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NEW RESULTS FOR THE DESCARTES-FRENICLE-SORLI CONJECTURE ON ODD PERFECT NUMBERS

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Abstract

If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then the Descartes-Frenicle-Sorli conjecture predicts that $k = 1$. Brown [5] has recently announced a proof for the inequality $q < n$, and a partial proof that $q^k < n$ holds under many cases. In this article, we give a strategy for strengthening Brown's result to $q^2 < n$.

1. Introduction

If N is a positive integer, then we write $\sigma(N)$ for the sum of the divisors of N. A number N is perfect if $\sigma(N) = 2N$. It is currently unknown whether there are infinitely many even perfect numbers, or whether any odd perfect numbers (OPNs) exist. Ochem and Rao recently proved [12] that, if N is an odd perfect number, then $N > 10^{1500}$ and that the largest component (i.e., divisor p^a with p prime) of N is bigger than 10^{62}. This improves on previous results by Brent, Cohen and te Riele [3] in 1991 $(N > 10^{300})$ and Cohen [7] in 1987 (largest component $p^a > 10^{20}$).

An odd perfect number $N = q^k n^2$ is said to be given in Eulerian form if q is prime with $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$. (The number q is called the Euler prime, while the component q^k is referred to as the Euler factor. Note that, since q is prime and $q \equiv 1 \pmod 4$, then $q \geq 5$.)

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We denote the abundancy index I of the positive integer x as

$$I(x) = \frac{\sigma(x)}{x}.$$

In his Ph.D. thesis, Sorli [13] conjectured that $k = 1$, after testing large numbers with 8 distinct prime factors for perfection. (More recently, Beasley [2] points out that Descartes was the first to conjecture $k = 1$ "in a letter to Mersenne in 1638, with Frenicle's subsequent observation occurring in 1657".)

In the M.Sc. thesis [11], it was conjectured that the components q^k and n are related by the inequality $q^k < n$. This conjecture was made on the basis of the result $I(q^k) < I(n)$. Recently, Brown [5] announced a proof for the inequality $q < n$, and a partial proof that $q^k < n$ holds under many cases.

2. Conditions Sufficient for Sorli's Conjecture

Some sufficient conditions for Sorli's conjecture were given in [9]. We reproduce these conditions here.

Lemma 1. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. If* $n < q$, *then* $k = 1$.

Remark 2. The proof of Lemma 1 follows from the inequality $q^k < n^2$ and the congruence $k \equiv 1 \pmod 4$ (see [9]). (Note the related inequality

$$I(q^k) < I(n^2)$$

for the abundancy indices of the components q^k and n^2).

Lemma 3. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. If*

$$\sigma(n) \le \sigma(q),$$

then $k = 1$.

Lemma 4. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. If*

$$\frac{\sigma(n)}{q} < \frac{\sigma(q)}{n},$$

then $k = 1$.

Remark 5. Notice that, if

$$\frac{\sigma(n)}{q} < \frac{\sigma(q)}{n},$$

then it follows that

$$\frac{\sigma(n)}{q^k} = \frac{\sigma(n)}{q} < \frac{\sigma(q)}{n} = \frac{\sigma(q^k)}{n}.$$

Consequently, by the contrapositive, if

$$\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k},$$

then

$$\frac{\sigma(q)}{n} \leq \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k} \leq \frac{\sigma(n)}{q}.$$

Remark 6. Let $N = q^k n^2$ be an odd perfect number given in Eulerian form. Suppose that

$$\frac{\sigma(q)}{n} = \frac{\sigma(n)}{q}.$$

Then we know that:

$$q\sigma(q) = n\sigma(n).$$

Since $\gcd(q, n) = 1$, then $q \mid \sigma(n)$ and $n \mid \sigma(q)$. Therefore, it follows that $\frac{\sigma(q)}{n}$ and $\frac{\sigma(n)}{q}$ are equal positive integers.

