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CERTAIN SEMIGROUP C^*- ALGEBRAS INDUCED BY *p*-ADIC ANALYSIS

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Abstract

In this paper, we construct certain multiplicative semigroups S_p in the C^*- algebras M_p generated by the $*$-probability space $(\mathcal{M}_p, \varphi_p)$ of the $*$-algebra $\mathcal{M}_p$ consisting of all measurable functions on the p-adic number fields $\mathbb{Q}_p$, and the p-adic integration φ_p, for primes p. We study operator-algebraic properties of the corresponding semigroup C^*- subalgebra $\mathfrak{S}_p$ of M_p, generated by S_p, and spectral properties of $\mathfrak{S}_p$, by computing free distributins of generating operators of $\mathfrak{S}_p$, for all primes p. More generally, we construct free product C^*- probability spaces of $\mathfrak{S}_p$'s, and study free probability on these C^*- probability spaces. Our main results illustrate another connection between number theory and operator algebra theory, via free probability.

1. Introduction

The main purpose of this paper is to consider connections among the number-theoretic results from p-adic analysis, operator-algebraic structures induced by p-adic analysis, and operator-theoretic (especially, spectral-

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theoretic) properties over p-adic number fields $\mathbb{Q}_p$, for primes p. Main tools to figure out such connections are free probability and representation theory.

1.1. Preview and Motivation. The relations between primes and operator algebras have been studied in various different approaches. For instance, we studied how primes act on certain von Neumann algebras generated by p-adic, and Adelic measure spaces (e.g., [3] and [4]). Meanwhile, in [5] and [6], primes are regarded as linear functionals acting on arithmetic functions. In such a case, one can understand arithmetic functions as Krein-space operators (for fixed primes), under certain Krein-space representations (e.g., [8]). Also, in [1], [2] and [7], we considered free-probabilistic structures on a Hecke algebra $\mathcal{H}(GL_2(\mathbb{Q}_p))$ for a fixed prime p.

In this paper, we considered free-probabilistic models on the $*$-algebra $\mathcal{M}_p$, consisting of all Haar-measurable functions over $\mathbb{Q}_p$, for primes p, and its Hilbert-space representation. i.e., we concentrate on investigating p-adic analysis in terms of suitable operator-algebraic settings. Under our representation, corresponding C^*-algebras M_p of $\mathcal{M}_p$ are constructed, and free probability on M_p is considered. In particular, for all $j \in \mathbb{Z}$, we define C^*-probability spaces (M_p, φ_j^p), where φ_j^p are kind of sectionized linear functionals implying the p-adic-analytic data on $\mathcal{M}_p$, in terms of the usual p-adic integration on $\mathbb{Q}_p$.

In particular, we are interested in a semigroup S_p in M_p generated by certain projections of M_p, and the corresponding semigroup C^*-subalgebra $\mathfrak{S}_p$ of M_p. By restricting our interests to the sub-C^*-probability spaces $(\mathfrak{S}_p, \varphi_j^p)$, for $p \in \mathcal{P}$, $j \in \mathbb{Z}$, we study free-distributional data of generating elements of $\mathfrak{S}_p$. Also, by establishing free products of $(\mathfrak{S}_p, \varphi_j^p)$, we study spectral data of operators generating $\mathfrak{S}_p$, for all $p \in \mathcal{P}$.

1.2. Overview. In Sections 2, we introduce backgrounds and a motivation of our works.

In Section 3, our free-probabilistic models on $\mathcal{M}_p$ is established, and considered based on p-adic analysis.

In Section 4, Hilbert-space representations of our free-probabilistic models of $\mathcal{M}_p$ are constructed. They preserve the free-distributional data implying number-theoretic information from $\mathcal{M}_p$. Under representation, corresponding C^*-algebras M_p are constructed from $\mathcal{M}_p$.

In Section 5, free probability on M_p is studied. In particular, we compute free distributions of generating operators of M_p.

In Section 6, semigroups S_p of M_p are introduced, and the structure theorem for the C^*-subalgebras $\mathfrak{S}_p$ of M_p is shown. Depending on our structure theorem, suitable free-probabilistic models on S_p are constructed. We study free probability on $\mathfrak{S}_p$.

In Section 7, we study free probability on the free products of $\mathfrak{S}_p$ over primes.

2. Preliminaries

In this section, we briey mention about backgrounds of our proceeding works.

2.1. Free Probability. Readers can check fundamental analytic-and-combinatorial free probability from [12] and [14] (and the cited papers therein). Free probability is understood as the noncommutative operator-algebraic version of classical probability theory (covering commutative cases). The classical independence is replaced by the freeness, by replacing measures to linear functionals. It has various applications not only in pure mathematics (e.g., [11]), but also in related applied topics (for example, see [3], [4], [6] and [8]). In particular, we will use combinatorial approach of Speicher (e.g., [12]).

Especially, in the text, without introducing detailed definitions and combinatorial backgrounds, free moments and free cumulants will be computed. Also, we use free product of C^*-probability spaces in the sense of

[12] and [14], without detailed introduction. However, rough introduction would be given whenever they are needed in text.

2.2. Calculus on $\mathbb{Q}_p$. Let $\mathbb{Q}_p$ be the p-adic number fields for $p \in \mathcal{P}$, equipped with the non-Archimedean p-norms $|\cdot|_p$ (on $\mathbb{Q}$), where $\mathcal{P}$ is the set of all primes in the natural numbers (or the positive integers) $\mathbb{N}$. This Banach space $\mathbb{Q}_p$ is also understood as a measure space

$$\mathbb{Q}_p = (\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p),$$

equipped with the left-and-right additive invariant Haar measure μ_p on the σ- algebra $\sigma(\mathbb{Q}_p)$. Recall also that, $\mathbb{Q}_p$ is a well-defined ring algebraically. If $x \in \mathbb{Q}_p$, then

$$x = \sum_{n=-N}^{\infty} x_n p^n \text{ with } x_n \in \{0, 1, \ldots, p-1\}$$

for some $N \in \mathbb{N}$, i.e.,

$$x = \left(\sum_{k=-N}^{-1} x_k p^k \right) + \left(\sum_{n=0}^{\infty} x_n p^n \right) \text{ in } \mathbb{Q}_p.$$

If $N \geq 0$, and hence, if $x = \sum_{n=0}^{\infty} x_n p^n$ in $\mathbb{Q}_p$, then x is said to be a p-adic integer of $\mathbb{Q}_p$.

As a topological space, the p-adic number field $\mathbb{Q}_p$ contains its basis elements

$$U_k = p^k \mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}, \tag{2.2.1}$$

satisfying the basis property,

$$\mathbb{Q}_p = \bigcup_{k \in \mathbb{Z}} U_k,$$

and the chain property,

$$\ldots \subset U_2 \subset U_1 \subset U_0 = \mathbb{Z}_p \subset U_{-1} \subset U_{-2} \subset \ldots,$$

and the measure-theoretic data,

$$\mu_p(U_k) = \frac{1}{p^k} = \mu_p(x + U_k), \text{ for all } k \in \mathbb{Z},$$

for all $x \in \mathbb{Q}_p$, where

$$U_0 = \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p = 1\}$$

$$= \bigcup_{l=0}^{\infty} U_l$$

is the unit disk of $\mathbb{Q}_p$, consisting of all p-adic integers.

By understanding $\mathbb{Q}_p$ as a measure space, one can establish a $*$-algebra $\mathcal{M}_p$ over $\mathbb{C}$ as a $*$-algebra consisting of all μ_p- measurable functions f,

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \text{ with } t_S \in \mathbb{C},$$

where the sum Σ means a finite sum, and χ_S are the usual characteristic functions of S. Of course, the adjoint f^* of f is defined to be

$$f^* = \sum_{S \in \sigma(\mathbb{Q}_p)} \overline{t_S} \chi_S \text{ in } \mathcal{M}_p,$$

where $\bar{z}$ mean the conjugates of z, for all $z \in \mathbb{C}$.

On $\mathcal{M}_p$, one can naturally define a linear functional φ_p,

$$\varphi_p(f) = \int_{\mathbb{Q}_p} f d\mu_p, \text{ for all } f \in \mathcal{M}_p, \tag{2.2.2}$$

and hence, the pair $(\mathcal{M}_p, \varphi_p)$ forms a well-determined $*$-probability space. Remark that it is a "commutative" $*$-probability space (and hence, it is well-covered by (non-commutative) free probability theory).

Define now subsets ∂_k of $\mathbb{Q}_p$ by

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}. \tag{2.2.3}$$

We call such μ_p- measurable subsets ∂_k, the k-th boundaries of the basis elements U_k of (2.2.1), which are also μ_p- measurable subsets, for all $k \in \mathbb{Z}$. By the basis property in (2.2.1), one obtains that

$$\mathbb{Q}_p = \bigcup_{k \in \mathbb{Z}} \partial_k, \tag{2.2.4}$$

where $\bigcup$ means the disjoint union, and

$$\mu_p(\partial_k) = \mu_p(U_k) - \mu_p(U_{k+1}) = \frac{1}{p^k} - \frac{1}{p^{k+1}},$$

by the measure-theoretic information in (2.2.1), for all $k \in \mathbb{Z}$.

Now, let $\mathcal{M}_p$ be the vector space of all μ_p- measurable functions on $\mathbb{Q}_p$, i.e.,

$$\mathcal{M}_p = \{f : \mathbb{Q}_p \to \mathbb{C} : f \text{ is } \mu_p\text{- measurable}\}. \tag{2.2.5}$$

So, $f \in \mathcal{M}_p$, if and only if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \, \chi_S \text{ with } t_S \in \mathbb{C},$$

where Σ means the finite sum, and χ_S are the usual characteristic functions of $S \in \sigma(\mathbb{Q}_p)$.

