}{10}\lambda\right\}$. Let G be a graph with at least ξn internally disjoint paths of length at most 6 between each pair of vertices. Let $v \in V(G)$, and let X be a set of $\gamma n = \left\lfloor \frac{\xi n+5}{6} \right\rfloor$ vertices in $G \setminus \{v\}$. Then there are γn paths, each of length at most 6, from v to X that are disjoint except for v.*

Proof. Let v be a vertex in G, and let $X = \{x_1, \dots, x_{\gamma n}\}$ be a set of γn vertices in $G \setminus \{v\}$. By assumption, we know there exist at least ξn paths, each of length at most 6, from v to each vertex in X. For each $x_i \in X$, let $\mathcal{P}_i = \{P \mid P$ is a (v,x_i)-path of length at most $6\}$. We use an inductive construction to build the desired set of paths $\{P_1, P_2, \dots, P_{\gamma n}\}$. For P_1, choose any v,x_1-path in $\mathcal{P}_1$.

Now suppose we have selected $t < \gamma n$ paths from v to $\{x_1, \dots, x_t\}$. By assumption, we know $|\mathcal{P}_{t+1}| \geq \xi n$. There are a total of at most $6t$ vertices (not counting v) used in the paths $P_1, \dots, P_t$. Each such vertex could be in at most one path in $\mathcal{P}_{t+1}$. Hence, there are at least $\xi n - 6t$ paths in $\mathcal{P}_{t+1}$, each of length at most 6, that do not intersect $\{P_1, P_2, \dots, P_t\}$. Since we have

$$\begin{aligned}
\xi n - 6t &\geq \xi n - 6\left(\left\lfloor \frac{\xi n+5}{6} \right\rfloor - 1\right) \\
&\geq \xi n - (\xi n + 5 - 6) \\
&= 1,
\end{aligned}$$

there is at least one path in $\mathcal{P}_{t+1}$ that is internally disjoint from $P_1 \cup P_2 \cup \cdots \cup P_t$. Setting P_{t+1} to be this path completes the induction step of the construction. Thus, there are at least γn internally disjoint paths, each of length at most 6, from v to X. $\square$

We now prove Lemma 2.3.

Lemma 2.3 *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \geq 14\}$ be a sequence of integers. Let G be a graph of order $n \geq n(H,w)$ with $\delta(G) \geq \frac{n}{2}$ and reduced graph R. If R is connected and not bipartite, then G is (H,w)-linked.*

Proof. For a multigraph H, let $w = \{w_e \mid e \in E(H), w_e \geq 14\}$ be a sequence of integers. Consider a sufficiently small $\delta \in (0,1)$. Choose parameters $\delta^* = \frac{\delta}{2}$ and Δ as in Lemma 3.2 with $\sum_{e \in E(H)} w_e |V(H)| e(H) \ll \Delta$. If T_1, T_2, and T_3 are independent sets, then let $\mathcal{T}(T_1,T_2,T_3)$ denote the complete tripartite graph on (T_1,T_2,T_3). By Lemma 3.2, there exists a value $\varepsilon_0 = \varepsilon_0(\delta^*, \Delta, 3)$ such that the following is true: if (T_1,T_2,T_3) is an $(\varepsilon_0, \delta^*)$-super-regular triple on sufficiently many vertices, then (T_1,T_2,T_3) contains all subgraphs of maximum degree at most Δ that are contained in $\mathcal{T}(T_1,T_2,T_3)$. Since this result is true for all $\varepsilon \leq \varepsilon_0$, it suffices to choose $\varepsilon \leq \varepsilon_0$ that also satisfies $\varepsilon \ll \delta^*$.

Apply Lemma 2.1 with parameters ε and δ on the graph G of order $n \geq n(\varepsilon,\delta)$ with $\delta(G) \geq \frac{n}{2}$ to obtain the reduced graph $R = R(G)$, which by Proposition 2.2 satisfies

$$\delta(R) \geq \Big(\frac{1}{2} - (\delta + 2\varepsilon)\Big) r.$$

Suppose R is connected and not bipartite. Andrásfai, Erdős, and Sós's result in [1] guarantees that R contains a triangle. Since R contains a triangle, there exists a corresponding (ε,δ)-regular triple of clusters $(V_1,V_2,V_3) \subset G$. By Fact 3.1, there exists an $(\varepsilon,\delta-2\varepsilon)$-super-regular triple $(T_1,T_2,T_3) \subset (V_1,V_2,V_3)$. However, this implies that (T_1,T_2,T_3) is also an (ε,δ^*)-super-regular triple since $\delta^* < \delta - 2\varepsilon$.

