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SHIFTED CHEBYSHEV-TAU METHOD FOR SOLVING AN INVERSE TIME-DEPENDENT SOURCE PROBLEM

S. AKBARPOUR, A. SHIDFAR and H. SABERI NAJAFI

Department of Mathematics
Lahijan Branch
Islamic Azad University
Lahijan, Iran
E-mail: samanehakbarpoor@gmail.com
shidfar@iust.ac.ir
hnajafi@guilan.ac.ir

Abstract

In this article, a numerical method for solving parabolic inverse problem with an unknown time-dependent source parameter is considered. This method is based upon Chebyshev Tau approximation and using Chebyshev operational matrix. Such approach has the advantage of reducing the problem to the solution of a system of algebraic equations. By solving this system of equations, the unknown Chebyshev coefficients can be determined. Numerical results show that the proposed method is of high accuracy and is efficient for solving an inverse parabolic problem with unknown time dependent parameter.

1. Introduction

Parabolic partial differential equations describe a wide range of problems in various fields of science including heat diffusion [1], ocean acoustic propagation [2], population dynamics [3], dynamics of nuclear reactors [4], adsorption of pollutants in soil and the diffusion of neutrons. The parabolic partial differential problem is concerned with the calculation of unknown solution while the initial and boundary conditions are given. But in the inverse parabolic partial differential problem with over specified condition, the determination of unknown solution and unknown source term are required. Inverse problems (IPs) have been appeared in many important

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applications in heat transfer, thermoelasticity, control theory, population dynamics, nuclear reactor dynamics, medical sciences, biochemistry and etc. [5-19]. Most often, the analytical solution for inverse problem is difficult to obtain. The important goal in IPs is their solvability and description of a constructive algorithm for finding a solution. Several numerical methods have been introduced to obtain the solutions of inverse problems, see for example [20-37]. In this paper, we consider the inverse problem with an unknown time-dependent source parameter. Over the last few years, it has become increasingly apparent that many physical phenomena can be described in terms of parabolic partial differential equations with source control parameters. This type of equations arise, for example, in the study of heat conduction processes, thermoelasticity, chemical diffusion and control theory [38-41]. Growing attention is being paid to the development, analysis and implementation of accurate methods for the numerical solution of parabolic inverse problems, i.e. for the determination of unknown function $p(t)$ in the parabolic partial differential equations.

In this paper, we consider the following parabolic equation:

$$u_t(x, t) = u_{xx}(x, t) + p(t)u(x, t) + q(x, t), \quad 0 < x < L, 0 < t \leq \tau, \tag{1}$$

with initial condition

$$u(x, 0) = f(x), \quad 0 < x < L, \tag{2}$$

and boundary conditions

$$\alpha_1(t)u_x(0, x) + \beta(t)u(0, t) = g_1(t), \quad 0 < t \leq \tau, \tag{3}$$

$$\alpha_2(t)u_x(L, t) + \gamma(t)u(L, t) = g_2(t), \quad 0 < t \leq \tau, \tag{4}$$

where $q(x, t), f(x), \beta(t), \gamma(t), \alpha_i(t), g_i(t), i = 1, 2$ are known functions.

If the function $p(t)$ is known, the problem of finding $u(x, t)$ from (1)-(4) is called the direct problem. However, the problem here is that the source parameter $p(t)$ is unknown, which needs to be determined by energy condition

$$\int_0^{s(t)} u(x, t)dx = E(t), \quad 0 < t \leq \tau, 0 < s(t) < L, \tag{5}$$

where $E(t), s(t)$ are given functions. This problem (1)-(5) is called the inverse problem.

The integral condition (5) can be used as supplementary information in the determination of the source parameter. Such type of condition can model various physical phenomena in context of chemical engineering [7], heat conduction [8], diffusion process [42, 43], thermoelasticity [44], fluid flow in porous media [45]. The existence and uniqueness and continuous dependence of the solutions to this problem and also some more applications are discussed in [5, 20]. In [20], the authors of the Sinc-collocation method for solving problems (1)-(5) used on interval $0 < x < 1$, $0 < t \leq \tau$ by $s(t) = 1$.

The main concern of this work is to extend the application of the shifted Chebyshev-Tau method to numerically solve the equations (1)-(5). We have developed some efficient Tau approximations based on a truncated series of shifted Chebyshev polynomials together with the Chebyshev operational matrices. This approach has the advantage of reducing such problems to the solution of a system of algebraic equations. Moreover, we apply the proposed algorithm to the numerical examples, in order to confirm the accuracy of this algorithm.

The rest of this article is organized as follows. In section 2 we present some necessary definitions and properties of the shifted Chebyshev polynomials. In Section 3 we have constructed and developed an algorithm for the solution of the inverse problems of parabolic partial differential (1)-(5), by using shifted Chebyshev-Tau method. In Section 4, some numerical experiments are provided and also a comparison of our method with another one has been shown. Finally, the paper ends with some conclusions in Section 5.