Then it forms a $*$-algebra over $\mathbb{C}$. Indeed, the vector space $\mathcal{M}_p$ of (2.2.5) is an algebra under the usual functional addition, and multiplication. Also, this algebra $\mathcal{M}_p$ has the adjoint,

$$\left(\sum_{S \in \sigma(G_p)} t_S \, \chi_S \right)^* \underline{\underline{\text{def}}} \sum_{S \in \sigma(G_p)} \overline{t_S} \, \chi_S,$$

where $t_S \in \mathbb{C}$, having their conjugates $\overline{t_S}$ in $\mathbb{C}$.

Let $f \in \mathcal{M}_p$. Then one can define the p-adic integral of f by

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \mu_p(S). \tag{2.2.6}$$

Note that, by (2.2.4), if $S \in \sigma(\mathbb{Q}_p)$, then there exists a subset Λ_S of $\mathbb{Z}$, such that

$$\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \varnothing\}, \tag{2.2.7}$$

satisfying the following result.

Proposition 2.1. *Let* $S \in \sigma(\mathbb{Q}_p)$, *and let* $\chi_S \in \mathcal{M}_p$. *Then there exist* $r_j \in \mathbb{R}$, *such that*

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_S, \tag{2.2.8}$$

and

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right),$$

where Λ_S *is in the sense of* (2.2.7).

Proof. By the basis property (2.2.1) and the boundary property (2.2.4), if $S \in \sigma(\mathbb{Q}_p)$, then

$$S = S \cap \mathbb{Q}_p = S \cap \left(\bigcup_{j \in \mathbb{Z}} \partial_j \right) = \bigcup_{j \in \mathbb{Z}} (S \cap \partial_j)$$

in $\mathbb{Q}_p$.

So, one obtains that

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \mu_p(S) = \mu_p \left(\bigcup_{j \in \mathbb{Z}} (S \cap \partial_j) \right)$$

$$= \sum_{j \in \mathbb{Z}} \mu_p(S \cap \partial_j) = \sum_{j \in \Lambda_S} \mu_p(S \cap \partial_j),$$

where $\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \varnothing\}$ in $\mathbb{Z}$, and hence, there exist

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_S,$$

such that

$$= \sum_{j \in \Lambda_S} r_j \, \mu_p(\partial_j) = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right).$$

Therefore, the formula (2.2.8) holds, for any $S \in \sigma(\mathbb{Q}_p)$.

By (2.2.8), one obtains that if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \in \mathcal{M}_p, \text{ with } t_S \in \mathbb{C},$$

then

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \left(\sum_{j \in \Lambda_S} r_j^S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right), \tag{2.2.9}$$

where r_j^S are in the sense of (2.2.8), for all $j \in \Lambda_S$, for all $S \in \sigma(\mathbb{Q}_p)$, whenever Λ_S of (2.2.7) is nonempty in $\mathbb{Z}$. i.e., one obtains the following p-adic integration, by (2.2.8).

Corollary 2.2. Let $f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \in \mathcal{M}_p$, with $t_S \in \mathbb{C}$. Then there exist $r_j^S \in \mathbb{R}$, such that

$$0 \leq r_j^S \leq 1, \textit{ for all } j \in \Lambda_S, \textit{ for all } S \in \sigma(\mathbb{Q}_p),$$

and

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \left(\sum_{j \in \Lambda_S} r_j^S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right).$$

Proof. The proof of the corollary is done by (2.2.8) and (2.2.9).

3. Free Probability on $\mathcal{M}_p$

Throughout this section, fix a prime $p \in \mathcal{P}$, and let $\mathbb{Q}_p$ be the corresponding p-adic number field, and let $\mathcal{M}_p$ be the $*$-algebra consisting of all μ_p- measurable functions on $\mathbb{Q}_p$. In this section, let's establish a suitable free-probabilistic model on the $*$-algebra $\mathcal{M}_p$. Remark that free probability provides a universal tool to study free distributions on "noncommutative" algebras, and hence, it covers the cases where given algebras are "commutative." Remark that $\mathcal{M}_p$ is a commutative $*$-algebra, but, for our later purposes, we construct free-probabilistic settings on $\mathcal{M}_p$.

Let U_k be the basis elements (2.2.1) of the topology for $\mathbb{Q}_p$, i.e.,

$$U_k = p^k \mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}, \tag{3.1}$$

with their boundaries

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}.$$

Define a linear functional $\varphi_p : \mathcal{M}_p \to \mathbb{C}$ by

$$\varphi_p(f) = \int_{\mathbb{Q}_p} f d\mu_p, \text{ for all } f \in \mathcal{M}_p. \tag{3.2}$$

The linear functionals φ_p of (3.2) on $\mathcal{M}_p$ are nothing but p-adic integrations on $\mathcal{M}_p$, for all $p \in \mathcal{P}$, and hence, they are well-defined unbounded linear functional on $\mathcal{M}_p$.

Then, by (3.2), one obtains that

$$\varphi_p(\chi_{U_j}) = \frac{1}{p^j}, \text{ and } \varphi_p(\chi_{\partial_j}) = \frac{1}{p^j} - \frac{1}{p^{j+1}},$$

since

$$\Lambda_{U_j} = \{k \in \mathbb{Z} : k \geq j\}, \text{ and } \Lambda_{\partial_j} = \{j\},$$

with help of (2.2.8) and (2.2.9), for all $j \in \mathbb{Z}$.

Moreover, by the commutativity on $\mathcal{M}_p$,

$$\varphi_p(f_1 f_2) = \varphi_p(f_2 f_1), \text{ for all } f_1, f_2 \in \mathcal{M}_p,$$

and hence, this linear functional φ_p of (3.2) is a trace on $\mathcal{M}_p$.

Definition 3.1. The free probability space $(\mathcal{M}_p, \varphi_p)$ is called the p-adic free probability space, for $p \in \mathcal{P}$, where φ_p is the linear functional (3.2) on $\mathcal{M}_p$.

Let U_k be in the sense of (3.1) in $\mathbb{Q}_p$, and $\chi_{U_k} \in \mathcal{M}_p$, for all $k \in \mathbb{Z}$. Then

$$\chi_{U_{k_1}} \chi_{U_{k_2}} = \chi_{U_{k_1} \cap U_{k_2}} = \chi_{U_{\max\{k_1, \dots, k_2\}}},$$

by (2.2.3), where $\max\{k_1, k_2\}$ means the maximum in $\{k_1, k_2\}$.

Say $k_1 \leq k_2$ in $\mathbb{Z}$. Then $U_{k_1} \supseteq U_{k_2}$ in $\mathbb{Q}_p$, by (2.2.3). Therefore, $U_{k_1} \cap U_{k_2} = U_{k_2}$ in $\mathbb{Q}_p$. So, if $k_1 \leq k_2$ in $\mathbb{Z}$, then

$$\chi_{U_{k_1}} \chi_{U_{k_2}} = \chi_{U_{k_1} \cap U_{k_2}} = \chi_{U_{k_2}} \text{ in } \mathcal{M}_p.$$

Thus, one can verify that

$$\varphi_p(\chi_{U_{k_1}} \chi_{U_{k_2}}) = \frac{1}{p^{\max\{k_1, k_2\}}}. \tag{3.3}$$

Inductive to (3.3), we obtain the following result.

Proposition 3.1. *Let* $(j_1, \dots, j_N) \in \mathbb{Z}^N$, *for* $N \in \mathbb{N}$. *Then*

$$\prod_{l=1}^{N} \chi_{U_{j_l}} = \chi_{U_{\max\{j_1, \dots, j_N\}}} \text{ in } \mathcal{M}_p, \tag{3.4}$$

and hence,

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{U_{j_l}}\right) = \frac{1}{p^{\max\{j_1, \dots, j_N\}}}.$$

Proof. The proof of (3.4) is done by induction on (3.3). Indeed, one can have that

$$S = \bigcap_{l=1}^{N} U_{j_l} = U_{\max\{j_1, \ldots, j_N\}},$$

by the chain property in (2.2.1). Moreover,

$$\chi_S = \chi_{\bigcap_{l=1}^{N} U_{j_l}} = \prod_{l=1}^{N} \chi_{U_{jl}} \text{ in } \mathcal{M}_p,$$

and hence,

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{U_{j_l}}\right) = \chi_p(\chi_S) = \varphi_p(\chi_{U_{\max\{j_1, \ldots, j_N\}}}).$$

Therefore, the joint free-moment formula (3.4) holds.

Now, let ∂_k be the k-th boundary $U_k \setminus U_{k+1}$ of U_k in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then, for $k_1, k_2 \in \mathbb{Z}$, one obtains that

$$\chi_{\partial_{k_1}} \chi_{\partial_{k_2}} = \chi_{\partial_{k_1} \cap \partial_{k_2}} = \delta_{k_1, k_2} \chi_{\partial_{k_1}}, \tag{3.5}$$

where δ means the Kronecker delta, and hence,

$$\varphi_p(\chi_{\partial_{k_1}} \chi_{\partial_{k_2}}) = \delta_{k_1, k_2} \varphi_p(\chi_{\partial_{k_1}})$$

$$= \delta_{k_1, k_2}\left(\frac{1}{p^{k_1}} - \frac{1}{p^{k_1+1}}\right).$$

So, we obtain the following computations.