By Lemma 3.2, there exists a complete tripartite graph $\mathcal{T}(X_1,X_2,X_3) \subset (T_1,T_2,T_3)$ with $|X_1| = |X_2| = |X_3| = \Delta$. Let $V(H) = \{v_1,\ldots,v_h\}$ and consider a map $f : V(H) \hookrightarrow V(G)$.

For each edge $e \in E(H)$ with end-points u and v, do the following. Consider a set $X_e \supset (X_1,X_2,X_3)$ of Δ vertices. By Lemma 3.5, there exist both $f(u),X_e$- and $f(v),X_e$-fans, each consisting of Δ internally disjoint paths of length at most 6. Choose a vertex $x_{e,1} \in X_1$ whose path in the $f(u),X_e$-fan has length b. Similarly, choose a vertex $x_{e,3} \in X_3$ whose path in the $f(v),X_e$-fan has length c. Then $w_e - b - c \geq 2$.

Case 1. $w_e - b - c \equiv 0 \bmod 3$ and is odd.

Choose vertices in X_1 and X_2 so that our $f(u),f(v)$-path starts at $f(u)$, passes through $x_{e,1}$, alternates between $w_e - b - c + 1$ vertices in X_1 and X_2 (the last of which is in X_2), passes through $x_{e,3}$ and then ends at $f(v)$.

Case 2. $w_e - b - c \equiv 0 \bmod 3$ and is even.

Choose vertices in X_1 and X_2 so that our $f(u),f(v)$-path starts at $f(u)$, passes through $x_{e,1}$, alternates between $w_e - b - c + 1$ vertices in X_1 and X_2 (the last of which is in X_1), passes through $x_{e,3}$ and then ends at $f(v)$.

Case 3. $w_e - b - c \equiv 1 \bmod 3$.

Choose vertices in X_1, X_2 and X_3 so that our $f(u),f(v)$-path starts at $f(u)$, passes through $x_{e,1}$, cycles through $w_e - b - c + 1$ vertices between X_2, X_3 and then X_1 (the last of which is in X_1), passes through $x_{e,3}$ and then ends at $f(v)$.

Case 4. $w_e - b - c \equiv 2 \bmod 3$.

Choose vertices in X_1, X_2 and X_3 so that our $f(u),f(v)$-path starts at $f(u)$, passes through $x_{e,1}$, cycles through $w_e - b - c + 1$ vertices between X_1, X_3 and then X_2 (the last of which is in X_2), passes through $x_{e,3}$ and then ends at $f(v)$.

See Figure 1 for an example of a path construction in G. Also note that $\mathcal{T}(X_1,X_2,X_3)$

is large enough that we can choose each $f(u), f(v)$-path to be internally disjoint from all other such edge-paths. Hence, G is (H,w)-linked. $\square$

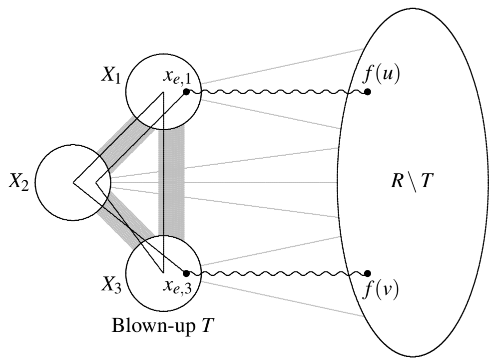

Figure 1. Construction of an $f(u), f(v)$-path.

We would like to discover a sharp minimum degree condition that would still allow a minimum edge-weight of 14.

4 R is Disconnected

In this section, we prove Lemma 2.4.

Given a graph G of sufficiently large order n with $\delta(G) \geq \frac{n+b(H)-2}{2}$ and a connected reduced graph, we show G is (H,w)-linked by partitioning the vertices of G into two relatively dense sets with relatively few (but at least $b(H)$) edges in between. We then use Williamson's Theorem [18] to create short paths between ground vertices.

When using the following results in the proofs of Theorems 1.1 and 1.2, we will let $b = b(H) - 2$ or $b = \max\{b(H) - 2, e(H) + h_0\}$, depending on the situation.

When R is disconnected, the next lemma provides a bipartition of G into almost equally sized vertex sets with few edges in between.

