**Applied Mathematical and Computational Sciences**
Volume 7, Issue 1, May 2015, Pages 1-14


# PROBABILISTIC SOLUTION OF RANDOM RICCATI-TYPE DIFFERENTIAL EQUATIONS APPEARING IN EPIDEMIOLOGICAL MODELS

**M.-C. CASABÁN, J.-C. CORTÉS, A. NAVARRO-QUILES and M.-D. ROSELLÓ**

Instituto Universitario de Matemática Multidisciplinar
Universitat Politècnica de València, Spain
E-mail: jccortes@imm.upv.es

## Abstract

This paper deals with the probabilistic solution of random Riccati-type differential equations that appear in a class of epidemiological models usually referred to as SIS-type models. Taking advantage of Random Variable Transformation technique, the first probabilistic density function of the solution stochastic process of that class of random differential equations is determined by two different ways. This permits to characterize, from a probabilistic point of view, the solution in every time instant and to compute its expectation, variance and confidence intervals. The obtained results are very general since all the model input parameters are assumed to be random variables with an arbitrary joint probability density function.

## 1. Introduction

Deterministic ordinary differential equations have played a key role to model many infectious diseases [11, 1]. The application of these models requires setting model input parameters such as coefficients, forcing terms and initial/boundary conditions. In practice, these values are set from measurements, which often involve measurement and modelling errors. From this simple but realistic perspective, it is natural to consider model input parameters as random variables or stochastic processes rather than deterministic quantities. Under this approach, two main classes of differential equations, which consider into their formulation uncertainty,

2010 Mathematics Subject Classification: 60H25, 34K50, 47B80, 47H40.
Keywords: First Probability Density Function, Random Variable Transformation Method, Riccati-type Random Differential Equations, SIS-type Epidemiological Model.
Received January 16, 2015; Revised April 27, 2015
Accepted April 27, 2015

have been proposed to formulate continuous models, namely, stochastic differential equations (SDEs) and random differential equations (RDEs).

On the one hand, the uncertainty or noise considered in SDEs is forced by an irregular process, usually termed white noise [3, pp. 40-44]. This process is the derivative of the Wiener process or Brownian motion (in a generalized sense of distributions). The Wiener process is characterized because its increments are Gaussian, stationary and independent. As a result, its increments are implicitly assumed to be statistically uncorrelated [14]. Thus, randomness in SDEs is limited to Gaussian type. This approach leads to Ito-type SDEs. Handling SDEs requires a special stochastic calculus, usually referred to as Ito-Calculus [3, Part II]. Ito's Lemma is the cornerstone result to solve SDEs [3, Ch.3]. In some cases the consideration of SDEs from its deterministic counterpart, can nicely be motivated via the perturbation of the involved deterministic input parameters [5, Ch.1]. Additionally, SDEs can also be seen as continuous version of auto-regressive models, which are widely applied in statistics [21]. Apart from using Wiener process to model uncertainty in SDEs, some authors have also considered another class of stochastic processes, termed coloured noise. This approach enables to take into account the short-term correlation often encountered in applications. A good account of SDEs with coloured noise and their applications can be found in [3, p.259], [6] and in [8, Ch.3]. Finally, we point out a generalization of Ito-type SDEs referred to as Ito-type stochastic fuzzy differential equations. This class of equations extends the notions of real-valued Ito SDE and set-valued Ito SDE using, in its integral form, fuzzy stochastic Lebesgue-Aumann integral and fuzzy-stochastic Ito integral driven by Wiener process [16]. None of the above types of SDEs is considered throughout this paper.

On the other hand, the random character of RDEs is manifested directly through input parameters, which are assumed to have regular behavior described by standard probabilistic distributions. For this choice, a wide range of potential probabilistic distributions are allowed including the classical ones such as exponential, gamma, beta, Gaussian, etc. In this manner, it can be considered that uncertainty is introduced in a more natural way in dealing with RDEs than SDEs. Unlike SDEs, RDEs permit considering other probabilistic behaviours apart from Gaussian randomness. An excellent introduction to RDEs can be found in [19]. Generalizations to

RDEs using the so-called fuzzy approach has been made recently leading to random fuzzy-differential equations [17].