Proposition 3.2. *Let* $(j_1, \ldots, j_N) \in \mathbb{Z}^N$, *for* $n \in \mathbb{N}$. *Then*

$$\prod_{l=1}^{N} \chi_{\partial_{j_l}} = \delta_{(j_1, \ldots, j_N)} \chi_{\partial_{j_1}} \text{ in } \mathcal{M}_p, \tag{3.6}$$

and hence,

$$\varphi_p\left(\prod_{l=1}^{N}\chi_{\partial_{j_l}}\right) = \delta_{(j_1, \ldots, j_N)}\left(\frac{1}{p^{j_1}} - \frac{1}{p^{j_1+1}}\right),$$

where

$$\delta_{(j_1, \ldots, j_N)} = \left(\prod_{l=1}^{N-1}\delta_{j_l, j_{l+1}}\right)(\delta_{j_N, j_1}).$$

Proof. The proof of (3.6) is done by (3.5).

Thus, one can get that, for any $S \in \sigma(\mathbb{Q}_p)$,

$$\varphi_p(\chi_S) = \varphi_p\left(\sum_{j\in\Lambda_S}\chi_{S\cap\partial_j}\right),$$

where Λ_S is in the sense of (2.2.8)

$$= \sum_{j\in\Lambda_S}\varphi_S(\chi_{S\cap\partial_j}) = \sum_{j\in\Lambda_S}\mu_p(S\cap\partial_j)$$

$$= \sum_{j\in\Lambda_S} r_j\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{3.7}$$

where $0 \leq r_j \leq 1$ are in the sense of (2.2.8), for all $j \in \mathbb{Z}$.

Also, if $S_1, S_2 \in \sigma(\mathbb{Q}_p)$, then

$$\chi_{S_1}\chi_{S_2} = \left(\sum_{k\in\Lambda_{S_1}}\chi_{S_1\cap\partial_k}\right)\left(\sum_{k\in\Lambda_{S_2}}\chi_{S_2\cap\partial_j}\right)$$

$$= \sum_{(k, j)\in\Lambda_{S_1}\times\Lambda_{S_2}}(\chi_{S_1\cap\partial_k}\chi_{S_2\cap\partial_j})$$

$$= \sum_{(k, j)\in\Lambda_{S_1}\times\Lambda_{S_2}}\delta_{k, j}\chi_{(S_1\cap S_2)\cap\partial_j}$$

$$= \sum_{j\in\Lambda_{S_1, S_2}}\chi_{(S_1\cap S_2)\cap\partial_j}, \tag{3.8}$$

where

$$\Lambda_{S_1, S_2} = \Lambda_{S_1} \cap \Lambda_{S_2},$$

because $\partial_k \cap \partial_j = \delta_{k,j}\partial_j$.

In (3.8), it is clear that, if Λ_{S_1, S_2} is empty, then

$$\chi_{S_1}\chi_{S_2} = 0_{\mathcal{M}_p} = \chi_{\varnothing}, \text{ the zero element of } \mathcal{M}_p,$$

where $\varnothing$ is the empty set in $\sigma(\mathbb{Q}_p)$.

Thus, one can get that there exist $w_j \in \mathbb{R}$, such that

$$0 \le w_j \le 1, \text{ for all } j \in \Lambda_{S_1, S_2}, \tag{3.9}$$

where

$$\varphi_p(\chi_{S_1}\chi_{S_2}) = \sum_{j \in \Lambda_{S_1, S_2}} w_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right),$$

by (3.8) and (2.2.10), for all $S_1, S_2 \in \sigma(\mathbb{Q}_p)$.

By (3.9), we obtain the following general result under induction.

Theorem 3.3. *Let* $S_l \in \sigma(\mathbb{Q}_p)$, *and let* $\chi_{S_l} \in (\mathcal{M}_p, \varphi_p)$, *for* $l = 1, \ldots, N$, *for* $N \in \mathbb{N}$. *Let*

$$\Lambda_{S_1, \ldots, S_N} = \bigcap_{l=1}^{N} \Lambda_{S_l} \text{ in } \mathbb{Z},$$

where Λ_{S_l} *are in the sense of* (2.2.7), for $l = 1, \ldots, N$. *Then there exist* $r_j \in \mathbb{R}$, *such that*

$$0 \le r_j \le 1 \text{ in } \mathbb{R}, \tag{3.10}$$

and

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{S_l} \right) = \sum_{j \in S_1, \ldots, S_N} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right).$$

Proof. The proof of (3.10) is done by induction on (3.9). □

Of course, if $\Lambda_{S_1, \ldots, S_N}$ is empty in $\mathbb{Z}$, then the formula (3.10) vanishes. By (3.10), we obtain that, if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \in (\mathcal{M}_p, \varphi_p), \text{ with } t_S \in \mathbb{C},$$

then

$$\varphi_p(f) = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \varphi_p(\chi_S)$$

$$= \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \left(\sum_{j \in \Lambda_S} r_j^S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right), \quad (3.11)$$

where r_j^S are in the sense of (2.2.10), for all $j \in \Lambda_S$.

The above joint free-moment formula (3.11) provides a universal tool to compute the free-distributional data of free random variables in our p-adic free probability space $(\mathcal{M}_p, \varphi_p)$.

4. Representations of $(\mathcal{M}_p, \varphi_p)$

Fix a prime $p \in \mathcal{P}$. Let $(\mathcal{M}_p, \varphi_p)$ be the p-adic free probability space. Now, we construct a representation of $\mathcal{M}_p$. By understanding Q_p as a measure space, construct the L^2- space,

$$H_p \underset{=\!=}{def} L^2(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p) = L^2(\mathbb{Q}_p), \quad (4.1)$$

over $\mathbb{C}$, consisting of all square-integrable μ_p- measurable functions on $\mathbb{Q}_p$. Then this L^2- space is a well-defined Hilbert space equipped with its inner product $\langle \ , \ \rangle_2$,

$$\langle f_1, f_2 \rangle_2 \underset{=\!=}{def} \int_{\mathbb{Q}_p} f_1 f_2^* \, d\mu_p, \text{ for all } f_1, f_2 \in H_p. \quad (4.2)$$

Naturally, H_p is the $\| \cdot \|_2$- norm completion in $\mathcal{M}_p$, where

$$\| f \|_2 \underset{=}{def} \sqrt{\langle f, f \rangle_2}, \text{ for all } f \in H_p, \tag{4.2}$$

where $\langle \, , \rangle_2$ is the inner product (4.2) on H_p.

Definition 4.1. We call the Hilbert space H_p of (4.1), the p-adic Hilbert space.

By the very construction (4.1) of the p-adic Hilbert space H_p, our $*$-algebra $\mathcal{M}_p$ acts on H_p, via an algebra-action α^p,

$$\alpha^p(f)(h) = fh, \text{ for all } h \in H_p, \tag{4.3}$$

for all $f \in \mathcal{M}_p$. i.e., the morphism α^p of (4.3) is an action of $\mathcal{M}_p$ acting on the Hilbert space H_p. i.e., for any $f \in \mathcal{M}_p$, the image $\alpha^p(f)$ is a multiplication operator on H_p with its symbol f contained in the operator algebra $B(H_p)$ of all (bounded linear) operators on H_p.

Notation. Denote $\alpha^p(f)$ by α_f^p, for all $f \in \mathcal{M}_p$. Also, for convenience, denote $\alpha_{\chi_S}^p$ simply by α_S^p, for all $S \in \sigma(\mathbb{Q}_p)$. For instance,

$$\alpha_{U_k}^p = \alpha_{\chi_{U_k}}^p = \alpha^p(\chi_{U_k}),$$

and

$$\alpha_{\partial_k}^p = \alpha_{\chi_{\partial_k}}^p = \alpha^p(\chi_{\partial_k}),$$

for all $k \in \mathbb{Z}$, where U_k are in the sense of (3.1), and ∂_k are the corresponding boundaries of U_k in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$.

It is not difficult to check that

$$\alpha_{f_1 f_2}^p = \alpha_{f_1}^p \alpha_{f_2}^p \text{ on } H_p, \text{ for all } f_1, f_2 \in \mathcal{M}_p,$$

and

$$(\alpha_f^p)^* = \alpha_{f^*} \text{ on } H_p, \text{ for all } f \in \mathcal{M}_p.$$

Therefore, one obtains that:

Proposition 4.1. *The pair* (H_p, α^p) *is a well-determined Hilbert-space representation of* $\mathcal{M}_p$.

Proof. To show the pair (H_p, α^p) is a Hilbert-space representation of $\mathcal{M}_p$, it suffices to show that the linear morphism α^p of (4.3) is a $*$-homomorphism from $\mathcal{M}_p$ into an operator algebra $B(H_p)$, which is the $*$-algebra consisting of all linear transformations (or operators) on H_p.

One can check that

$$\alpha^p_{f_1 f_2}(h) = f_1 f_2 h = f_1(f_2 h)$$

$$= f_1(\alpha^p_{f_2}(h)) = \alpha^p_{f_1}(\alpha^p_{f_2}(h))$$

$$= \alpha^p_{f_1}\alpha^p_{f_2}(h),$$

by (4.3), for all $h \in H_p$, for all $f_1, f_2 \in \mathcal{M}_p$, and hence,

$$\alpha^p_{f_1 f_2} = \alpha^p_{f_1}\alpha^p_{f_2}, \text{ on } H_p, \text{ for all } f_1, f_2 \in \mathcal{M}_p.$$

Also, one has that

$$\langle \alpha^p_f(h_1), h_2 \rangle_2 = \langle fh_1, h_2 \rangle_2 = \int_{\mathbb{Q}_p} f h_1 h_2^* d\mu_p$$

$$= \int_{\mathbb{Q}_p} h_1(f h_2^*) d\mu_p = \int_{\mathbb{Q}_p} h_1(h_2 f^*)^* d\mu_p$$

$$= \int_{\mathbb{Q}_p} h_1(f^* h_2)^* d\mu_p = \langle h_1, \alpha^p_{f^*}(h_2) \rangle_2,$$

for all $h_1, h_2 \in H_p$, for all $f \in \mathcal{M}_p$. Thus, we have

$$(\alpha^p_f)^* = \alpha^p_{f^*} \text{ on } H_p, \text{ for all } f \in \mathcal{M}_p.$$

Therefore, the linear morphism α^p of (4.3) is a well-determined $*$-homomorphism from $\mathcal{M}_p$ into $B(H_p)$, equivalently, it is a well-defined $*$-

algebra action of $\mathcal{M}_p$ acting on H_p. So, the pair (H_p, α^p) is a Hilbert-space representation of $\mathcal{M}_p$.