Lemma 4.1 *Let b be a positive integer, let $\varepsilon, \delta \in (0,1)$, and let G be a graph $G = G(\varepsilon, \delta)$ of order $n \leq n(\varepsilon, \delta)$ with $\delta(G) \geq \frac{n+b}{2}$ and disconnected reduced graph $R = R(G)$. There is a bipartition $G = A \cup B$ satisfying*

$$\frac{n+b}{2} - (\delta + 3\varepsilon)n \leq |A|, |B| \leq \frac{n+b}{2} + (\delta + 3\varepsilon)n, \tag{3}$$

and

$$e(A,B) < (\delta + 3\varepsilon)n^2. \tag{4}$$

Proof. Let $b, \varepsilon, \delta, G$ and n be as given in the statement. Note that this implies n is large and $\delta, \varepsilon \ll 1$ since $R \neq G$ (as R is disconnected and G is connected). Apply Lemma 2.1 to obtain the spanning subgraph $G' = G'' \cup V_0$. By Lemma 2.1, we have $\deg_{G'}(v) > \frac{n}{2} + b - (\delta + \varepsilon)n$ for all n vertices $v \in G$. Since $|V_0| \leq \varepsilon n$, it follows that

$$\delta(G'') > \frac{n+b}{2} - (\delta + 2\varepsilon)n. \tag{5}$$

Since R is disconnected, we see that G'' must be disconnected as well. From (5), we see G'' has two components; call them C_1 and C_2. It follows that

$$\frac{n+b}{2} - (\delta + 2\varepsilon)n + 1 < |C_i| < \frac{n+b}{2} + (\delta + 2\varepsilon)n - 1$$

for $i = 1, 2$. Since G'' can have as many as n vertices, each having at most $(\delta + 2\varepsilon)n$ edges in $G \setminus G''$, there must be fewer than $(\delta + 2\varepsilon)n^2$ edges in G between C_1 and C_2.

Lemma 2.1 implies that there are fewer than εn^2 edges in G incident to vertices of V_0. Then there are fewer than $(\delta + 3\varepsilon)n^2$ edges between the components C_1, C_2, and V_0. Now partition V_0 into two sets C_A and C_B of equal size. Create a bipartition of G by adjoining C_A with C_1 and C_B with C_2. Let

$$A = C_1 \cup C_A$$
$$B = C_2 \cup C_B.$$

It follows that A and B satisfy (3) and (4). $\square$

Although $(\delta + 3\varepsilon)n^2$ may seem large, it is small compared to the minimum of $\frac{n^2}{4}$ edges in G. Also note that we may assume vertices in A have at least $\frac{n}{4}$ neighbors in A – otherwise, such a vertex should be put in B. A similar statement is true for vertices in B. This ensures that we have the smallest number of paths between A and B possible.

For our next result, recall that a graph G is *panconnected* if for every pair of vertices $w_i, w_j \in G$ and every integer t satisfying $2 \leq t \leq n-1$, there exists a w_i, w_j-path of length t.

Theorem 4.2 (Williamson [18]) *If $\delta(G) \geq \frac{n+2}{2}$, then G is panconnected.*

The following lemma, a generalization of Claim 5 in [4], establishes an upper bound for the vertex sets $D_A \subset A$ and $D_B \subset B$ with many edges to the opposite set and shows the density of $G[A \setminus D_A]$ and $G[B \setminus D_B]$.

Lemma 4.3 *Consider $G = A \cup B$ as in Lemma 4.1. Let $D_A = \left\{v \in A \mid \deg_B(v) \geq \frac{n}{100b^2}\right\}$, and let $D_B \subset B$ be defined symmetrically, we have*

$$|D_A \cup D_B| < 200b^2(\delta + 2\varepsilon)n. \tag{6}$$

Furthermore, for each set $A' \subseteq A \setminus D_A$ and $B' \subseteq B \setminus D_B$ with $|A'|, |B'| \geq \frac{n}{10b^2}$, the graphs $G[A']$ and $G[B']$ are panconnected.

Proof. Let $|D_A| = \xi n$. There are at least $\xi n \cdot \frac{n}{100b^2}$ edges between D_A and B. From (4), we know

$$\frac{\xi}{100b^2} n^2 < (\delta + 3\varepsilon) n^2.$$

Solving for ξ, we get $\xi < 100b^2(\delta + 3\varepsilon)$. Using the same logic for D_B, we have (6).

It is immediate that G is panconnected for $b \leq 2$; consider $b > 2$. By Lemma 4.1, we know $\delta(G[A \setminus D_A]) \geq \frac{n+b}{2} - (\delta + 3\varepsilon)n - \frac{n}{100b^2} - 18b^2(\delta + 2\varepsilon)n \geq \left(\frac{1}{2} - \frac{1}{50b^2}\right)n$. By symmetry, we know $\delta(G[B \setminus D_B]) \geq \left(\frac{1}{2} - \frac{1}{50b^2}\right)n$ as well. Then we have

$$\begin{aligned}
\delta(G[A \setminus D_A]) &\geq \left(\frac{1}{2} - \frac{1}{50b^2}\right)n \\
&= \frac{n}{2} - \frac{n}{50b^2} \\
&= \frac{|A| + |B|}{2} - \frac{n}{50b^2} \\
&\geq \frac{2|A| - \frac{n}{25b^2}}{2} - \frac{n}{50b^2} \\
&= |A| - \frac{n}{25b^2} \\
&\geq |A \setminus D_A| - \frac{n}{25b^2}.
\end{aligned}$$