This is a contradiction, as:

$$1 < I(q) = \frac{\sigma(q)}{q} = 1 + \frac{1}{q} \leq \frac{6}{5} < \sqrt{\frac{5}{3}} < I(n) < I(q)I(n) = I(qn) < 2$$

which implies that:

$$1 < \sqrt{\frac{5}{3}} < I(n) < I(q)I(n) = I(qn) = \left[\frac{\sigma(q)}{q}\right]\left[\frac{\sigma(n)}{n}\right] = \left[\frac{\sigma(q)}{n}\right]\left[\frac{\sigma(n)}{q}\right] < 2.$$

Consequently,

$$\frac{\sigma(q)}{n} \neq \frac{\sigma(n)}{q}.$$

Similarly, we can prove that

$$\frac{\sigma(q^k)}{n} \neq \frac{\sigma(n)}{q^k}.$$

Lemma 7. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. Then* $n < q$ *if and only if* $N < q^3$.

Proof. Suppose that $N = q^k n^2$ is an odd perfect number given in Eulerian form. If $n < q$, then assuming to the contrary that $q^3 < N$, we get that

$$q^3 < N = qn^2 < q \cdot q^2 = q^3$$

since $n < q$ implies $k = 1$, by Lemma 1. For the other direction, if $N < q^3$, then $q^k n^2 < q^3$, so that we have

$$n^2 < q^{3-k} \leq q^2$$

since $k \equiv 1 \pmod 4$ implies that $k \geq 1$. Consequently, $n < q$, and we are done. □

Corollary 8. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. Then* $n < q^{5/2}$ *if and only if* $N < q^6$.

Proof. First we show that $n < q^{5/2}$ implies $k = 1$. To this end, assuming $n < q^{5/2}$, since $q^k < n^2$ (see [9]), we then have that:

$$q \leq q^k < n^2 < q^5.$$

The last chain of inequalities implies that

$$1 \leq k < 5.$$

This inequality, together with the condition $k \equiv 1 \pmod 4$, implies that $k = 1$.

We now prove the claim in Corollary 8. If $n < q^{5/2}$, then assuming to the contrary that $q^6 < N$, we get that

$$q^6 < N = qn^2 < q \cdot q^5 = q^6.$$

This is a contradiction. For the other direction, if $N < q^6$, then $q^k n^2 < q^6$, so that we have

$$n^2 < q^{6-k} \leq q^5$$

since $k \equiv 1 \pmod 4$ implies that $k \geq 1$. Consequently, $n < q^{5/2}$, and we are done. □

Remark 9. A recent result by Acquaah and Konyagin [1] almost disproves $n < q$. They obtained the estimate $y < (3N)^{1/3}$ for all the prime factors y of an odd perfect number N. In particular, if $N = q^k n^2$ is an odd perfect number given in Eulerian form, then letting $y = q$ and assuming $k = 1$ gives:

$$q < (3N)^{1/3} = (3qn^2)^{1/3} \Rightarrow q^3 < 3qn^2 \Rightarrow q < n\sqrt{3}.$$

Since the contrapositive of the implication $n < q \Rightarrow k = 1$ is $k > 1 \Rightarrow q < n$, it follows that the inequality

$$q < n\sqrt{3}$$

holds unconditionally, regardless of the status of Sorli's conjecture.

More recently, Brown [5] claims a proof for the inequality $q < n$, and a partial proof that $q^k < n$ holds under many cases.

We now give a condition that is weaker than $n < q$, which also implies $k = 1$.

Lemma 10. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. Then*

$$n < \left(\frac{3}{2} q^5\right)^{1/2}$$

implies $k = 1$.

Proof. Suppose that $N = q^k n^2$ is an odd perfect number given in Eulerian form. Let

$$n < \left(\frac{3}{2} q^5\right)^{1/2}$$

and assume to the contrary that $k \neq 1$. Since $k \equiv 1 (\mathrm{mod}\, 4)$, this means that $k \geq 5$. Additionally, from [9], we have that

$$q^k < \sigma(q^k) \leq \frac{2}{3} n^2.$$

Consequently, we have the following chain of inequalities:

$$q^5 \leq q^k < \frac{2}{3}\left(\left(\frac{3}{2} q^5\right)^{1/2}\right)^2 < q^5.$$

This is a contradiction. □

We also have the following corollary to Lemma 10, and this uses a result from [4].