2. Shifted Chebyshev Polynomial

In this section, we briefly review some definitions and properties of the shifted Chebyshev polynomials which are used further in this paper.

The shifted Chebyshev polynomials satisfy the following three-term recurrence relation:

$$T_{L,0}(x) = 1,\ T_{L,1}(x) = \frac{2x}{L} - 1,$$

$$T_{L,j}(x) = 2\left(\frac{2x}{L} - 1\right)T_{L,j-1}(x) - T_{L,j-2}(x) \qquad j = 2, 3, \ldots, n. \tag{6}$$

The following formula for the j-th degree of $T_{L,j}(x)$

$$T_{L,j}(x) = j\sum_{k=0}^{j}(-1)^{j-k}\,\frac{(j+k-1)!2^{2k}}{(j-k)!(2k)!\,L^{k}}\,x^{k},\ j = 1, 2, 3, \ldots, n \tag{7}$$

where $T_{L,j}(0) = (-1)^{j}$ and $T_{L,j}(L) = 1$.

The orthogonality condition is

$$\int_{0}^{L} T_{L,j}(x)T_{L,k}(x)w_{L}(x)dx = h_{j}, \tag{8}$$

where

$$w_{L}(x) = \frac{1}{\sqrt{Lx - x^{2}}}, \tag{9}$$

and

$$h_{j} = \begin{cases} \frac{\varepsilon_{j}}{2}\pi, & k = j, \\ 0, & k \neq j, \end{cases} \quad \varepsilon_{0} = 2,\ \varepsilon_{j} = 1;\ j \geq 1. \tag{10}$$

A function $u(x, t)$ of two independent variables defined for $0 < x < L, 0 < t \leq \tau$ may be expanded into the shifted Chebyshev polynomials as:

$$u(x, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}T_{\tau,i}(t)T_{L,j}(x). \tag{11}$$

If the infinite series in (11) is truncated, than it can be written as:

$$u_{m,n}(x, t) \simeq \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}T_{\tau,i}(t)T_{L,j}(x) = \psi^{T}(t)A\phi(x), \tag{12}$$

where the shifted Chebyshev vectors $\psi(t)$ and $\phi(x)$ and the shifted Chebyshev coefficient matrix A are given as:

$$\psi(t) = [T_{\tau,0}(t), T_{\tau,1}(t), \ldots, T_{\tau,m}(t)]^{T},$$

$$\phi(x) = [T_{L,0}(x), T_{L,1}(x), \ldots, T_{L,n}(x)]^{T}, \tag{13}$$

$$A = \begin{bmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m0} & a_{m1} & \dots & a_{mn} \end{bmatrix},$$

where

$$a_{ij} = \frac{1}{h_i h_j} \int_0^{\tau} \int_0^{L} u(x, t) T_{\tau, i}(t) T_{L, j}(x) w_{\tau}(t) w_L(x) dx dt,$$

$$i = 0, 1, \dots m, \; j = 0, 1, \dots, n. \tag{14}$$

Theorem 1. *The first derivative of the shifted Chebyshev vector* $\phi(x)$ *may be expressed as*

$$\frac{d\phi(x)}{dx} = D^{(1)} \phi(x), \tag{15}$$

where $D^{(1)}$ *is the* $(n+1) \times (n+1)$ *operational matrix of derivative given by*

$$D^{(1)} = d_{ij} = \begin{cases} \dfrac{4i}{\varepsilon_j L} & j = i - k, \begin{cases} k = 1, 3, \dots, n \text{ if } (n) \text{ is odd} \\ k = 1, 3, \dots, n-1 \text{ if } (n) \text{ is even} \end{cases} \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where $\varepsilon_0 = 2, \varepsilon_j = 1, j \geq 1,$ *see* [46, 47].

For example, for odd n given as:

$$D = \frac{2}{L} \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 4 & 0 & \dots & 0 & 0 & 0 \\ 3 & 0 & 6 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2(n-1) & 0 & \dots & 2(n-1) & 0 & 0 \\ n & 0 & 2n & \dots & 0 & 2n & 0 \end{bmatrix}$$

and for even n given as:

$$D = \frac{2}{L}\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 4 & 0 & \dots & 0 & 0 & 0 \\ 3 & 0 & 6 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n-1 & 0 & 2(n-1) & \dots & 2(n-1) & 0 & 0 \\ 0 & 2n & 0 & \dots & 0 & 2n & 0 \end{bmatrix}$$