The above proposition shows that all $*$-measurable functions f in $\mathcal{M}_p$ can be regarded as operators α_f^p acting on H_p.

Definition 4.2. The Hilbert-space representation (H_p, α^p) is said to be the p-adic (Hilbert-space) representation of $\mathcal{M}_p$.

Depending on the p-adic representation (H_p, α^p) of $\mathcal{M}_p$, one can construct the C^*- subalgebra M_p, generated by $\mathcal{M}_p$, in the operator algebra $B(H_p)$. Recall that the operator algebra $B(H_p)$ is equipped with the operator-norm,

$$\| T \| = \sup\{\| Th \|_2 : h \in H_p, \| h \|_2 = 1\},$$

for all $T \in B(H_p)$, where $\| \cdot \|_2$ means the L^2- norm (4.2) on H_p.

Definition 4.3. Let M_p be the operator-norm closure of $\mathcal{M}_p$ in the operator algebra $B(H_p)$, i.e.,

$$M_p \underset{=}{\underline{def}} \overline{\alpha^p(\mathcal{M}_p)}^{\| \cdot \|} = \overline{\mathbb{C}[\alpha_f^p : f \in \mathcal{M}_p]}^{\| \cdot \|} \text{ in } B(H_p), \tag{4.4}$$

where $\overline{X}^{\| \cdot \|}$ mean the operator-norm closures of subsets X of $B(H_p)$. Then the C^*- algebra M_p is called the p-adic C^*- algebra of $(\mathcal{M}_p, \varphi_p)$.

5. Free Probability on M_p

Throughout this section, let's fix a prime $p \in \mathcal{P}$. Let $(\mathcal{M}_p, \varphi_p)$ be the corresponding p-adic free probability space, and let (H_p, α^p) be the p-adic representation of $\mathcal{M}_p$, inducing the corresponding p-adic C^*- algebra M_p of (4.4). In this section, we consider suitable free-probabilistic models on M_p. In particular, we are interested in a system $\{\varphi_j^p\}_{j \in \mathbb{Z}}$ of linear functionals on M_p, determined by the j-th boundaries $\{\partial_j\}_{j \in \mathbb{Z}}$ of $\mathbb{Q}_p$.

Define a linear functional $\varphi_j^p : M_p \to \mathbb{C}$ by a linear morphism,

$$\varphi_j^p(a) \underset{=\!=}{def} \langle \alpha_a^p(\chi_{\partial_j}), \chi_{\partial_j} \rangle_2, \text{ for all } a \in M_p, \tag{5.1}$$

for all $j \in \mathbb{Z}$, where $\langle \, , \rangle_2$ is the inner product (4.2) on the p-adic Hilbert space H_p of (4.1).

First, remark that, if $a \in M_p$, then

$$a = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \text{ in } M_p,$$

where Σ is finite or infinite (limit of finite) sum (s), under C^*-topology of M_p.

Definition 5.1. Let $j \in \mathbb{Z}$, and let φ_j^p be the linear functional (5.1) on the p-adic C^*-algebra M_p. Then the C^*-probability space (M_p, φ_j^p) is said to be the j-th (p-adic) C^*-probability space.

So, one can get the system

$$\{(M_p, \varphi_j^p) : j \in \mathbb{Z}\}$$

of C^*-probability spaces for a fixed C^*-algebra M_p.

Now, fix $j \in \mathbb{Z}$, and take the corresponding j-th C^*-probability space (M_p, φ_j^p). For $S \in \sigma(\mathbb{Q}_p)$, and an element $\chi_S \in M_p$, one has that

$$\begin{aligned}\varphi_j^p(\chi_S) &= \langle \alpha_S^p(\chi_{\partial_j}), \chi_{\partial_j} \rangle_2 = \langle \chi_{S \cap \partial_j} \chi_{\partial_j} \rangle_2 \\ &= \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} \chi_{\partial_j}^* d\mu_p = \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} \chi_{\partial_j} d\mu_p \\ &= \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} d\mu_p = \mu_p(S \cap \partial_j) = r_S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \end{aligned} \tag{5.2}$$

for some $0 \leq r_S \leq 1$ in $\mathbb{R}$.

Proposition 5.1. *Let* $S \in \sigma(\mathbb{Q}_p)$, *and* $\alpha_S^p = \alpha_{\chi_S}^p \in (M_p, \varphi_j^p)$, *for a fixed* $j \in \mathbb{Z}$. *Then there exists* $r_S \in \mathbb{R}$, *such that*

$$0 \leq r_S \leq 1 \text{ in } \mathbb{R}, \tag{5.3}$$

and

$$\varphi_j((\sigma_S^p)^n) = r_S\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \text{ for all } n \in \mathbb{N}.$$

Proof. Remark that the element α_S is a projection in M_p, in the sense that:

$$(\alpha_S^p)^* = \alpha_S^p = (\alpha_S^p)^2, \text{ in } M_p.$$

Indeed,

$$(\alpha_S^p)^* = (\alpha_{\chi_S}^p)^* = \alpha_{\chi_S^*}^p = \alpha_{\chi_S}^p = \alpha_S^p,$$

and

$$(\alpha_S^p)^2 = \alpha_{(\chi_S)^2}^p = \alpha_{\chi_S}^p = \alpha_S^p,$$

and hence, the operator α_S^p is a projection on H_p, in M_p.

Since α_S^p is a projection in M_p,

$$(\alpha_S^p)^n = \alpha_S^p, \text{ for all } n \in \mathbb{N}.$$

So,

$$\varphi_p((\alpha_S^p)^n) = \varphi_p(\alpha_S^p) = \mu_p(S).$$

Therefore, by (5.2), we obtain (5.3).

The above free-moment formula (5.3) characterizes the free distributions of α_S^p in the j-th C^*-probability space (M_p, φ_j^p), for $j \in \mathbb{Z}$. More precisely, we obtain the following theorem.

Theorem 5.2. *Let* $S_l \in \sigma(\mathbb{Q}_p)$ *and* $\alpha_{S_l}^p = \alpha^p(\chi_{S_l}) \in (M_p, \varphi_j^p)$, *for a fixed* $j \in \mathbb{Z}$, *for* $l = 1, \ldots, N$, *for* $N \in \mathbb{N}$. *Then there exists* $r_{(S_1, \ldots S_N)} \in \mathbb{R}$, *such that*

$$0 \leq r_{(S_1, \ldots, S_N)} \leq 1 \text{ in } \mathbb{R}, \tag{5.4}$$

and

$$\varphi_j\left(\left(\prod_{l=1}^{N} \alpha_{S_l}^p\right)^n\right) = r_{(S_1, \ldots, S_N)}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

for all $n \in \mathbb{N}$.

Proof. Let $S_1, \ldots, S_N$ be μ_p-measurable subsets of $\mathbb{Q}_p$, for $N \in \mathbb{N}$, and let

$$S = \bigcap_{l=1}^{N} S_l \in \sigma(\mathbb{Q}_p).$$

Then, one has that

$$\alpha_S^p = \prod_{l=1}^{N} \alpha_{S_l}^p \text{ in } M_p,$$

satisfying

$$(\alpha_S^p)^* = \alpha_S = (\alpha_S^p)^2 \text{ in } M_p.$$

Therefore, by (5.3), the formula (5.4) holds.

The above joint free moment formula (5.4) characterizes the free-distributions of finitely many projections $\alpha_{S_1}^p, \ldots, \alpha_{S_N}^p$ in the j-th C^*-probability space (M_p, φ_j^p), for $j \in \mathbb{Z}$, for all $N \in \mathbb{N}$.

As a corollary of (5.4), we obtain the following results.

Corollary 5.3. *Let* U_k *be in the sense of* (3.1), *and* ∂_k, *the k-th boundaries of* U_k *in* $\mathbb{Q}_p$, *for all* $k \in \mathbb{Z}$. *Then*

$$\varphi_j^p((\alpha_{U_k}^p)^n) = \begin{cases} \frac{1}{p^j} - \frac{1}{p^{j+1}} & \text{if } k \leq j \\ 0 & \text{otherwise}, \end{cases} \tag{5.5}$$

and

$$\varphi_j^p((\alpha_{\partial_k}^p)^n) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)$$

for all $n \in \mathbb{N}$, for $k \in \mathbb{Z}$.

6. Semigroup C^*-Subalgebras $\mathfrak{S}_p$ of M_p

Throughout this section, $p \in \mathcal{P}$. Let M_p be the p-adic C^*-algebra for a fixed prime p, as in Section 5. Take operators

$$P_{p,k} = \alpha_{\partial_k}^p \in M_p, \tag{6.1}$$

for all $k \in \mathbb{Z}$. As we have seen in Section 5, such operators $P_{p,k}$ of (6.1) are projections on the p-adic Hilbert space H_p, satisfying

$$(P_{p,k})^* = P_{p,k} = (P_{p,k})^2, \text{ in } M_p.$$

Also, by (5.3) and (5.4), we obtain that

$$\varphi_j^p(P_{p,k}) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \text{ for all } k \in \mathbb{Z} \tag{6.2}$$

(see (5.5)), in the j-th p-adic C^*-probability space (M_p, φ_j^p), for all $j \in \mathbb{Z}$. Now, define a set of projections $\mathfrak{P}_p$ in M_p by

$$\mathfrak{P}_p = \{P_{p,k} \in M_p : k \in \mathbb{Z}\}. \tag{6.3}$$

This family $\mathfrak{P}_p$ consists of "mutually-orthogonal" projections in the sense that

$$P_{p,k_1}P_{p,k_2} = \delta_{k_1,k_2}P_{p,k_1}.$$

So, under the inherited operator-multiplication on M_p, one can define the corresponding sub-semigroup

$$S_p = \langle \mathfrak{P}_p \rangle \text{ of } M_p \tag{6.4}$$

generated by the family $\mathfrak{P}_p$ of (6.3), where $\langle Y \rangle$ mean the semigroups (under inherited operator-multiplication) generated by subsets Y of M_p.