Corollary 11. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. Then*

$$n < \left(\frac{315}{2} q^5\right)^{1/2}$$

implies $k = 1$.

Proof. The proof is very similar to that of Lemma 10, except that it uses the improved bound

$$\sigma(q^k) \leq \frac{2}{315} n^2$$

(see [4]) instead of

$$\sigma(q^k) \leq \frac{2}{3} n^2$$

(see [9]). □

Remark 12. Similar to the proofs of Lemma 7 and Corollary 8, we can show that the following biconditionals are true:

$$n < \left(\frac{3}{2} q^5\right)^{1/2} \Leftrightarrow N < \frac{3}{2} q^6$$

$$n < \left(\frac{315}{2} q^5\right)^{1/2} \Leftrightarrow N < \frac{315}{2} q^6.$$

Remark 13. Chen and Chen [6] has a relatively recent paper which further improves on Broughan et al.'s results (see [4]). They also pose a related open problem.

3. New Results Related to Sorli's Conjecture

First, we reproduce the following lemma from [9], as we will be using these results later.

Lemma 14. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. The following series of inequalities hold*:

- *If* $k = 1$, *then* $1 < I(q^k) = I(q) \leq \frac{6}{5} < \sqrt{\frac{5}{3}} < I(n) < 2.$
- *If* $k \geq 1$, *then* $1 < I(q^k) < \frac{5}{4} < \sqrt{\frac{8}{5}} < I(n) < 2.$

We have the following (slightly) stronger inequality from [9].

Lemma 15. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. Then* $(I(q^k))^2 < I(n^2)$.

Proof. The proof follows from the inequality $I(q^k) < \sqrt[3]{2}$ and the equation $2 = I(q^k) I(n^2)$. □

Remark 16. Another proof of Lemma 15 is as follows:

$$I(q^k) < \frac{5}{4} \Rightarrow (I(q^k))^2 < \frac{25}{16} = 1.5625 < 1.6 = \frac{8}{5} < I(n^2).$$

In fact, if

$$(I(q^k))^y < \left(\frac{5}{4}\right)^y \leq \frac{8}{5} < I(n^2)$$

then

$$y \leq \frac{3\log 2 - \log 5}{\log 5 - 2\log 2}.$$

Thus, if we let

$$z = \frac{3\log 2 - \log 5}{\log 5 - 2\log 2} \approx 2.1062837195$$

then

$$(I(q^k))^z \leq \frac{8}{5} < I(n^2).$$

Next, we derive a lower bound for $I(q^k) + I(n)$.

Lemma 17. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. The following inequality holds*:

$$I(q^k) + I(n) \geq I(q) + I(n) > 1 + \sqrt{2}.$$

Proof. Let $N = q^k n^2$ be an odd perfect number given in Eulerian form. Then we have the following:

$$I(q^k) + I(n) \geq I(q) + I(n) \geq 1 + \frac{1}{q} + \sqrt{\frac{2(q-1)}{q}}.$$

But

$$f(q) = 1 + \frac{1}{q} + \sqrt{\frac{2(q-1)}{q}}$$

is a decreasing function of q. Consequently,

$$f(q) > \lim_{q\to\infty}\left(1 + \frac{1}{q} + \sqrt{\frac{2(q-1)}{q}}\right) = 1 + \sqrt{2}.$$ □

Remark 18. The following result was communicated to the author (via e-mail, by Pascal Ochem) in April of 2013. If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then

$$I(n) > \left(\frac{8}{5}\right)^{\frac{\ln(4/3)}{\ln(13/9)}} \approx 1.44440557$$

(Note that $\left(\frac{8}{5}\right)^{\frac{\ln(4/3)}{\ln(13/9)}} > \sqrt{2}$.)