Remark 1. The operational matrix for the nth derivative can be derived as [24, 46]

$$\frac{d^n \phi(x)}{dx^n} = (D^{(1)})^n \phi(x), \tag{17}$$

where $n \in N$ and the superscript in $D^{(1)}$, denotes matrix powers. Thus

$$D^n (D^{(1)})^n, \ n = 1, 2, \dots. \tag{18}$$

Theorem 2. *The integration of* $\psi_{\tau, m}(t)$ *may be written as* [46, 48]

$$\int_0^t \psi(t')dt' \simeq P\psi(t), \tag{19}$$

where P *is the* $(m+1) \times (m+1)$ *shifted Chebyshev operational matrix of integration and is given by*

$$p = \begin{bmatrix} w_0 & \delta_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ w_1 & 0 & \lambda_1 & 0 & 0 & \dots & 0 & 0 \\ w_2 & \delta_2 & 0 & \lambda_2 & 0 & \dots & 0 & 0 \\ w_3 & 0 & \delta_3 & 0 & \lambda_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ w_{m-2} & 0 & 0 & 0 & \ddots & \ddots & \lambda_{m-2} & 0 \\ w_{m-1} & 0 & 0 & 0 & 0 & \ddots & 0 & \lambda_{m-1} \\ w_m & 0 & 0 & 0 & 0 & \dots & \delta_m & 0 \end{bmatrix}, \tag{20}$$

where

$$w_k = \begin{cases} \dfrac{\tau}{2} & k = 0 \\ \dfrac{-\tau}{8} & k = 1 \\ \dfrac{(-1)^{k+1}\tau}{2(k-1)(k+1)} & k = 2, 3, \ldots \end{cases}, \quad \delta_k = \begin{cases} \dfrac{\tau}{2} & k = 0 \\ 0 & k = 1 \\ \dfrac{-\tau}{4(k-1)} & k = 2, 3, \ldots \end{cases},$$

$$\lambda_k = \begin{cases} 0 & k = 0 \\ \dfrac{\tau}{8} & k = 1 \\ \dfrac{\tau}{4(k+1)} & k = 2, 3, \ldots \end{cases}. \tag{21}$$

Obviously similar to (19) we have

$$\int_0^x \phi(x')dx' \simeq G\phi(x), \tag{22}$$

where G is the $(n+1)\times(n+1)$ shifted Chebyshev operational matrix of integration and is defined similar to (20).

3. Shifted Chebyshev-Tau Method

In this part, we will use the tau approximation together with the shifted Chebyshev operational matrix for solving inverse parabolic problems (1)-(5). We approximate $u(x, t)$, $q(x, t)$ and $f(x)$ by using the shifted Chebyshev operational matrix as:

$$u_{m,n}(x, t) = \psi^T(t)A\phi(x), \tag{23}$$

$$q_{m,n}(x, t) \simeq \sum_{i=0}^{m}\sum_{j=0}^{n} q_{ij}T_{\tau,i}(t)T_{L,j}(x) = \psi^T(t)Q\phi(x),$$

$$f(x) \simeq \sum_{j=0}^{n} f_j T_{L,j}(x) = \psi^T(t)F\phi(x),$$

where A is an unknown $(m+1)\times(n+1)$ matrix, Q and F are known $(m+1)\times(n+1)$ matrices as;

$$Q = \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0n} \\ q_{01} & q_{11} & \cdots & q_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ q_{m0} & q_{m1} & \cdots & q_{mn} \end{bmatrix}, \qquad F = \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-1} & f_n \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{24}$$

where

$$q_{ij} = \frac{1}{h_i h_j}\int_0^{\tau}\int_0^{L} q(x, t)T_{\tau, i}(t)T_{L, j}(x)w_{\tau}(t)w_L(x)dxdt, \; i = 0, 1, \ldots, m, \, j = 0, 1, \ldots n \tag{25}$$

and

$$f_j = \frac{1}{h_j}\int_0^L f(x)T_{L, j}(x)w_L(x)dx, \qquad j = 0, 1, \ldots, n. \tag{26}$$

Integrating equation (1) from 0 to t and using equation (2) (see [24, 46]), we have

$$u(x, t) - f(x) = \int_0^t u_{xx}(x, t')dt' + \int_0^t p(t')u(x, t')dt' + \int_0^t q(x, t')dt'. \tag{27}$$

Using equations (12), (17) and (19) we get

$$\int_0^t u_{xx}(x, t')dt' = \left(\int_0^t \psi^T(t')dt'\right)A\left(\frac{d^2\phi(x)}{dx^2}\right) = \psi^T(t)P^T AD^2\phi(x). \tag{28}$$

The function $p(t)$ may be expanded in terms of $m + 1$ shifted Chebyshev series as

$$p(t) = \sum_{k=0}^{m} b_k T_{\tau, k}(t) = B^T\psi(t), \tag{29}$$

where $B = [b_0, b_1, \ldots, b_m]^T$ is an unknown vector.