In this paper, a Riccati-type differential equation that appears in dealing with an important epidemiological model, termed SIS model, is solved from a probabilistic standpoint using the RDE approach. SIS model considers the spread of a disease which develops over time. This model assumes that individuals of population are classified in two types of individuals, Susceptible (S) and Infected (I). Transitions can be from Susceptible to Infected or vice versa, which usually is represented as $S \to I \to S.$ SIS-type epidemiological models have been extensively used to describe the spread of diseases such as, gonorrhea, meningitis, etc. [11, 1, 5]. We will introduce randomness in the SIS model by considering in its deterministic formulation that all the involved model input parameters (coefficients and initial conditions) are random variables rather than constant quantities. As it shall see later, this leads to a Riccati-type RDE. The main finding of this paper is the determination, through a closed-form expression, of the first probability density function (1-PDF) of the solution of the Riccati-type RDE associated to the SIS model. This will be done taking advantage of the so-called Random Variable Transformation (RVT) method. This method, that will be introduced in the next section, has been successfully applied to solve some types of RDEs both by means of abstract formulations [10, 2, 3] and in applications [20, 12]. As it will be seen later, the results presented in [2] concerning to first-order random linear differential equations will play an important role in our subsequent development. In the context of epidemiological models based on RDEs, the RVT method has been recently applied to study the so-called SI model [15, 4]. Under the SI model transitions from infected to susceptible subpopulations are not contemplated, thus it is simpler than the SIS model.

It is important to point out that the computation of the 1-PDF is advantageous because from it, a full probabilistic description in each time instant of the solution of the SIS model is achieved. In particular, the computation of the mean, the variance and confidence intervals follows straightforwardly from the 1-PDF. This fact constitutes valuable information from a practical point of view because it permits providing more realistic answers than its deterministic counterpart where predictions are just punctual values.

SIS-type models considering uncertainty in their formulation have been previously studied but using the SDE approach, see for example [9, 8]. Therefore, the randomness under this approach is assumed to be of Gaussian type, and as a consequence its application to real data is limited. As it has been previously pointed out, the approach proposed in this paper permits considering further probabilistic distributions for the model input parameters besides Gaussian patterns and also including the possibility that model input parameters be statistically dependent, i.e., they have an arbitrary joint probability density function.

This paper is organized as follows. Section 2 is devoted to introduce SIS model through a system of differential equations whose inputs are random variables. An important technique that we will need for solving this type of problems, the RVT method, is presented in Section 3. In Section 4, the 1-PDF of the solution of that random system is computed taking advantage of RVT technique. Conclusions are drawn in Section 5.

## 2. Description of the SIS Model

SIS model is a classical mathematical representation to describe the dynamics of diseases for which infection does not confer immunity. The total number of individuals of the population are divided into two subpopulations, Susceptibles ($S$) and Infected (I). The SIS-model can be formulated by the following nonlinear system of differential equations

$$\begin{cases} S'(t) = -\beta S(t)I(t) + \gamma I(t), \\ I'(t) = \beta S(t)I(t) - \gamma I(t), \end{cases} \quad t > 0, \tag{1}$$

with initial conditions

$$S(0) = S_0,\ I(0) = I_0. \tag{2}$$

$S(t)$ and $I(t)$ denote the percentage of susceptibles and infected at the time instant $t$, respectively. At the beginning these values correspond to $S_0$ and $I_0$. As the total population is classified as either susceptible or infected, then

$$S(t) + I(t) = 1,\ \forall t \geq 0. \tag{3}$$

The parameters $\beta$ and $\gamma$ denote the rate of decline in the percentage of susceptibles and the recovery rate (infected that are again susceptible), respectively. The model can be represented by means of a compartmental diagram (see Figure 1). From this graphical description and using an argument based on the physical mass law it is straightforward setting the model (1)-(2), [1].

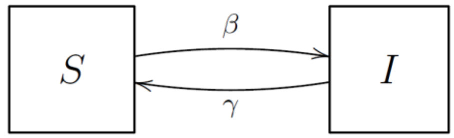


**Figure 1.** Flow diagram representation of the SIS model (1)-(2).

As it has been previously motivated, throughout this paper we will assume that parameters $S_0$, $\beta$ and $\gamma$ are random variables. As a result, from (3), $I_0$ is also a random variable.