Thus, given $A' \subseteq A \setminus D_A$ with $|A'| \geq \frac{n}{10b^2}$, we have

$$\begin{aligned}
\delta(G[A']) &\geq |A'| - \frac{n}{25b^2} \\
&\geq \frac{|A'| + 2}{2} + \frac{n}{25b^2} - \frac{n}{25b^2} \\
&\geq \frac{|A'| + 2}{2}.
\end{aligned}$$

By Theorem 4.2, A' and likewise B' are panconnected. □

We use Lemma 4.3 in the proof of Lemma 2.4 to create edge-paths of arbitrary length between appropriate ground vertices. We do not actually need the full strength of Theorem 4.2 – it suffices to show that small subsets of $A \setminus D_A$ and $B \setminus D_B$ induce graphs of diameter 2. We could then string together these various subsets to create edge-paths of arbitrary length. However, it is simpler to use Theorem 4.2.

We now prove Lemma 2.4.

Lemma 2.4 *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \geq 8\}$ be a sequence* *of integers. Consider a graph G of order $n \geq n(H,w)$ with $\delta(G) \geq \frac{n+b(H)-2}{2}$ and reduced graph R. If R is disconnected, then G is (H,w)-linked.*

Proof. Let H be a multigraph, let $w = \{w_e \mid e \in E(H),\, w_e \geq 8\}$ be a sequence of integers and consider a map $f : V(H) \hookrightarrow V(G)$. Consider a (simple) graph G of sufficiently large order $n \gg |H|$ with $\delta(G) \geq \frac{n+b(H)-2}{2}$ with small enough $\varepsilon, \delta \in (0,1)$ to satisfy the conditions of Lemma 4.1. Apply Lemma 2.1 on G to obtain the disconnected reduced graph R of G.

By Lemma 4.1, we can bipartition G into sets A and B so that A and B satisfy (3) and (4). Define D_A and D_B as in Lemma 4.3. We consider several different scenarios depending on the locations of $f(u)$ and $f(v)$. In each case, we construct an $f(u), f(v)$-path in G that is internally disjoint from all other such paths. If without loss of generality $f(u), f(v) \in A$, then since $\delta(G[A]) \geq \frac{n}{5}$, we can create a set $A_e \subset A$ of order $\frac{n}{10(b(H)-2)^2}$ containing both $f(u)$ and $f(v)$. By Lemma 4.3, there exists an $f(u), f(v)$-path in $G[A_e]$ of length w_e. The set A_e is small enough to be disjoint from all other such sets if necessary. A similar technique may be employed for the case where $f(u) \in D_A$ and $f(v) \in B$. The only remaining case is that in which $f(u) \in A \setminus D_A$ and $f(v) \in B \setminus D_B$. Since $\delta(G) \geq \frac{n+b(H)-2}{2}$, there exists a set M of at least $b(H)$ disjoint paths from $A \setminus D_A$ to $B \setminus D_B$, each having length at most 2. Call these paths *transportation paths*. By the definition of $b(H)$, there must be at least as many transportation paths as there are edges in H with corresponding ground vertices in $A \setminus D_A$ and $B \setminus D_B$. Hence, it suffices to show that we use only one transportation path for each $f(u), f(v)$-path with $f(u) \in A \setminus D_A$ and $f(v) \in B \setminus D_B$. However, this is straightforward, as we may create sets $A_e \subset A \setminus D_A$ and $B_e \subset B \setminus D_B$ containing $f(u)$ and $f(v)$, respectively, and create paths within their induced graphs to the end-points of a single transportation path. Hence, we have an $f(u), f(v)$-path of length w_e that is disjoint from all other such paths.

Note that w_e must be at least 5 for this to be guaranteed. □

5 *R* is Bipartite

Recall that a bipartite graph is *balanced* if its partite sets have the same order.

A bipartite graph $U \cup V$ is *bipanconnected* if for every pair of vertices $x \in U$ and $y \in V$, there exist x, y-paths of every possible odd length except 1, and for every pair of vertices $x, y \in U$ (and V), there exist x, y-paths of every even length. Note that we must exclude the value 1 from our definition in order to allow graphs $U \cup V$ that are not complete bipartite.

To prove Lemma 2.5, we simply need to show that small balanced bipartite graphs in a large graph with high minimum degree are bipanconnected.