Further to Remark 18 and Lemma 15, we have the following related result.

Lemma 19. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. Then* $(I(q))^2 < I(n)$.

Proof. By Lemma 14,

$$I(q) \leq \frac{6}{5} \Rightarrow (I(q))^2 \leq \frac{36}{25} = 1.44.$$

The conclusion follows from the result $I(n) > 1.44440557$ in Remark 18.

In fact, if

$$I(q)^u < \left(\frac{6}{5}\right)^u \leq \left(\frac{8}{5}\right)^{\frac{\ln(4/3)}{\ln(13/9)}}$$

Then

$$u \leq -\frac{(2\log(2) - \log(3))(3\log(2) - \log(5))}{(\log(2) + \log(3) - \log(5))(2\log(3) - \log(13))}.$$

Thus, if we let

$$v = -\frac{(2\log(2) - \log(3))(3\log(2) - \log(5))}{(\log(2) + \log(3) - \log(5))(2\log(3) - \log(13))} \approx 2.0168$$

then

$$(I(q))^{v} \le \left(\frac{8}{5}\right)^{\frac{\ln(4/3)}{\ln(13/9)}} < I(n). \qquad \square$$

Remark 20. As pointed out by Ochem to the author (via the same e-mail mentioned in Remark 18), a case-by-case analysis yields a sharper lower bound for $I(q^k) + I(n)$:

- If $q = 5$, then $I(q^k) + I(n) \ge I(q) + I(n) \ge (6/5) + (8/5)^{\ln(4/3)/\ln(13/9)} \approx 2.6444055$

- If $q \ge 13$, then $I(q^k) + I(n) \ge I(q) + I(n) \ge (14/13) + (24/13)^{\ln(4/3)/\ln(13/9)} \approx 2.6924318$

Therefore, we have the lower bound

$$I(q^k) + I(n) \ge I(q) + I(n) \ge \frac{6}{5} + \left(\frac{8}{5}\right)^{\frac{\ln(4/3)}{\ln(13/9)}} \approx 2.6444055$$

We now state and prove the following theorem, which provides conditions equivalent to the conjecture mentioned in the introduction.

Theorem 21. *If* $N = q^k n^2$ *is an odd perfect number given in Eulerian form, then the following biconditional is true*:

$$q^k < n \Leftrightarrow \sigma(q^k) < \sigma(n).$$

In preparation for the proof of Theorem 21, we derive the following results.

Lemma 22. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. If*

$$I(q^k) + I(n) < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k},$$

then

$$q^k < n \Leftrightarrow \sigma(q^k) < \sigma(n).$$

Proof. Let $N = q^k n^2$ be an odd perfect number given in Eulerian form. Assume that

$$I(q^k) + I(n) < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}.$$

It follows that

$$I(q^k) + I(n) < \left(\frac{q^k}{n}\right) I(q^k) + \left(\frac{n}{q^k}\right) I(n).$$

Consequently,

$$q^k n(I(q^k) + I(n)) < q^{2k} I(q^k) + n^2 I(n).$$

Thus,

$$n[q^k - n]I(n) < q^k [q^k - n] I(q^k).$$

If $q^k < n$, then $q^k - n < 0$. Hence,

$$q^k < n \Rightarrow q^k I(q^k) < nI(n) \Rightarrow \sigma(q^k) < \sigma(n).$$

If $n < q^k$, then $0 < q^k - n$. Hence,

$$n < q^k \Rightarrow nI(n) < q^k I(q^k) \Rightarrow \sigma(n) < \sigma(q^k).$$

Consequently, we have

$$q^k < n \Leftrightarrow \sigma(q^k) < \sigma(n),$$

as desired. □

Lemma 23. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. If*

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n),$$

then

$$q^k < n \Leftrightarrow \sigma(n) < \sigma(q^k).$$

Proof. The proof of Lemma 23 is very similar to the proof of Lemma 22. □

Now, assume that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n).$$

Consider the conclusion of the implication in Lemma 23 in light of the result $I(q^k) < I(n)$:

$$q^k < n \Leftrightarrow \sigma(n) < \sigma(q^k).$$

If $q^k < n$, then since $I(q^k) < I(n)$ implies that

$$\frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n},$$

we have

$$\frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n} < 1,$$

which further implies that $\sigma(q^k) < \sigma(n)$. This contradicts Lemma 23. Similarly, if $\sigma(n) < \sigma(q^k)$, then

$$1 < \frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n},$$

from which it follows that $n < q^k$. Again, this contradicts Lemma 23. Hence, we know that

$$n < q^k < \sigma(q^k) < \sigma(n)$$

must hold, under the given assumption. Assuming Brown's proof for $q^k < n$ is completed, this case is ruled out. Consequently, the inequality

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n)$$

cannot be true. Therefore, the reverse inequality

$$I(q^k) + I(n) \leq \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

must be true.

It remains to consider the case when

$$I(q^k) + I(n) = \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}.$$

Notice that this is true if and only if

$$\sigma(q^k) = \sigma(n),$$

(because $q^k \neq n$). Thus, since $I(q^k) < I(n)$, this implies that $n < q^k$.

Again, assuming Brown's proof for $q^k < n$ is completed, this case is ruled out.

In other words (by Lemma 22), we have Theorem 21 (and the corollary that follows).

Corollary 24. *If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then the following biconditional is true*:

$$q^k < n \Leftrightarrow \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}.$$

We now give another condition that is equivalent to the author's conjecture (mentioned in the introduction).

Theorem 25. *If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then the following biconditional is true*:

$$\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k} \Leftrightarrow \frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)}.$$

Proof. Let N be an odd perfect number given in Eulerian form. Then $N = q^k n^2$ where $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$.

First, we show that

$$\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}$$

implies

$$\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)}.$$

Since $I(q^k) < I(n)$, we have that

$$\frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n}.$$

On the other hand, the inequality

$$\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}$$

gives us that

$$\frac{\sigma(q^k)}{\sigma(n)} < \frac{n}{q^k}.$$

This in turn implies that

$$\frac{q^k}{n} < \frac{\sigma(n)}{\sigma(q^k)}.$$

Putting these inequalities together, we have the series

$$\frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n} < \frac{\sigma(n)}{\sigma(q^k)}.$$

Now consider the product

$$\left(\frac{\sigma(q^k)}{\sigma(n)} - \frac{q^k}{n}\right)\left(\frac{\sigma(n)}{\sigma(q^k)} - \frac{q^k}{n}\right).$$

This product is negative. Consequently we have

$$\left(\frac{\sigma(q^k)}{\sigma(n)}\right)\left(\frac{\sigma(n)}{\sigma(q^k)}\right) - \left(\frac{q^k}{n}\right)\left(\frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)}\right) + \left(\frac{q^k}{n}\right)^2 < 0,$$

from which it follows that

$$1 + \left(\frac{q^k}{n}\right)^2 < \left(\frac{q^k}{n}\right)\left(\frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)}\right).$$

Therefore, we obtain

$$\frac{n}{q^k} + \frac{q^k}{n} < \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)}$$

as desired.

Next, assume that

$$\frac{\sigma(n)}{q^k} < \frac{\sigma(q^k)}{n}.$$

Since $I(q^k) < I(n)$, we obtain

$$\frac{n}{q^k} < \frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n}.$$

Now consider the product

$$\left(\frac{n}{q^k} - \frac{\sigma(q^k)}{\sigma(n)}\right)\left(\frac{q^k}{n} - \frac{\sigma(q^k)}{\sigma(n)}\right).$$