Although the main goal of this contribution is to determine the 1-PDF of the underlying Riccati-type RDE associated to model (1)-(2), it is important to point out some issues regarding its applications. First, since random variable $S_0$ represents the initial percentage of susceptibles, it lies between 0 and 1, hence a consistent probability distribution to describe it is the beta distribution. Secondly, random input parameters $\beta$ and $\gamma$ determine the contagion and the recovery rate from the disease, respectively, and, as a consequence, both are positive. Therefore, consistent probability distributions to be assigned to them are, for example, exponential and gamma, as well as another distribution, like Gaussian, truncated on positive intervals. All these types of distributions can be handled under the proposed approach.

## 3. Some Results about RVT Method

As it has been pointed out previously, the main goal of this paper is to obtain the 1-p.d.f. of the solution of the Riccati-type associated to the SIS model (1)-(2) taking advantage of RVT method. RVT is a powerful method to determine the PDF of a random variable which comes from mapping another random variable whose PDF is known.

The next result establishes a general version of RVT method.

**Theorem 1** (RVT multidimensional version [10, pp. 24-25]). *Let* $\mathbf{U} = (U_1, \ldots, U_n)^T$ *and* $\mathbf{V} = (V_1, \ldots, V_n)^T$ *be two n-dimensional absolutely continuous random vectors. Let* $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ *be a one-to-one deterministic transformation of* $\mathbf{U}$ *into* $\mathbf{V}$, *i.e.,* $\mathbf{V} = g(\mathbf{U})$. *Let us assume that g is continuous in* $\mathbf{U}$ *and has continuous partial derivatives with respect to* $\mathbf{U}$. If $f_{\mathbf{U}}(\mathbf{u})$ *denotes the joint PDF of vector* $\mathbf{U}$, *and* $\mathbf{h} = \mathbf{g}^{-1} = (h_1(v_1, \ldots, v_n), \ldots, h_n(v_1, \ldots v_n))^T$ *represents the inverse mapping of* $\mathbf{g} = (g_1(u_1, \ldots, u_n), \ldots, g_n(u_1, \ldots u_n))^T$, *then the joint PDF of vector* $\mathbf{V}$ *is given by*

$$f_{\mathbf{V}}(\mathbf{v}) = f_{\mathbf{U}}(\mathbf{h}(\mathbf{v}))\,|\,J\,|,$$

*where* $|\,J\,|$ *is the absolute value of the Jacobian defined by*

$$J = \det\left(\frac{\partial \mathbf{u}^T}{\partial \mathbf{v}}\right) = \det\begin{pmatrix} \dfrac{\partial h_1(v_1, \ldots, v_n)}{\partial v_1} & \cdots & \dfrac{\partial h_n(v_1, \ldots, v_n)}{\partial v_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial h_1(v_1, \ldots, v_n)}{\partial v_n} & \cdots & \dfrac{\partial h_n(v_1, \ldots, v_n)}{\partial v_n} \end{pmatrix}.$$

Now, we establish several results from Theorem 1 that will be required in the next section.

**Proposition 1.** *Let* $\mathbf{U} = (U_1, U_2, U_3)^T$ *be an absolutely continuous random vector defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$ *and with joint PDF* $f_{U_1, U_2, U_3}(u_1, u_2, u_3)$ *such that* $\mathbb{P}[\{w \in \Omega : U_2(w) \neq 0\}] = 1$. *Then, the PDF* $f_L(l)$ *of the transformation* $L = U_1 + \dfrac{U_3}{U_2}$ *is given by*

$$f_L(l) = \int_{\mathcal{D}(U_2)} \int_{\mathcal{D}(U_3)} f_{U_1, U_2, U_3}\left(l - \frac{u_3}{u_2}, u_2, u_3\right) du_3 du_2. \tag{4}$$

**Proof.** Let us consider the transformation $(v_1, v_2, v_3) = \mathbf{g}(u_1, u_2, u_3) = \left(u_1 + \dfrac{u_3}{u_2}, u_2, u_3\right)$. Its inverse mapping is given by $\mathbf{h}(v_1, v_2, v_3) = \left(v_1 - \dfrac{v_3}{v_2}, v_2, v_3\right)$, being its Jacobian $J = 1 \neq 0$. Then, applying Theorem 1, we get the joint PDF of the random vector $\mathbf{V} = \left(U_1 + \frac{U_3}{U_2}, U_2, U_3\right)$

$$f_{V_1, V_2, V_3}(v_1, v_2, v_3) = f_{U_1, U_2, U_3}\left(v_1 - \frac{v_3}{v_2}, v_2, v_3\right).$$