The following lemma is a generalization of Claim 1 in [4] that establishes the bipanconnectivity of sufficiently dense balanced bipartite graphs.

Lemma 5.1 *If $U \cup V$ is a balanced bipartite graph of order $2m$ with $\delta(U \cup V) \geq \frac{3m}{4}$, then $U \cup V$ is bipanconnected.*

Proof. Consider a graph $U \cup V$ of order $2m$ with $\delta(U \cup V) \geq \frac{3m}{4}$. First let $u \in U$ and $v \in V$. We prove this result using induction on the desired length k of a u,v-path. The initial step is straightforward since, given the high minimum degree of $U \cup V$, the vertices u and v must have adjacent neighbors. Now suppose there is a u,v-path P of length $2k-1$ for $3 \leq k < m$.

First assume $k \leq \frac{m}{2}$. For an edge $xy \in P$, there exists a vertex $w \in N(x)$ such that $(N(w) \cup N(y)) \setminus P \neq \emptyset$. Call a vertex in this set z. Replacing xy with $wxyz$ in P gives a path of length $2k+1$. Next, suppose $k > \frac{m}{2}$ and further suppose there exists an edge xy with $x \in U \setminus P$ and $y \in V \setminus P$. Since $\delta(U \cup V) > \frac{3m}{4}$, the vertex x must be adjacent to more than half of $P \cap U$ and the vertex y must be adjacent to more than half of $P \cap V$. Then there must be some edge $wz \in P$ such that $xw, yz \in E(U \cup V)$. Replacing wz with $wxyz$ in P gives a path of length $2k+1$. Finally, suppose $k > \frac{m}{2}$ and there is no edge xy outside P. Since $\delta(U \cup V) > \frac{3m}{4}$, we must have $k > \frac{3m}{4}$. Consider vertices $x \in U \setminus P$ and $y \in V \setminus P$. Then x has $\delta(U \cup V)$ edges into $V \cap P$ and y has $\delta(U \cup V)$ edges into $U \cap P$. Call a vertex $z \in P$ *replaceable* if either x or y is adjacent to both neighbors of z in P (meaning this vertex can be replaced in P by x or y). Then at least $\frac{n}{2}$ vertices in $U \cap P$ and $\frac{m}{2}$ vertices in $V \cap P$ are replaceable. It follows then that there must be an edge $wz \in E(U \cap V)$ such that both w and z are replaceable. Replacing w with x and z with y gives a path of length $2k-1$ with an edge outside of P. We consider the previous scenario and find a path of length $2k+1$.

Now, without loss of generality, let $u_1, u_2 \in U$. We again use induction on the desired length k of a u_1,u_2-path. The initial step is straightforward, since given the high minimum degree of $U \cup V$, the vertices u_1 and u_2 must share a neighbor. Now suppose there is a u_1,u_2-path P of length $2k$ for $2 \leq k < m$. By an argument identical to the previous case above, we get a u_1,u_2-path of length $2k+2$.

Hence, $U \cup V$ is bipanconnected. □

We now prove Lemma 2.5.

Lemma 2.5 *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \geq 3\}$ be a sequence of integers. Consider a graph G of order $n \geq n(H,w)$ with $\delta(G) \geq \frac{n}{2}$ and reduced graph R. Let $f : V(H) \hookrightarrow V(G)$ be a vertex map. If R is bipartite, then G is $(H,w,1)$-linked.*

Proof. Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \geq 3\}$ be a sequence of integers. Consider a sufficiently large graph G of order n with $\delta(G) \geq \frac{n}{2}$ whose reduced graph R is bipartite (after applying Lemma 2.1, of course). Let A_R and B_R be the independent sets composing R, and let A and B be the sets of clusters in G corresponding to A_R and B_R, respectively. We see by Proposition 2.2 that

$$\Big(\frac{1}{2} - (\delta + 2\varepsilon)\Big)n + 1 \leq |A| \leq \Big(\frac{1}{2} + (\delta + 2\varepsilon)\Big)n - 1,$$

with a symmetric inequality for $|B|$. For each edge $e \in E(H)$, define the set $T_e \subset G$ satisfying

- $|T_e| = 5(\delta + 2\varepsilon)n$,
- T_e induces a balanced bipartite graph in G (i.e., T_e has equally many vertices in A and in B), and
- $T_e \cap f(V(H)) = \{f(u), f(v)\}$.

Note that sets T_e and $T_{e'}$ can be chosen to be disjoint if necessary. By Lemma 5.1, $G[T_e]$ is bipanconnected. Hence, regardless of the locations of $f(u)$ and $f(v)$, we can construct an $f(u), f(v)$-path in G (disjoint from all other such paths) of length w_e.