Proposition 6.1. *Let* S_p *be the sub-semigroup* (6.4) *of the p-adic* C^**-algebra* M_p *generated by the family* $\mathfrak{P}_p$ *of* (6.3). *Then*

$$S_p = \{P_{p,k} : k \in \mathbb{Z}\} = \mathfrak{P}_p, \text{ in } M_p, \tag{6.5}$$

set-theoretically.

Proof. By the very definition (6.4), one has that

$$S_p = \overline{\bigcup_{N \in \mathbb{N}} \left\{ \prod_{l=1}^{N} P_{p,j_l}^{n_l} : n_l \in \mathbb{N},\ j_l \in \mathbb{Z} \right\}},$$

where $\overline{Y}$ mean the C^*-topology closures of subsets Y in M_p.

However, by the mutual orthogonality of the generating family $\mathfrak{P}_p$ of (6.3), since all generating elements $P_{p,k}$ are projections,

$$S_p = \bigcup_{N \in \mathbb{N}} \left\{ \prod_{l=1}^{N} P_{p,j_l} : j_l \in \mathbb{Z} \right\} = \{P_{p,k} : k \in \mathbb{Z}\},$$

in M_p, i.e.,

$$S_p = \mathfrak{P}_p \text{ in } M_p,$$

set-theoretically.

The above structure theorem (6.5) shows that the family $\mathfrak{P}_p$ of (6.3), itself, is regarded as a semigroup S_p of (6.4) in the p-adic C^*-algebra M_p under operator-multiplication. From below, we use the terms $\mathfrak{P}_p$ and S_p as an identical sub-semigroup of M_p, in the sense of (6.4), by (6.5).

One can also verify that the semigroup S_p does not contain its semigroup-identity from (6.5). So, it is a pure semigroup in M_p. Now, we construct the semigroup C^*-subalgebra $\mathfrak{S}_p$ generated by the semigroup S_p of (6.4) in the p-adic C^*-algebra M_p.

Definition 6.1. Fix $p \in \mathcal{P}$. Let $\mathfrak{S}_p$ be the C^*-subalgebra

$$\mathfrak{S}_p = C^*(S_p) = \overline{\mathbb{C}[S_p]} \text{ of } M_p, \tag{6.6}$$

where S_p is the semigroup (6.4). We call this semigroup C^*-subalgebra $\mathfrak{S}_p$, the p-adic boundary $(C^*$-$)$ subalgebra of M_p.

By the structure theorem (6.5), we obtain the following structure theorem of $\mathfrak{S}_p$.

Proposition 6.2. *Let $\mathfrak{S}_p$ be the p-adic boundary subalgebra* (6.6) *of the p-adic C^*-algebra M_p. Then*

$$\mathfrak{S}_p \underset{\overline{\overline{\quad}}}{* -iso} \underset{j \in \mathbb{Z}}{\oplus} (\mathbb{C} \cdot P_{p,j}) \underset{\overline{\overline{\quad}}}{* -iso} C^{\oplus \mathbb{Z}}, \tag{6.7}$$

in M_p where $\oplus$ means the (*topological*) *direct product of C^*-algebras.*

Proof. Observe that

$$\mathfrak{S}_p = C^*(S_p) = \overline{\mathbb{C}[S_p]}$$

by (6.6)

$$= \overline{C[\mathfrak{P}_p]}$$

by (6.5)

$$\underline{\underline{*\text{-iso}}} \overline{\underset{j \in \mathbb{Z}}{\oplus} \mathbb{C}[P_{p,j}]} \underline{\underline{*\text{-iso}}} \overline{\underset{j \in \mathbb{Z}}{\oplus} (\mathbb{C} \cdot P_{p,j})}$$

since $P_{p,j}$ are projections in M_p, for all $j \in \mathbb{Z}$, where $\underline{\oplus}$ means (pure-algebraic) direct product of algebras

$$\underline{\underline{*\text{-iso}}} \underset{j \in \mathbb{Z}}{\oplus} (\mathbb{C} \cdot P_{p,j}),$$

where $\oplus$ means the direct product of C^*-algebras under product topology. Therefore,

$$\mathfrak{S}_p \ \underline{\underline{*\text{-}iso}} \ \underset{j \in \mathbb{Z}}{\oplus} (\mathbb{C} \cdot P_{p,j}) \underline{\underline{*\text{-}iso}} \ \mathbb{C}^{\oplus |\mathbb{Z}|},$$

in M_p.

By the structure theorem (6.7) of $\mathfrak{S}_p$, one can realize that this semigroup C^*-subalgebra $\mathfrak{S}_p$ acts like a diagonal subalgebra inside M_p. Since p-adic boundary subalgebras $\mathfrak{S}_p$ are C^*-subalgebras of M_p, one can naturally get the corresponding C^*-probability spaces,

$$(\mathfrak{S}_p, \varphi_j^p), \text{ for all } j \in \mathbb{Z},$$

for any fixed $p \in \mathcal{P}$.

i.e., we have a family

$$\{(\mathfrak{S}_p, \varphi_j^p), : p \in \mathcal{P}, \ j \in \mathbb{Z}\}, \tag{6.8}$$

where the linear functionals φ_j^p in (6.8) are restrict linear functionals of the linear functionals φ_j^p of (5.1) on M_p, for all $j \in \mathbb{Z}$, for $p \in \mathcal{P}$. For convenience, we denote these restricted linear functionals simply by φ_j^p, too, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Definition 6.2. We call C^*-probability spaces $(\mathfrak{S}_p, \varphi_j^p)$, the j-th p-adic diagonal C^*-probability spaces of the j-th p-adic C^*-probability spaces (M_p, φ_j^p), for all $p \in \mathcal{P}$, and $j \in \mathbb{Z}$.

7. Free Product C^*-Probability Spaces of $\{(\mathfrak{S}_p, \varphi_j^p)\}_{p\in\mathcal{P}, j\in\mathbb{Z}}$

For an arbitrarily fixed $p \in \mathcal{P}$, let

$$\mathfrak{S}(p) = \{(\mathfrak{S}_p, \varphi_j^p) : j \in \mathbb{Z}\} \tag{7.0.1}$$

by the family (6.8) of j-th p-adic diagonal C^*-probability spaces.

From a C^*-probability space $(\mathfrak{S}_p, \varphi_j^p)$ in the family $\mathfrak{S}(p)$ of (7.0.1), for $j \in \mathbb{Z}$, we have that

$$\varphi_j^p(\alpha_{\partial_k}^p) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{7.0.2}$$

by (6.2), where $\alpha_{\partial_k}^p$ are the generating projections of the p-adic diagonal subalgebra $\mathfrak{S}_p$ of M_p, for all $k \in \mathbb{Z}$.

By the structure theorem (6.5) of $\mathfrak{S}_p$, the above free-moment formula (7.0.2) characterizes the free distributions of all elements of $(\mathfrak{S}_p, \varphi_j^p)$, for $j \in \mathbb{Z}$.

7.1. Free Product C^*-Probability Spaces. Let (A_k, φ_k), be arbitrary C^*-probability spaces, consisting of C^*-algebras A_k, and corresponding linear functionals φ_k, for $k \in \Delta$, where Δ is an arbitrary countable (finite or infinite) index set. The free product C^*-algebra A,

$$A = \underset{l\in\Delta}{*} A_l$$

is the C^*-algebra generated by the "non-commutative" reduced words in $\bigcup_{Nl=1}^{N} A_l$, having a free product linear functional,

$$\varphi = \underset{l \in \Delta}{*} \varphi_l.$$

The new C^*-probability space (A, φ) is said to be the free product C^*-probability space of $\{(A, \varphi_k)\}_{k \in \Delta}$ (e.g., see [12], [14] and cited papers therein). So, by the very definition, even though each free blocks A_l are commutative for $l \in \Delta$, the free product C^*-algebra A is highly noncommutative.

The C^*-algebra A is understood as a Banach space,

$$\mathbb{C} \oplus \left(\overset{\infty}{\underset{n=1}{\oplus}} \left(\underset{(i_1, \ldots, i_n) \in all(\Delta^n)}{\oplus} \left(\overset{n}{\underset{k=1}{\oplus}} A_{i_k}^o \right) \right) \right) \tag{7.1.1}$$

with

$$A_{i_k}^o = A_{i_k} \Theta \, \mathbb{C}, \text{ for all } k = 1, \ldots, n,$$

where

$$alt(\Delta^n) = \left\{ (i_1, \ldots i_n) \,\middle|\, \begin{array}{r} (i_1, \ldots i_n) \in \Delta^n \\ i_1 \neq i_2, \, i_2 \neq i_3, \\ \ldots, \, i_{n-1} \neq i_n \end{array} \right\},$$

for all $n \in \mathbb{N}$, and where $\oplus$, and $\otimes$ are the (topological) direct product, respectively, tensor product of Banach spaces.