Now, the PDF of random variable $L$ given by (4) is straightforwardly obtained by marginalizing $f_{V_1, V_2, V_3}(v_1, v_2, v_3)$ with respect to $V_2$ and $V_3$. □

**Proposition 2.** *Let* $c \in \mathbb{R}$ *and X be an absolutely continuous real random variable defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$, *with PDF* $f_X(x)$. *Assume that X is a non-zero random variable and let us denote by* $\mathcal{D}(X)$ *the domain of X, where*

$$\mathcal{D}(X) = I_x^- \cup I_x^+, \begin{cases} I_x^- = \{x = X(\omega) \in \mathbb{R} : x < 0, \omega \in \Omega\}, \\ I_x^+ = \{x = X(\omega) \in \mathbb{R} : x > 0, \omega \in \Omega\}. \end{cases}$$

*Then, the PDF* $f_Y(y)$ *of the inverse-vertical translation transformation* $Y = \frac{1}{X} + c$ *is given by*

$$f_Y(y) = \frac{1}{(y-c)^2} f_X\left(\frac{1}{y-c}\right), \; y \in \mathcal{D}(Y), \tag{5}$$

*where*

$$\mathcal{D}(Y) = I_y^- \cup I_y^+, \begin{cases} I_y^- = \{y \in \mathbb{R} : y < c\}, \\ I_y^+ = \{y \in \mathbb{R} : y > c\}. \end{cases}$$

**Proof.** Let us consider the transformation $y = g(x) = \frac{1}{x} + c$. Its inverse mapping is given by $x = h(y) = \frac{1}{y-c}$, being its Jacobian $J = -1/(y-c)^2 \neq 0$. Then, expression (5) corresponding to the PDF of random variable $Y = \frac{1}{X} + c$ is immediately obtained by applying Theorem 1. The determination of the domain $\mathcal{D}(Y)$ follows easily since the transformation $g(x)$ is decreasing monotone in each subinterval. □

**Proposition 3.** *Let* $\mathbf{U} = (U_1, U_2, U_3)^T$ *be an absolutely continuous random vector defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$ *and with joint PDF* $f_{U_1, U_2, U_3}(u_1, u_2, u_3)$ *such that* $\mathbb{P}[\{w \in \Omega : U_1(w) \neq 1\}] = 1$. *Then, the PDF* $f_{V_1, V_2, V_3}(v_1, v_2, v_3)$ *of the three-dimensional transformation*

$$V_1 = \frac{1}{U_1 - 1}, \; V_2 = -U_3, \; V_3 = U_2 - U_3,$$

*is given by*

$$f_{V_1, V_2, V_3}(v_1, v_2, v_3) = f_{U_1, U_2, U_3}\left(\frac{1}{v_1} + 1, \; v_3 - v_2, \; -v_2\right)\frac{1}{(v_1)^2}. \tag{6}$$

**Proof.** Let us consider the transformation $(v_1, v_2, v_3) = \mathbf{g}(u_1, u_2, u_3) = \left(\frac{1}{u_1 - 1}, -u_3, u_2 - u_3\right)$. Its inverse mapping is given by $\mathbf{h}(v_1, v_2, v_3) = \left(\frac{1}{v_1} + 1, v_3 - v_2, -v_2\right)$, being its Jacobian $J = 1/(v_1)^2 \neq 0$. Then, applying Theorem 1, we get the joint PDF of $\mathbf{V} = \left(\frac{1}{U_1 - 1}, -U_3, U_2 - U_3\right)$ given by (6).

## 4. Main Result: Determination of the 1-PDF of the SIS Model

Notice that from the relationship (3) one gets $I(t) = 1 - S(t)$. Hence, initial value problem (IVP) (1)-(2) can be rewritten as the following Riccati RDE with initial condition

$$\begin{cases} S'(t) = \beta(S(t))^2 - (\gamma + \beta)S(t) + \gamma, \; t > 0, \\ S(0) = S_0. \end{cases} \tag{7}$$

In this section we will determine the probabilistic solution of (7) by computing its 1-PDF. In this manner a full probabilistic description of the percentage of susceptibles, $S(t)$, of the SIS model (1)-(2) will be done. This computation will be performed in two different ways. It will lead to two representations of the 1-PDF, which, although in appearance seem to be distinct are equivalent. At this point we underline that having several available representations of the 1-PDF is very useful from a computational point in order to reduce the computational burden. This also exhibits the flexibility of RVT technique in dealing with the determination of the 1-PDF of RDEs.