It follows that G is $(H, w, 1)$-linked. □

We believe that the degree assumption used in the previous lemma is likely far from sharp under the assumption that the reduced graph is bipartite.

To prove Lemma 2.6, we must ensure that every edge-path in G has the precise length. To do this, we show the existence of either a matching or a few vertices with large degree.

Lemma 5.2 *If G is a graph of order $n \gg \delta(G) \geq k$, then either*

1. *there exist $2k$ independent edges in G, or*
2. *there exist k vertices of degree at least $\frac{n}{5k}$; in particular, there exist k independent edges in G.*

Proof. Consider a graph G of order $n \gg \delta(G) \geq k$. If there exist $2k$ independent edges in G, then we are done, so suppose not. Consider the largest collection of $\ell < 2k$ independent edges, i.e., the largest perfect matching in G. Call this set of edges M_ℓ. Then $G = M_\ell \cup A$, where A must induce an independent set of $n - 2\ell$ vertices. Since $\delta(G) \geq k$, each vertex in A must be adjacent to k vertices in M_ℓ. This means there are at least $k(n-2\ell)$ edges from A to M_ℓ. Therefore, there are at least k vertices in M_ℓ with degree at least $\frac{kn-2\ell k}{2\ell} = \left(\frac{k}{\ell}\right)\frac{n}{2} - k > \frac{n}{5k}$. □

When G is close to being bipartite, it may be difficult to construct paths having length with the correct parity. For example, two vertices x and y in the same partite set of a bipartite graph can only have paths of even length between them, so all x, y-path lengths have even parity. All path lengths between a fixed pair of vertices in a bipartite graph must have the same parity. Since our main results involve specific path lengths between arbitrary pairs within any sufficiently large and dense graph (including potentially bipartite graphs), we sometimes will only be able to get within 1 of the desired length of a path.

Lastly, we prove Lemma 2.6.

Lemma 2.6 *Let H be a multigraph with h_0 isolated vertices, and let $w = \{w_e \mid e \in E(H), w_e \geq 3\}$ be a sequence of integers. Consider a graph G of order $n \geq n(H,w)$ with $\delta(G) \geq \frac{n+e(H)+h_0}{2}$ and reduced graph R. If R is bipartite, then G is (H,w)-linked.*

Proof. Let H be a multigraph. We can assume H has no isolated vertices since the images of these vertices can be removed from G, preserving the integrity of the minimum degree condition and the result. Let $w = \{w_e \mid e \in E(H), w_e \geq 3\}$ be a sequence of integers. Consider a sufficiently large graph G of order n with $\delta(G) \geq \frac{n+e(H)}{2}$ whose reduced graph R is bipartite after applying Lemma 2.1. Let A_R and B_R be the independent sets composing R, and let A and B be the sets of clusters in G corresponding to A_R and B_R, respectively. By Proposition 2.2 we have

$$\Big(\frac{1}{2} - (\delta + 2\varepsilon)\Big)n + 1 \leq |A| \leq \Big(\frac{1}{2} + (\delta + 2\varepsilon)\Big)n - 1,$$

and similarly for $|B|$. By Lemma 5.2, A and B have either $2e(H)$ independent edges or $e(H)$ stars, each of size at least $\frac{n}{5e(H)}$.

Let e be an edge in H with end-points u and v. Without loss of generality, if $f(u), f(v) \in A$ and w_e is even, or if $f(u) \in A$ and $f(v) \in B$ and w_e is odd, then create a set T_e of order $5(\delta + 2\varepsilon)n$ containing both $f(u)$ and $f(v)$ whose induced graph is balanced bipartite. By Lemma 5.1, there exists an $f(u), f(v)$-path in G of length w_e.

Otherwise, we need to insert an independent edge within either A or B into the path to correct the parity problem. By Lemma 5.2, we know without loss of generality that $G[A]$ contains at least $e(H)$ independent edges. Without loss of generality, if $f(u), f(v) \in A$ and w_e is odd, then create two sets $T_{e,1}$ and $T_{e,2}$ of order $5(\delta + 2\varepsilon)n$ whose induced graphs are balanced bipartite. Furthermore, create $T_{e,1}$ and $T_{e,2}$ so that each set contains an end-point of an independent edge $x_e y_e$. By Lemma 5.1, we may create a path within $G[T_{e,1}]$ starting at $f(u)$ and ending at x_e, and we may create another path within $T_{e,2}$ starting at y_e and ending at $f(v)$. Choosing the appropriate lengths of these paths results in an $f(u), f(v)$-path of length w_e. An analogous argument works when $f(u) \in A$ and $f(v) \in B$ and w_e is even.