In particular, if an element $a \in A$ is of the form of free reduced word,

$$a = \prod_{l=1}^{n} a_{i_l} \text{ in } A,$$

then one can understand a as an equivalent Banach-space vector, $\overset{n}{\underset{l=1}{\oplus}} a_{i_l}$ in the Banach space A of (7.1.1), contained in a direct summand, $\overset{n}{\underset{k=1}{\oplus}} A_{i_k}^o$ of

(7.1.1). Remark that, under the above equivalence, this free reduced word a of A is regarded as the operator $\underset{l=1}{\overset{n}{\oplus}} a_{i_l}$ in

$$\mathbb{C} \oplus \left(\underset{k=1}{\overset{n}{\oplus}} A_{i_k}^o \right) = \underset{k=1}{\overset{n}{\oplus_{\mathbb{C}}}} A_{i_k},$$

where $\oplus_{\mathbb{C}}$ means the (topological) tensor product of C^*-algebras. i.e., the free reduced word a, understood as a Banach-space vector $\underset{k=1}{\overset{n}{\oplus}} a_{i_l}$ in A of (7.1.1), is regarded as an operator $\underset{k=1}{\overset{n}{\oplus}} a_{i_l}$ in the C^*-subalgebra $\underset{k=1}{\overset{n}{\oplus_{\mathbb{C}}}} A_{i_k}$ of A.

We denote this relation by

$$a \underset{=\!=}{\underline{equi}} \underset{l=1}{\overset{n}{\oplus}} a_{i_l} \text{ in } A, \text{ as operators.} \tag{7.1.2}$$

Remark that, if a is a free reduced word in A, then

$$a^k \underset{=\!=}{\underline{equi}} \left(\underset{l=1}{\overset{n}{\oplus}} a_{i_l} \right)^k = \underset{l=1}{\overset{n}{\oplus}} a_{i_l}^k \underset{=\!=}{\underline{equi}} \prod_{l=1}^{n} a_{i_l}^k \text{ in } A, \tag{7.1.3}$$

for all $k \in \mathbb{N}$.

Notation and Remark 7.1.1 (in short, NR 7.1.1 below). Let $a = \prod_{l=1}^{n} a_{i_l}$ be a free reduced word in A, as above. The power a^k in (7.1.3) means the k-th power of a as an operator of A in the sense of (7.1.2), which is also understood as a vector in $\underset{l=1}{\overset{n}{\otimes}} A_{i_l}^o \subset A$ in the sense of (7.1.1).

To avoid the confusion, we may use the notation $a^{(k)}$, as a construction of new free "non-reduced" word,

$$a^{(k)} = \underbrace{a \cdot a \dots a}_{k\text{-times}}$$

in A.

For example, let $a = a_{i_1} a_{i_2} a_{i_1}$ be a free reduced word with

$$(i_1,\ i_2,\ i_1) \in alt(\Delta^3),$$

as an operator,

$$a_{i_1} \otimes a_{i_2} \otimes a_{i_1} \text{ in } A, \text{ by (7.1.2).}$$

Then

$$a^3 \underline{\underline{equi}} \ (a_{i_1} \otimes a_{i_2} \otimes a_{i_3})^3 \underline{\underline{equi}} \ a_{i_1}^3 a_{i_2}^3 a_{i_1}^3,$$

in A, but

$$\begin{aligned} a^{(3)} &= (a_{i_1} a_{i_2} a_{i_1})^{(3)} \\ &= a_{i_1} a_{i_2} a_{i_1} a_{i_1} a_{i_2} a_{i_1} a_{i_1} a_{i_2} a_{i_1} \quad \text{(non-reduced word)} \\ &= a_{i_1} a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}, \quad \text{(reduced word)} \end{aligned}$$

in A; i.e.,

$$a^{(3)} = a_{i_1} a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1},$$

as a new free reduced word in A.

So, in the text below, if we use the term "a^k" for a fixed free reduced word a, then it is in the sense of (7.1.3). In the following text, we will not use the concept "$a^{(k)}$." However, we want to emphasize at this very moment the differences between a^k and $a^{(k)}$ in the free product algebra A, for $k \in \mathbb{N}$. Of course, $a^1 = a = a^{(1)}$ in A.

Similar to a^k and $a^{(k)}$, one can understand the adjoints a^* and $a^{(*)}$ of a fixed free reduced word a in A. i.e.,

$$a^* \ \underline{\underline{equi}} \left(\overset{n}{\underset{l=1}{\otimes}} a_{i_l} \right)^* = \overset{n}{\underset{l=1}{\otimes}} a_{i_l}^* \ \underline{\underline{equi}} \prod_{l=1}^{n} a_{i_l}^* \text{ in } A,$$

but

$$a^{(*)} = \left(\prod_{l=1}^{n} a_{i_l}\right)^{(*)} = (a_{i_1} a_{i_2}, \ldots, a_{i_n})^{(*)}$$

$$= a_{i_n}^* a_{i_{n-1}}^* \ldots a_{i_2}^* a_{i_1}^* = \prod_{l=1}^{n} a_{i_{n-l+1}}^*,$$

in A. In the following text, if we use the term a^*, then it is determined under equivalence (7.1.3). Again, we want to emphasize the differences between a^* and $a^{(*)}$ in A, at this moment.

So, the free product linear functional φ on A satisfies that, whenever a is a reduced free word in A satisfying (7.1.2), then

$$\varphi(a^k) = \varphi\left(\prod_{l=1}^{n} a_{i_l}^k\right) = \prod_{l=1}^{n} (\varphi_{i_l}(a_{i_l}^k)), \tag{7.1.4}$$

by (7.1.3), for all $k \in \mathbb{N}$. Sometimes, by abusing (7.1.3), one can / may re-write (7.1.4) by

$$\varphi(a^k) \underline{\underline{equi}} \varphi\left(\underset{l=1}{\overset{n}{\otimes}} a_{i_l}^k\right) = \prod_{l=1}^{n} \varphi(a_{i_l}^k),$$

whenever $a = \prod_{l=1}^{n} a_{i_l}$ ail is a "free reduced word" in A, for all $k \in \mathbb{N}$.

Now, let

$$b = \sum_{l=1}^{n} b_{i_l} \in (A, \varphi),$$

where $b_{i_1}, \ldots, b_{i_n}$ are free reduced words in A.

We say that a is a free sum in A, if all summands $b_{i_1}, \ldots, b_{i_n}$ of b are contained in "mutually-distinct" direct summands of a Banach space A of (7.1.1), as equivalent Banach-space vectors (or corresponding operators) of the free reduced words. Then, similar to the above observation, one can realize that

$$b \underset{=\!=}{equi} \overset{n}{\underset{l=1}{\oplus}} b_{i_l} \text{ in the Banach space } A \text{ of (7.1.1)}, \tag{7.1.5}$$

satisfying

$$\varphi(b^k) \underset{=\!=}{equi} \varphi\left(\left(\overset{n}{\underset{l=1}{\oplus}} b_{i_l}\right)^k\right) = \varphi\left(\overset{n}{\underset{l=1}{\oplus}} b_{i_l}^k\right)$$

$$\underset{=\!=}{equi} \varphi\left(\sum_{l=1}^{n} b_{i_l}^k\right) = \sum_{l=1}^{n} \varphi(b_{i_l}^k),$$

for all $k \in \mathbb{N}$.

Here, remark that each summand $\varphi(b_{i_l}^k)$ of (7.1.5) satisfies (7.1.4), for all $l = 1, \ldots, n$, for all $n \in \mathbb{N}$.

Notation and Remark 7.1.2 (in short, NR 7.1.2 below). Similar to the free-reduced-word case, if b is a free sum in the sense of (7.1.5), then one can consider

$$b^{(k)} = \left(\sum_{l=1}^{n} b_{i_l}\right)^{(k)}$$

$$= \sum_{(l_1, \ldots, l_k) \in \{1, \ldots, n\}^k} (b_{i_{l_1}} b_{i_{l_2}} \ldots b_{i_{l_k}}),$$

where the summands $b_{i_{l_1}} b_{i_{l_2}} \ldots b_{i_{l_k}}$ are free "non-reduced" words in A. In the following text, we will not use the concept "$b^{(k)}$" in A. But, as before, we emphasize the differences between b^k and $b^{(k)}$ for a fixed free sum b of A.

Similar to **NR 7.1.1**, remark also the differences between

$$b^* \underset{=\!=}{equi} \left(\overset{n}{\underset{l=1}{\oplus}} b_{i_l}\right)^* = \overset{n}{\underset{l=1}{\oplus}} b_{i_l}^* \underset{=\!=}{equi} \sum_{l=1}^{n} b_{i_l}^*,$$

and

$$b^{(*)} = \sum_{l=1}^{n} b_{i_l}^{(*)},$$

in (A, φ), where the adjoints $b_{i_l}^*$ and $b_{i_l}^*$ are in the sense of **NR 7.1.1**, for all $l = 1, \ldots, n$.

For more about (free-probabilistic) free product algebras, and corresponding free probability spaces, see [11], [12] and [14].

7.2. Free Product C^*-Probability Space $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$. Let $p \in \mathcal{P}$ be fixed, and let $\mathfrak{S}(p)$ be the family (7.0.1)

$$\mathfrak{S}(p) = \{\mathfrak{S}_p(j) \underline{\underline{denote}} (\mathfrak{S}_p, \varphi_j^p) : j \in \mathbb{Z}\}$$

of p-adic diagonal C^*-probability spaces.