Our main result is

**Theorem 2.** *Let us consider the Riccati RDE* (7) *where* $(S_0, \gamma, \beta)$ *is assumed to be a random vector defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$ *and with joint PDF* $f_{S_0,\gamma,\beta}(s_0, \gamma, \beta)$. *Then, the* 1-*PDF of the solution stochastic process* $S(t)$ *of* (7) *can be represented in the two following ways*:

$$f_S(s) = \frac{1}{(s-1)^2} \times \int_{\mathcal{D}(Z_3)} \int_{\mathcal{D}(Z_2)} f_{S_0,\gamma,\beta}\left( \frac{-\eta}{\left(\frac{1}{s-1} - \eta - \xi\right)\xi} + 1, \frac{1+\xi}{t} \ln\left(\frac{-\eta}{\xi}\right), \frac{\xi}{t} \ln\left(\frac{-\eta}{\xi}\right) \right)$$

$$\times \frac{(s-1)^2}{(1 - s\eta + \eta - s\xi + \xi)^2 |\xi| t^2} \left| \ln\left(-\frac{\eta}{\xi}\right) \right| d\eta d\xi, \tag{8}$$

*where* $Z_2$ *and* $Z_3$, *that appear in the domains of integration, are defined by* $Z_2 = -\frac{\beta}{\gamma - \beta} e^{(\gamma - \beta)t}$ *and* $Z_3 = \frac{\beta}{\gamma - \beta}$, *and*

$$f_S(s) = \frac{1}{(s-1)^2} \times \int_{\mathcal{D}(\gamma-\beta)} \int_{\mathcal{D}(\beta)} f_{S_0,\gamma,\beta}\left( \frac{\eta(\eta+\xi)(s-1)e^{\eta t} + \eta^2 - \xi\eta(s-1)}{1 - \eta\xi(s-1) + \xi\eta(s-1)e^{\eta t}}, \eta + \xi, \xi \right)$$

$$\times \left( \frac{(s-1)\eta^2}{1 - \xi\eta(s-1) + \xi\eta(s-1)e^{\eta t}} \right) e^{\eta t} d\xi d\eta. \tag{9}$$

**Proof.** In order to demonstrate the first representation, let us introduce the following change of variable

$$Q(t) = \frac{1}{S(t) - 1}. \tag{10}$$

This permits rewriting the Riccati RDE (7) as the linear IVP

$$\begin{cases} Q'(t) = (\gamma - \beta)Q(t) - \beta, \; t > 0, \\ Q(0) = \dfrac{1}{S_0 - 1}. \end{cases} \tag{11}$$

Let us suppose that the 1-PDF, $f_Q(q)$, of the solution $Q = Q(t)$ of IVP (11) has been obtained. Then, by applying Proposition 2 to $Y = S$, $X = Q$ and $c = 1$, the PDF of the number of susceptibles, $S = S(t)$, will be given by

$$f_S(s) = \frac{1}{(s-1)^2} f_Q\left(\frac{1}{s-1}\right). \tag{12}$$

Now, we will determine the 1-PDF $f_Q(q)$, and as a result, the 1-PDF of $S(t)$ will be obtained using (12). To that end, let us apply expression [14, Equation (157)] taking

$$Z = Q, \; A = \gamma - \beta, \; B = -\beta, \; Z_0 = \frac{1}{S_0 - 1} \text{ and } t_0 = 0.$$

Then, according to [14, Equation (157)] the expression of the PDF for the random variable $Q$ is given by

$$f_Q(q) = \int_{\mathcal{D}(Z_3)} \int_{\mathcal{D}(Z_2)} f_{Z_0, B, A}\left(-\frac{(q - \eta - \xi)\xi}{\eta}, -\frac{\xi}{t} \ln\left(-\frac{\eta}{\xi}\right), \frac{1}{t} \ln\left(-\frac{\eta}{\xi}\right)\right)$$

$$\times \frac{|\xi|}{\eta^2} \frac{1}{t^2} \left| \ln\left(-\frac{\eta}{\xi}\right) \right| d\eta d\xi,$$

where $Z_2 = \dfrac{B}{A} \mathrm{e}^{At}$ and $Z_3 = \dfrac{-B}{A}$. Since the PDF of random vector $(S_0, \gamma, \beta)$ is assumed to be known, applying Proposition 3 with the following identification: $U_1 = S_0$, $U_2 = \gamma$ and $U_3 = \beta$, one gets the PDF of the random vector $\left(\dfrac{1}{S_0 - 1}, -\beta, \gamma - \beta\right)$ in terms of our data