It follows that G is (H,w)-linked. □

6 Appendix: Proof of Proposition 2.2

Fix $\varepsilon > 0$ and $\delta > 0$. If a graph $G = G(\varepsilon, \delta)$ of order $n \geq n(\varepsilon, \delta)$ satisfies $\delta \geq \frac{n}{2}$, then the reduced graph $R = R(G)$ satisfies

$$\delta(R) \geq \Big(\frac{1}{2} - (\delta + 2\varepsilon)\Big)r.$$

The proof follows from a simple edge count argument.

Proof. Fix ε and δ, and apply Lemma 2.1 on G to create the subgraph $G' = G'' \cup V_0$ and the reduced graph R. Since $\delta(G) \geq \frac{n}{2}$, we have $\delta(G') > \left(\frac{1}{2} - (\delta+\varepsilon)\right)n$. From $|V_0| \leq \varepsilon n$, it follows that

$$\delta(G'') \geq \left(\frac{1}{2} - (\delta+2\varepsilon)\right)n.$$

Using Item 6 of Lemma 2.1, we see that if a vertex $v \in V_i$ is adjacent to vertices in another cluster V_j, then $d(V_i,V_j) > \delta$, and hence $v_i v_j \in E(R)$. Since each cluster in G'' has order L,

$$\delta(R) \geq \frac{\delta(G)}{L} \geq \left(\frac{1}{2} - (\delta+2\varepsilon)\right)r.$$

□

7 Appendix: Proof of Proposition 3.3

Proposition 3.3 *Let $\varepsilon,\delta \in (0,1)$, and suppose $G = G(\varepsilon,\delta)$ is a graph of order $n \geq n(\varepsilon,\delta)$ with $\delta(G) \geq \frac{n}{2}$. If $R(G)$ has order r and is connected, then G is $\left(\frac{\varepsilon(1-\varepsilon)}{r}n\right)$-connected.*

Proof. Apply Lemma 2.1 on G to obtain $G' = G'' \cup V_0$. We first prove G'' is $\left(\frac{\varepsilon(1-\varepsilon)}{r}n\right)$-connected and then extend this to G' and G.

Suppose $\kappa(G'') < \frac{\varepsilon(1-\varepsilon)}{r}n$ and let S be a minimum cutset of G''. Since $\delta(G) \geq \frac{n}{2}$, we know from Lemma 2.1 that $\delta(G'') > \left(\frac{1}{2} - (\delta+2\varepsilon)\right)n$. It is straightforward that G'' has cutset C with components A and B satisfying

$$\left(\frac{1}{2} - (\delta+2\varepsilon)\right)n - |C| + 1 \leq |A| \leq \left(\frac{1}{2} + (\delta+2\varepsilon)\right)n - 1, \quad \text{and}$$
$$\left(\frac{1}{2} - (\delta+2\varepsilon)\right)n - |C| + 1 \leq |B| \leq \left(\frac{1}{2} + (\delta+2\varepsilon)\right)n - 1.$$

From Lemma 2.1, we have $|V_0| \leq \varepsilon n$ and $|V_i| = L \geq \frac{\varepsilon(1-\varepsilon)}{r}n$ for all $1 \leq i \leq r$. Note, however, that a cluster in G'' could have vertices in A, B, and C.

Claim *For all i satisfying $1 \leq i \leq r$, we have either*

$$|V_i \cap A| < \varepsilon L, \quad \text{or}$$
$$|V_i \cap B| < \varepsilon L.$$

Proof. Let $V_1^A = V_1 \cap A$ and $V_1^B = V_1 \cap B$. Suppose without loss of generality that $|V_1^A| \geq \varepsilon L$ and $|V_1^B| \geq \varepsilon L$. From the minimum degree condition on G'', there must be some cluster V_2 such that (V_1,V_2) forms an ε-regular pair with $d(V_1,V_2) > \delta$. Since $|S| < \frac{\varepsilon(1-\varepsilon)}{r}n < (1-\varepsilon)\varepsilon L$, we must have more than εL vertices of V_2 in either A or B. Without loss of generality, we will assume these vertices are in A. Call this set of vertices V_2^A. Then the pair (V_1^B, V_2^A) has no edges, contradicting the (ε,δ)-regularity of (V_1,V_2) (see Figure 2). Hence, either $|V_1 \cap A| < \varepsilon L$ or $|V_1 \cap B| < \varepsilon L$. $\square$

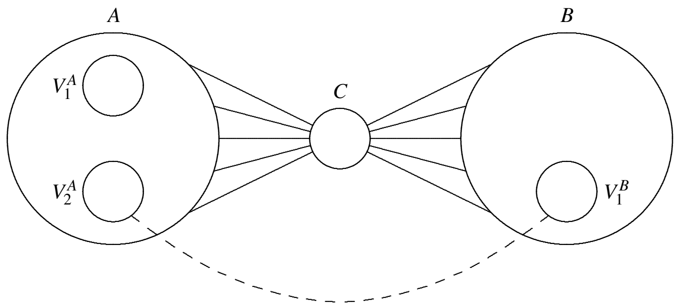

Figure 2. There are no edges between V_2^A and V_1^B.