This product is negative. Therefore, we obtain

$$\left(\frac{n}{q^k}\right)\left(\frac{q^k}{n}\right) - \left(\frac{\sigma(q^k)}{\sigma(n)}\right)\left(\frac{n}{q^k} + \frac{q^k}{n}\right) + \left(\frac{\sigma(q^k)}{\sigma(n)}\right)^2 < 0,$$

from which we get

$$1 + \left(\frac{\sigma(q^k)}{\sigma(n)}\right)^2 < \left(\frac{\sigma(q^k)}{\sigma(n)}\right)\left(\frac{n}{q^k} + \frac{q^k}{n}\right).$$

Consequently, we have

$$\frac{\sigma(n)}{\sigma(q^k)} + \frac{\sigma(q^k)}{\sigma(n)} < \frac{n}{q^k} + \frac{q^k}{n}.$$

Together with the result in the previous paragraph, this shows that

$$\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}$$

is equivalent to

$$\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)}. \qquad \square$$

Remark 26. Let $N = q^k n^2$ be an odd perfect number given in Eulerian form.

Note that, in general, it is true that

$$\frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k},$$

and

$$\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}.$$

Therefore,

$$\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}$$

is equivalent to

$$\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k},$$

while

$$\frac{\sigma(n)}{q^k} < \frac{\sigma(q^k)}{n}$$

is equivalent to

$$\frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)} < \frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}.$$

At this point, we dispose of the following lemma:

Lemma 27. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. Then at least one of the following sets of inequalities is true*:

- **A** : $q^k < \sigma(q^k) < n < \sigma(n)$
- **B** : $q^k < n < \sigma(q^k) < \sigma(n)$
- **C** : $n < q^k < \sigma(n) < \sigma(q^k)$
- **D** : $n < \sigma(n) \leq q^k < \sigma(q^k)$.

Lemma 27 is proved by listing all possible permutations of the set

$$\{q^k, n, \sigma(q^k), \sigma(n)\}$$

and then using Theorem 21.

Note that Brown's result that $q^k < n$, when completed, would rule out cases **C** and **D** in Lemma 27. Also, notice that by assuming $k = 1$, case **B** is also ruled out.

Consequently, we have the following theorem.

Theorem 28. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. If* $k = 1$, *then* $\sigma(q^k) < n$.

As a corollary, by the contrapositive to Theorem 28, we have:

Corollary 29. *Let* $N = q^k n^2$ *be an odd perfect number given in Eulerian form. If* $n < \sigma(q^k)$, *then* $k > 1$.

Remark 30. If one could show the biconditional

$$n < q^{k+1} \Leftrightarrow n < \sigma(q^k),$$

then one would be able to show that

$$n < q^{k+1} \Rightarrow k > 1.$$

By the contrapositive, one would then have

$$k = 1 \Rightarrow q^{k+1} < n \Rightarrow q^2 < n.$$

However, we know that

$$n < q^2 \Rightarrow k = 1.$$

Consequently,

$$n < q^2 \Rightarrow k = 1 \Rightarrow q^2 < n$$

which proves that $q^2 < n$, strengthening Brown's result.

4. Final Analysis of the New Results

The new results presented in this article seem to imply the following conjecture (see [10]).

Conjecture 31. Let $N = q^k n^2$ be an odd perfect number given in Eulerian form. Then the Descartes-Frenicle-Sorli conjecture is false. (That is, $k > 1$ must hold).

Remark 32. Notice how all of the implications in the Lemmas 1, 3 and 4 in Section 2 become vacuously true, given Brown's result that $q < n$. Also, notice that, in Section 3, we could specialize Theorem 21 (and its consequences) to the case $k = 1$ and still get the same results, as follows:

$$q < n \Leftrightarrow \sigma(q) < \sigma(n) \Leftrightarrow \frac{\sigma(q)}{n} < \frac{\sigma(n)}{q}.$$

5. Conclusion

An improvement to the currently known upper bound of $I(n) < 2$ will be considered a major breakthrough. In the sequel (http://arxiv.org/abs/1303.2329), a viable approach towards improving the inequality $I(n) < 2$ will be presented, which may necessitate the use of ideas from the paper [14].

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