$$f_Q(q) = \int_{\mathcal{D}(Z_3)} \int_{\mathcal{D}(Z_2)} f_{S_0, \gamma, \beta}\left(\frac{-\eta}{(q - \eta - \xi)\xi} + 1, \frac{1}{t} \ln\left(\frac{-\eta}{\xi}\right)(1 + \xi), \frac{\xi}{t} \ln\left(\frac{-\eta}{\xi}\right)\right)$$

$$\times \frac{1}{(q - \eta - \xi)^2 |\xi| t^2} \left| \ln\left(\frac{-\eta}{\xi}\right) \right| d\eta d\xi.$$

Notice that domains $\mathcal{D}(Z_2)$ and $\mathcal{D}(Z_3)$ are easily determined in terms of the domains of the inputs $\gamma$ and $\beta$ as follows

$$\mathcal{D}(Z_2) = \mathcal{D}\left(-\frac{\beta}{\gamma-\beta} e^{(\gamma-\beta)t}\right), \; \mathcal{D}(Z_3) = \mathcal{D}\left(\frac{\beta}{\gamma-\beta}\right).$$

Now, taking into account relationship (12), it is straightforward to obtain expression (8).

In order to establish (9), let us introduce the following change of variables

$$Q(t) = H(t) + \frac{\beta}{\gamma-\beta}. \tag{13}$$

It permits writing IVP (11) as follows

$$\begin{cases} H'(t) = (\gamma-\beta)H(t), \quad t > 0, \\ H(0) = \dfrac{1}{S_0 - 1} - \dfrac{\beta}{\gamma-\beta}, \end{cases} \tag{14}$$

whose solution is given by

$$H(t) = \left(\frac{1}{S_0 - 1} - \frac{\beta}{\gamma-\beta}\right) e^{(\gamma-\beta)t}.$$

Next, let us apply Theorem 1 to random vector $\mathbf{U} = (S_0, \gamma, \beta)$

$$V_1 = \left(\frac{1}{S_0 - 1} - \frac{\beta}{\gamma-\beta}\right) e^{(\gamma-\beta)t} = \mathbf{H}(t), \; \mathbf{V}_2 = \gamma-\beta, \; V_3 = \beta.$$

This yields

$$f_{V_1, V_2, V_3}(v_1, v_2, v_3) = f_{S_0, \gamma, \beta}\left(\frac{(v_2 + v_3)e^{v_2 t} + v_1 v_2}{v_1 v_2 + v_3 e^{v_2 t}}, v_2 + v_3, v_3\right)$$

$$\times \left(\frac{v_2}{v_1 v_2 + v_3 e^{v_2 t}}\right)^2 e^{v_2 t}. \tag{15}$$

Now, we compute the PDF of $Q = Q(t)$ taking into account the relationship (13) between $Q(t)$ and $H(t)$. To that end, let us apply Proposition 1 to $\mathbf{U} = (V_1, V_2, V_3)$ and $V = Q$ one gets

$$f_Q(q) = \int_{\mathcal{D}(\gamma-\beta)} \int_{\mathcal{D}(\beta)} f_{V_1, V_2, V_3}\left(q - \frac{\xi}{\eta}, \eta, \xi\right) d\xi d\eta.$$

Therefore, by (15) we obtain the PDF of $Q$

$$f_Q(q) = \int_{\mathcal{D}(\gamma-\beta)} \int_{\mathcal{D}(\beta)} f_{S_0,\gamma,\beta}\left( \frac{(\eta+\xi)e^{\eta t} + q\eta - \xi}{q\eta - \xi + \xi e^{\eta t}},\ \eta+\xi,\ \xi \right) \times \left( \frac{\eta}{q\eta - \xi + \xi e^{\eta t}} \right)^2 e^{\eta t} d\xi d\eta.$$

Finally, taking into account (12) it is straightforward to obtain expression (9). This finishes the proof. □

## 5. Conclusions

The SIS model plays an important role in modelling the spread of diseases over time. The consideration of uncertainty in this epidemiological model leads to a Riccati-type random differential equation. In this paper we have obtained the first probability density function (1-PDF) to the solution stochastic process of that random differential equation taking advantage of the so-called Random Variable Transformation technique. The computation of the 1-PDF has been done by two different ways assuming that all input parameters are random variables. The 1-PDF has been obtained under very general hypotheses since all model input parameters are assumed to be random variables with arbitrary joint PDF. This fact is a key issue in dealing with practical applications of the SIS model where the random input parameters could be statistically dependent with different joint PDFs. Moreover, we point out that from a practical standpoint, the computation of the 1-PDF is very useful since from it one can construct both punctual and probabilistic predictions by means of the mean and confidence intervals, respectively.