In this section, we construct a free product C^*-probability space $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$ of the family $\mathfrak{S}(p)$,

$$(\mathfrak{S}_p(\mathbb{Z}), \varphi^p) \underline{\underline{def}} \underset{X \in \mathfrak{S}(p)}{*} X. \tag{7.2.1}$$

i.e., by (7.2.1),

$$(\mathfrak{S}_p(\mathbb{Z}), \varphi^p) = \underset{j \in \mathbb{Z}}{*} (\mathfrak{S}_p(j)) \underset{j \in \mathbb{Z}}{*} (\mathfrak{S}_p, \varphi_j^p)$$

$$= \left(\underset{j \in \mathbb{Z}}{*} (\mathfrak{S}_p)_j, \underset{j \in \mathbb{Z}}{*} \varphi_j^p \right),$$

with

$$(\mathfrak{S}_p)_j = \mathfrak{S}_p, \text{ for all } j \in \mathbb{Z},$$

for a fixed $p \in \mathcal{P}$.

Definition 7.1. Let $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$ be the free product C^*-probability space

(7.2.1) of the family $\mathfrak{S}(p)$ of (7.0.1), for a fixed prime $p \in \mathcal{P}$. Then we call it the p-adic diagonal C^*-probability space.

Let T be a free reduced word,

$$T = \prod_{l=1}^{N} P_{p, j_l} \text{ in } (\mathfrak{S}_p(\mathbb{Z}), \varphi^p). \tag{7.2.2}$$

It guarantees that the corresponding integer-sequence $(j_1, \ldots, j_N)$ of T is an alternating sequence, i.e.,

$$(j_1, \ldots, j_N) \in alt(\mathbb{Z}^N), \text{ for } n \in \mathbb{N}.$$

Theorem 7.1. *Let T be a free reduced word* (7.2.2) *in the p-adic diagonal C^*-probability space $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$. Then*

$$\varphi^p(T^k) = \prod_{l=1}^{N}\left(\frac{1}{p^{j_l}} - \frac{1}{p^{j_l+1}}\right), \tag{7.2.3}$$

for all $k \in \mathbb{N}$.

Proof. Since T is a free reduced word $\prod_{l=1}^{N} P_{p, j_l}$ in the p-adic diagonal C^*-probability space $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$ of (7.2.1), we have

$$\varphi^p(T^k) \underline{\underline{equi}} \varphi^p\left(\left(\underset{l=1}{\overset{N}{\otimes}} P_{p, j_l}\right)^k\right) = \varphi^p\left(\underset{l=1}{\overset{N}{\otimes}} P_{p, j_l}^k\right)$$

$$\underline{\underline{equi}} \varphi^p\left(\prod_{l=1}^{N} P_{p, j_l}^k\right) = \varphi^p\left(\prod_{l=1}^{N} P_{p, j_l}\right),$$

since P_{p, j_l} are projections in $\mathfrak{S}_p(\mathbb{Z})$, for all $l = 1, \ldots, N$

$$= \varphi^p(T)$$

$$= \prod_{l=1}^{N} \varphi_{j_l}^p(P_{p, j_l}) = \prod_{l=1}^{N}\left(\frac{1}{p^{j_l}} - \frac{1}{p^{j_l+1}}\right),$$

since T is a free reduced word, for all $k \in \mathbb{N}$.

Remark again the differences between the notation T^k and $T^{(k)}$ for a free reduced word T in $\mathfrak{S}_p(\mathbb{Z})$, for $k \in \mathbb{N}$, see **NR 7.1.1**.

Also, by **NR 7.1.1**, one has that, if T is as above, then

$$T^* = T \text{ in } \mathfrak{S}_p(\mathbb{Z}).$$

Therefore, the free-moment formula (7.2.3) characterizes the free distribution of free reduced words T of (7.2.2) in $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$.

Theorem 7.2. *Let* $T_s = \prod_{l=1}^{N_s} P_{p,\, j_{j_{s_l}}}$ *be free reduced words* (7.2.2) *in the* p-adic *diagonal* C^*-*probability space* $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$, *inducing the corresponding integer-sequences* $J_s = (j_{s_1}, \ldots, j_{s_{N_s}})$, *for all* $s = 1, \ldots, n$, *for* $N_s, n \in \mathbb{N}$. *Let*

$$\Sigma = \sum_{s=1}^{N} T_s \in (\mathfrak{S}_p(\mathbb{Z}), \varphi^p), \tag{7.2.4}$$

for $N \in \mathbb{N}$. If the integer-sequences $\{J_s\}_{s=1}^{N}$ are mutually-distinct in $\mathbb{Z}^\infty$, in the sense that: $J_{s_1} \neq J_{s_2}$, whenever $s_1 \neq s_2$ in $\{1, \ldots, n\}$, then

$$\varphi^p(\Sigma^k) = \sum_{s=1}^{N} \left(\prod_{l=1}^{N_s} \left(\frac{1}{p^{j_l}} - \frac{1}{p^{j_l+1}} \right) \right), \tag{7.2.5}$$

for all $k \in \mathbb{N}$.

Proof. Let Σ be a free random variable (7.2.4) of the p-adic diagonal C^*-probability space $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$, where every summand T_s is a free reduced word, for all $s = 1, \ldots, n$.

Note that, by the assumption that the corresponding integer-sequences $J_1, \ldots, J_n$ are mutually-distinct from each other as sequences, all summands $T_1, \ldots, T_n$ are contained in the mutually-distinct direct summands of $\mathfrak{S}_p(\mathbb{Q})$ in the sense of (7.1.1). It guarantees that all summands $T_1, \ldots, T_n$ are free from each other in $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$. So, the free random variable Σ of (7.2.4) forms a free sum in $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$.

Therefore, the free sum Σ satisfies that

$$\Sigma^k \ \underline{\underline{equi}} \left(\underset{s=1}{\overset{N}{\oplus}} T_s \right)^k = \underset{s=1}{\overset{N}{\oplus}} T_s^k = \underset{s=1}{\overset{N}{\oplus}} T_s \ \underline{\underline{equi}} \ \Sigma,$$

for all $k \in \mathbb{N}$, because $T_s^k = T_s$, for all $k \in \mathbb{N}$, for all $s = 1, \ldots, N$. Thus, one has

$$\varphi^p(\Sigma^k) = \varphi^p(\Sigma) = \varphi^p\left(\sum_{s=1}^{N} T_s\right)$$

$$= \sum_{s=1}^{N} \varphi^p(T_s),$$

where $\varphi^p(T_s)$ satisfy (7.2.3), for all $s = 1, \ldots, N$.

Therefore, the free-moment formula (7.2.5) holds true.

Remark the differences between the notations Σ^k and $\Sigma^{(k)}$ in $\mathfrak{S}_p(\mathbb{Z})$, for all $k \in \mathbb{N}$, see **NR 7.1.2**. Also, by **NR 7.1.2**, one can have that if Σ is in the sense of (7.2.4), then

$$\Sigma^* = \Sigma \text{ in } (\mathfrak{S}_p(\mathbb{Z}), \varphi^p).$$

Thus, the above free-moment formula (7.2.5) completely characterizes the free distribution of free sums Σ in $\mathfrak{S}_p(\mathbb{Z})$.

7.3. The Adelic-Diagonal C^*-Probability Space $(\mathfrak{S}_p(\mathbb{Z}), \varphi)$. Now, let $(\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$ be the p-adic diagonal C^*-probability spaces, for all $p \in \mathcal{P}$. One can construct the family

$$\mathfrak{A} = \{\mathfrak{S}_p(\mathbb{Z}) \ \underline{\underline{denote}} \ (\mathfrak{S}_p(\mathbb{Z}), \varphi^p) : p \in \mathcal{P}\} \tag{7.3.1}$$

of C^*-probability spaces.

Then, one can construct the free product C^*-probability space $(\mathfrak{S}_p(\mathbb{Z}), \varphi)$ of the family $\mathfrak{A}$ of (7.3.1),

$$(\mathfrak{S}_p(\mathbb{Z}), \varphi) \underline{\underline{def}} \underset{Y \in \mathfrak{A}}{*} Y. \tag{7.3.2}$$

i.e.,

$$\begin{aligned}(\mathfrak{S}_p(\mathbb{Z}), \varphi) &= \underset{p \in \mathcal{P}}{*} \mathfrak{S}_p(Z) \\ &= \left(\underset{p \in \mathcal{P}}{*} \mathfrak{S}_p(\mathbb{Z}), \underset{p \in \mathcal{P}}{*} \varphi^p \right) \\ &= \underset{p \in \mathcal{P}, j \in \mathbb{Z}}{*} (\mathfrak{S}_p, \varphi_j^p),\end{aligned}$$

where $\mathfrak{S}_p(\mathbb{Z}) = (\mathfrak{S}_p(\mathbb{Z}), \varphi^p)$ are the p-adic diagonal C^*-probability spaces, and $(\mathfrak{S}_p, \varphi_j^p)$ are the j-th p-adic C^*-probability spaces, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Definition 7.2. The free product C^*-probability space $(\mathfrak{S}_{\mathcal{P}}(\mathbb{Z}), \varphi)$ of (7.3.1), satisfying (7.3.2), is called the Adelic-diagonal C^*-probability space.

Now, let T be an operator

$$T = \prod_{l=1}^{N} P_{p_l, j_l} = \prod_{l=1}^{N} \alpha_{\partial_{j_l}}^{p_l}, \text{ for } N \in \mathbb{N}, \tag{7.3.3}$$

in the Adelic-diagonal C^*-probability space $(\mathfrak{S}_{\mathcal{P}}(\mathbb{Z}), \varphi)$. Let

$$W_T = (p_l)_{l=1}^{N}, \text{ and } J_T = (j_l)_{l=1}^{N}$$

be the prime-sequence, respectively, the integer-sequence obtained from the free reduced word T of (7.3.3).