The statement of the above claim is equivalent to saying that for all i satisfying $1 \le i \le r$, we have either

$$|V_i \cap A| > (1-\varepsilon)L, \quad \text{or} \tag{7}$$

$$|V_i \cap B| > (1-\varepsilon)L. \tag{8}$$

Since R is connected and $r \ge 2$, there must be a cluster V_i satisfying (7) and a cluster V_j satisfying (8) such that (V_i,V_j) is an ε-regular pair. However, the pair (V_1^A, V_2^B) has no edges, contradicting the (ε,δ)-regularity of (V_1,V_2). This contradicts the bound on $|S|$, meaning that G'' is $\left(\frac{\varepsilon(1-\varepsilon)}{r}n\right)$-connected.

Now consider $G' = G'' \cup V_0$. Every vertex in V_0 has at least $\left(\frac{1}{2}-\varepsilon\right)n$ edges into G''. Since the addition of a k,J-star to a k-connected graph J yields a k-connected graph, we can add each vertex of V_0 to G'' preserving the connectivity. It follows that G', and therefore G as well, is $\left(\frac{\varepsilon(1-\varepsilon)}{r}n\right)$-connected. $\square$

References

[1] B. Andrásfai, P. Erdős, V. T. Sós, On the connection between chromatic number, maximal clique and minimal degree of a graph, *Discrete Math.* 8 (1974) 205–218.

[2] B. Bollobás, Modern graph theory, volume 184 of Graduate texts in mathematics. Springer, New York, Berlin, Paris, 1998.

[3] R. Diestel, Graph theory, volume 173 of Graduate Texts in Mathematics, Springer-Verlag, New York, second edition, 2000.

[4] R. Faudree, R. Gould, M. Jacobson, C. Magnant, Distributing vertices on a hamiltonian cycle, *J. Graph Theory* 69 (2011) 28–45.

[5] M. Ferrara, R. Gould, M. Jacobson, F. Pfender, J. Powell, T. Whalen, New Ore-type conditions for H-linked graphs, *J. Graph Theory* 71 (2012) 69–77.

[6] M. Ferrara, R. Gould, G. Tansey, T. Whalen, On H-linked graphs, *Graphs Combin.* 22 (2006) 217–224.

[7] M. Ferrara, C. Magnant, J. Powell, Pan-H-linked graphs, *Graphs Combin.* 26 (2010) 225–242.

[8] R. Gould, A. Kostochka, G. Yu, On minimum degree implying that a graph is H-linked, *SIAM J. Discrete Math.* 20 (2006) 829–840.

[9] R. Gould, T. Whalen, Subdivision extendibility, *Graphs Combin.* 23 (2007) 165–182.

[10] H. A. Jung, Eine Verallgemeinerung des n-fachen Zusammenhangs für Graphen, *Math. Ann.* 187 (1970) 95–103.

[11] J. Komlós, G. N. Sárközy, E. Szemerédi, Blow-up lemma, *Combinatorica* 17 (1997) 109–123.

[12] J. Komlós, M. Simonovits, Szemerédi's regularity lemma and its applications in graph theory, in: *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993)*, volume 2 of *Bolyai Soc. Math. Stud.*, pages 295–352, János Bolyai Math. Soc., Budapest, 1996.

[13] A. Kostochka, G. Yu, An extremal problem for H-linked graphs, *J. Graph Theory* 50 (2005) 321–339.

[14] A. Kostochka, G. Yu, Minimum degree conditions for H-linked graphs, *Discrete Appl. Math.* 156 (2008) 1542–1548.

[15] D. Kühn, D. Osthus, A. Taraz, Large planar subgraphs in dense graphs, *J. Combin. Theory Ser. B* 95 (2005) 263–282.

[16] E. Szemerédi, Regular partitions of graphs, in: *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401, CNRS, Paris, 1978.

[17] T. Whalen, Degree conditions and relations to distance, extendability, and levels of connectivity in graphs, PhD thesis, Emory University, August 2003.

[18] J. E. Williamson, Panconnected graphs. II, *Period. Math. Hungar.* 8 (1977) 105–116.