## Acknowledgements

This work has been partially supported by the Ministerio de Economía y Competitividad grants MTM2013-41765-P and TRA2012-36932. Ana Navarro Quiles acknowledges the doctorate scholarship granted by Programa de Ayudas de Investigación y Desarrollo (PAID), Universitat Politècnica de València.

## References

[1] F. Brauer and C. Castillo-Chávez, Mathematical Models in Population Biology and Epidemiology, Springer, New York, 2001.

[2] M. C. Casabán, J. C. Cortées, J. V. Romeró and M. D. Rosello, Determining the first probability density function of linear random initial value problems by the Random Variable Transformation (RVT) technique: A comprehensive study, Abstract and Applied Analysis, 2014-ID248512, 1-25, 10.1155/2014/248512.

[3] M. C. Casabán, J. C. Cortés, J. V. Romero and M. D. Roselló, Probabilistic solution of random homogeneous linear second-order difference equations, Applied Mathematics Letters 34 (2014), 27-32.

[4] M. C. Casabán, J. C. Cortés, J. V. Romero and M. D. Roselló, Probabilistic solution of random SI-type epidemiological models using the random variable transformation technique, Communications Nonlinear Science and Numerical Simulation 24(13) (2015), 86-97.

[5] C. Castillo-Chávez, W. Huang and J. Li, Competitive exclusion in gonorrhea models and other sexually-transmitted diseases, SIAM Journal Applied Mathematics 56 (1996), 494-508.

[6] W. M. Charles, E. van den Berg and A. W. Heemink, Coloured noise for dispersion of contaminants in shallow waters, Applied Mathematical Modelling 33(2) (2009), 1158-1171.

[7] W. M. Charles, Application of coloured noise as a driving force in the stochastic differential equations, In Stochastic Control (Chapter 3), Croatia, Intech. 2010.

[8] C. Chen and K. Yanmei, Dynamics of a stochastic SIS epidemic model with saturated incidence, Abstract and Applied Analysis, 2014-ID723825, 1-13, doi:10.1155/2014/723825.

[9] D. Clancy, A stochastic SIS infection model incorporating indirect transmission, Journal of Applied Probability, 331, 42(3) (2007), 726-737.

[10] M. A. El-Tawil, The approximate solutions of some stochastic differential equations using transformations, Applied Mathematics and Computation 164(1) (2005), 167-178.

[11] H. W. Hethcote, The mathematics of infectious diseases, SIAM Review 42 (2000), 599-653.

[12] A. Hussein and M. M. Selim, Solution of the stochastic radiative transfer equation with Rayleigh scattering using RVT technique, Applied Mathematics and Computation 218(13) (2012), 7193-7203.

[13] P. Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations, Series: Applied Mathematics, 23 3rd Ed., Springer, New York, 2000.

[14] I. Karatzas and S. E. Shreve, Brownian Motion and Stochastic Calculus, Springer, New York, 1998.

[15] B. Kegan and R. W. West, Modeling the simple epidemic with deterministic differential equations and random initial conditions, Mathematical Biosciences 195(5) (2005), 179-193.

[16] V. Lakshmikantham and R. N. Mohapatra, Theory of Fuzzy Differential Equations and Inclusions, London, Taylor and Francis, 2003.

[17] M. T. Malinowski, On random fuzzy-differential equations, Fuzzy Sets and Systems 160 (2009), 3152-3165.

[18] B. Øksendal, Stochastic Differential Equations, An Introduction with Applications, 5th Ed., Springer, Berlin, 1998.

[19] T. T. Soong, Random Differential Equations in Science and Engineering, Academic Press, New York, 1973.

[20] L. T. Santos, F. A. Dorini and M. C. C. Cunha, The probability density function to the random linear transport equation, Applied Mathematics and Computation 216(5) (2010), 1524-1530.

[21] H. Tong, Nonlinear Times Series: A Dynamical System Approach, Clarendon Press, Oxford, 1990.