Theorem 7.3. *Let T be an operator* (7.3.3) *of the Adelic-diagonal C^*-probability space* $(\mathfrak{S}_{\mathcal{P}}(\mathbb{Z}), \varphi)$ *of* (7.3.1). *Assume that either*

$$W_T \in alt(\mathcal{P}^N), \text{ or } J_T \in alt(\mathbb{Z}^N). \tag{7.3.4}$$

Then we obtain

$$\varphi(T^k) = \prod_{l=1}^{N} \left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}} \right), \tag{7.3.5}$$

for all $k \in \mathbb{N}$.

Proof. By the condition (7.3.4), the given operator T of (7.3.3) is a free reduced word in $(\mathfrak{S}_{\mathcal{P}}(\mathbb{Z}), \varphi)$, by (7.3.2). Since T forms a free reduced word in $\mathfrak{S}_{\mathcal{P}}(\mathbb{Z})$, one has

$$T^k \underline{\underline{equi}} \left(\underset{l=1}{\overset{N}{\otimes}} P_{p_l, j_l} \right)^k = \underset{l=1}{\overset{N}{\otimes}} P_{p_l, j_l}^k \underline{\underline{equi}}\, T$$

in $\mathfrak{S}_{\mathcal{P}}(\mathbb{Z})$, for all $k \in \mathbb{N}$.

So, we obtain that

$$\varphi(T^k) = \varphi(T) = \varphi\left(\prod_{l=1}^{N} P_{p_l, j_l} \right)$$

$$= \prod_{l=1}^{N} \varphi_{j_l}^{p_l}(P_{p_l, j_l})$$

by (7.3.2)

$$= \prod_{l=1}^{N} \left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}} \right),$$

for all $k \in \mathbb{N}$. Therefore, the free-distributional data (7.3.5) is obtained.

Let T be an operator (7.3.3) in the Adelic-diagonal C^*-probability space $(\mathfrak{S}_{\mathcal{P}}(\mathbb{Z}), \varphi)$, and suppose either the prime-sequence W_T, or the integer-sequence J_T is an alternating sequence. Then T is a free reduced word in $\mathfrak{S}_{\mathcal{P}}(\mathbb{Z})$, satisfying

$$T^* = T \text{ in } \mathfrak{S}_{\mathcal{P}}(\mathbb{Z}).$$

(See **NR 7.1.1**!)

So, the above free-moment formula (7.3.5) characterizes the free distributions of a free reduced word T in $\mathfrak{S}_{\mathcal{P}}(\mathbb{Z})$.

7.4. Discussion. Let $\mathfrak{S}_{\mathcal{P}}(\mathbb{Z}) = (\mathfrak{S}_{\mathcal{P}}(\mathbb{Z}), \varphi)$ be our Adelic-diagonal C^*-probability space (7.3.2). Recall that, by (7.3.5), if T is a free reduced word in the sense of (7.3.3) in $\mathfrak{S}_{\mathcal{P}}(\mathbb{Z})$, then

$$\varphi(T^k) = \prod_{l=1}^{N}\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right) = \prod_{l=1}^{N}\frac{1}{p_l^{j_l}}\left(1 - \frac{1}{p_l}\right)$$

$$= \prod_{l=1}^{N}\frac{p_l}{p_l^{j_l+1}}\left(1 - \frac{1}{p_l}\right),$$

i.e.,

$$\varphi(T^k) = \left(\prod_{l=1}^{N}\frac{1}{p_l^{j_l+1}}\right)\left(\prod_{l=1}^{N}p_l\left(1 - \frac{1}{p_l}\right)\right), \tag{7.4.1}$$

for all $k \in \mathbb{N}$.

Recall that an arithmetic function $\phi : \mathbb{N} \to \mathbb{C}$ is the Euler totient function, if

$$\phi(n) \underline{\underline{def}} \left| \left\{ k \in \mathbb{N} \,\middle|\, \begin{array}{c} 1 \le k \le n \ and \\ \gcd(n, k) = 1 \end{array} \right\} \right|, \tag{7.4.2}$$

where $|X|$ mean the cardinalities of sets X, and gcd means "the greatest common divisor."

It is well-know that, ϕ is the Euler totient function (7.4.2), if and only if

$$\phi(n) = n\left(\prod_{p\in\mathcal{P},\ p\mid n}\left(1 - \frac{1}{p}\right)\right), \text{ for all } n \in \mathbb{N}, \tag{7.4.3}$$

where "$p \mid n$" means "p divides n," or "p is a divisor of n." For instance,

$$\phi(q) = q - 1 = q\left(1 - \frac{1}{q}\right), \text{ for all } q \in \mathcal{P}. \tag{7.4.4}$$

Also, the Euler totient function ϕ is a multiplicative arithmetic function in the sense that:

$$\phi(n_1 n_2) = \phi(n_1)\phi(n_2), \tag{7.4.5}$$

whenever

$$\gcd\,(n_1,\ n_2 = 1,)$$

for $n_1,\ n_2 \in \mathbb{N}$.

Thus, one can figure out the following relation on our free-distributional data.

Theorem 7.4. *Let* $T = \prod_{l=1}^{N} P_{p_l, j_l}$ *be an operator* (7.3.3) *in the Adelic-diagonal* C^**-probability space* $\mathfrak{S}_{\mathcal{P}}(\mathbb{Z})$ *of* (7.3.2), *and let* $W_T = (p_1, \ldots, p_N)$ *be the corresponding prime-sequence for T in the sense of* (7.3.4). *Assume that all the entries* $p_1, \ldots, p_N$ *of* W_T *are mutually-distinct from each other in* $\mathcal{P}$. *Then there exists* $n_T \in \mathbb{N}$, *and* $N_T \in \mathbb{Q}$, *such that*

$$\varphi(T^k) = N_T\, \phi(n_T), \text{ for all } k \in \mathbb{N}, \tag{7.4.6}$$

where ϕ *is the Euler totient function.*

Proof. Let T be an operator (7.3.3) in the Adelic-diagonal C^*-probability space $\mathfrak{S}_{\mathcal{P}}(\mathbb{Z})$ and assume that all entries $p_1, \ldots, p_N$ of the corresponding prime-sequence $W_T \in (p_1, \ldots, p_N)$ for T are mutually-distinct from each other in $\mathcal{P}$. Then such a mutually-distinctness of $\{p_l\}_{l=1}^{N}$ guarantees that the sequence W_T is an alternating sequence, i.e.,

$$W_T \in all(\mathcal{P}^N),$$

and hence, T forms a free reduced word in $\mathfrak{S}_{\mathcal{P}}(\mathbb{Z})$.

Thus, one has that

$$\varphi(T^k) = \left(\prod_{l=1}^{N} \frac{1}{p_l^{j_l+1}}\right)\left(\prod_{l=1}^{N}\left(1 - \frac{1}{p_l}\right)\right)$$

by (7.4.1)

$$= N_T \left(\prod_{l=1}^{N} \phi(p_l)\right)$$

by (7.4.4), where $N_T = \prod_{l=1}^{N} \frac{1}{p_l^{j_l+1}} \in \mathbb{Q}$

$$= N_T \phi\left(\prod_{l=1}^{N} p_l\right)$$

since ϕ is multiplicative in the sense of (7.4.5), because $p_1, \ldots, p_N$ are mutually-distinct in $\mathcal{P}$

$$= N_T \phi(n_T),$$

where $n_T = \prod_{l=1}^{N} p_l \in \mathbb{N}$.

Therefore, there exist

$$N_T = \prod_{l=1}^{N} \frac{1}{p_l^{j_l+1}} \in \mathbb{Q}, \text{ and } n_T = \prod_{l=1}^{N} p_l \in \mathbb{N},$$

such that

$$\varphi(T^k) = N_T \phi(n_T), \text{ for all } k \in \mathbb{N}.$$

For example, if

$$T = P_{5,-3} P_{2,1} P_{7,1} \in \mathfrak{S}_{\mathcal{P}}(\mathbb{Z}).$$

Then we obtain that

$$\varphi(T^k)\underline{\underline{equi}}\,\varphi((P_{5,-3}\otimes P_{2,1}\otimes P_{7,1})^k)=\varphi(P_{5,-3}^k\otimes P_{2,1}^k\otimes P_{7,1}^k)\underline{\underline{equi}}\,\varphi(P_{5,-3}P_{2,1}P_{7,1})$$

$$=\varphi_{-3}^5(P_{5,-3})\varphi_1^2(P_{2,1})\varphi_1^7(P_{7,1})$$

$$=\left(\frac{1}{5^{-3}}-\frac{1}{5^{-3+1}}\right)\left(\frac{1}{2^1}-\frac{1}{2^{1+1}}\right)\left(\frac{1}{7^1}-\frac{1}{7^{1+1}}\right)$$

$$=(125-25)\left(\frac{1}{2}-\frac{1}{4}\right)\left(\frac{1}{7}-\frac{1}{45}\right)=\frac{150}{49},$$

by (7.3.5), for all $k \in \mathbb{N}$.

Also, one obtains that: for a fixed free reduced word T, since the corresponding prime-sequence (5, 2, 7) have mutually-distinct entries in $\mathcal{P}$, one can have that

$$N_T=\frac{1}{5^{-3+1}}\frac{1}{2^{1+1}}\frac{1}{7^{1+1}}=\frac{5^2}{2^2\cdot 7^2}=\frac{25}{4.49}=\frac{25}{196}\text{ in }\mathbb{Q},$$

and

$$n_T=5\cdot 2\cdot 7=70\text{ in }\mathbb{N}.$$

Note that

$$N_T\,\phi(n_T)=\frac{25}{196}\phi(70)=\frac{25\cdot 70}{196}\left(\frac{4}{5}\right)\left(\frac{1}{2}\right)\left(\frac{6}{7}\right)=\frac{150}{49},$$

which demonstrates that

$$\varphi(T^k)=N_T\,\phi(n_T),$$

for all $k \in \mathbb{N}$.

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