Now, using equations (10), (17) and (29) we have

$$\int_0^t p(t')u(x, t')dt' = \left(\int_0^t B^T\psi(t')\psi^T(t')dt'\right)A\phi(x). \tag{30}$$

Let

$$B^T\psi(t)\psi^T(t) = \psi^T(t)H, \tag{31}$$

where H is an $(m+1)\times(m+1)$ matrix. To find H, we rewrite equation (31) (see [46]) in the form

$$\sum_{k=0}^{m} b_k T_{\tau,k}(t) T_{\tau,j}(t) = \sum_{k=0}^{m} H_{kj} T_{\tau,k}(t), \quad j = 0, 1, \ldots m. \tag{32}$$

Multiplying both sides of (32) by $T_{\tau,i}(t)w_\tau(t)$, $i = 0, 1, \ldots, m$ and integrating from 0 to τ yields

$$\sum_{k=0}^{m} b_k \int_0^\tau T_{\tau,i}(t) T_{\tau,k}(t) T_{\tau,j}(t) w_\tau(t) dt$$

$$= \sum_{k=0}^{m} H_{kj} \int_0^\tau T_{\tau,k}(t) T_{\tau,i}(t) w_\tau(t) dt, \quad i, j = 0, 1, \ldots m. \tag{33}$$

By using equation (33) and employing the orthogonality relation (8) gives

$$\sum_{k=0}^{m} b_k \int_0^\tau T_{\tau,i}(t) T_{\tau,k}(t) T_{\tau,j}(t) w_\tau(t) dt = H_{ij} h_i,$$

or equivalently

$$H_{ij} = \frac{1}{h_i} \sum_{k=0}^{m} b_k \int_0^\tau T_{\tau,i}(t) T_{\tau,k}(t) T_{\tau,j}(t) w_\tau(t) dt, \quad i, j = 0, 1, \ldots, m. \tag{34}$$

Employing equations (19), (30) and equation (31) can be written as

$$\int_0^t p(t')u(x, t')dt' = \psi^T(t) P^T HA \phi(x). \tag{35}$$

Also by using equations (12), (19) and (23) (see [46]), we get

$$\int_0^t q(x, t')dt' = \left(\int_0^t \psi^T(t')dt'\right) Q\phi(x) = \psi^T(t) P^T Q\phi(x). \tag{36}$$

Applying equations (12), (23), (28), (35) and (36) the residual $R_{m,n}(x, t)$ for equation (27) can be written as

$$R_{m,n}(x, t) = \psi^T(t)[A - F - P^T HA - P^T AD^2 - P^T Q]\phi(x) = 0.$$

Let

$$Z = [A - F - P^T HA - P^T AD^2 - P^T Q],$$

than we have

$$\psi^T(t) Z \phi(x) = 0. \tag{37}$$

As in a typical Tau method we generate $(m+1) \times (n-1)$ linear algebraic equations using the following algebraic equations

$$Z_{ij} = 0, \qquad i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, n-2. \tag{38}$$

Also, by substituting equations (23) and (29) in equations (3)-(4) we get

$$\alpha_1(t)\psi^T(t) AD\phi(0) + \beta(t)\psi^T(t) A\phi(0) = g_1(t), \tag{39}$$

$$\alpha_2(t)\psi^T(t) AD\phi(L) + \gamma(t)\psi^T(t) A\phi(L) = g_2(t). \tag{40}$$

And applying (20), (23) in equation (5) we have

$$\psi^T(t) AG\phi(s(t)) = E(t). \tag{41}$$

Equations (39)-(41) are collocated at $m+1$ points. For suitable collocation points we use the shifted Chebyshev roots t_i, $i = 1, 2, \ldots, m+1$ of $T_{\tau, m+1}(t)$. The number of the unknown coefficients a_{ij}, $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n$ and b_k, $k = 0, 1, \ldots, m$ is equal to $(m+1)(n+1) + (m+1)$ and can be obtained from equations (38)-(41).Consequently $u(x, t)$ given in equation (12) and $p(t)$ given in equation (29) can be calculated.

4. Numerical Results and Comparisons

In order to verify the performance and functionality of the proposed method, two examples are examined in this section. We also drew a comparison between our method and Sinc-collocation method proposed by [20]. In this case the exact solution $u(x, t)$ and $p(t)$ to the problem is known, we will report the accuracy and efficiency of the new method based on absolute errors e_u and e_p defined as:

$$e_u = | u_{m,n}(x, t) - u(x, t) |,\ e_p = | p_m(t) - p(t) |.$$

Example 1. Consider the inverse problem (1)-(5) with the input data

$\tau = 1,\ L = 1$

$q(x,\ t) = 0,$

$f(x) = 1 + \cos x,$

$g_1(t) = t^2 e^{(t^2 - \sin t)}\{1 + e^{-t} \sin 1\},$

$g_2(t) = e^{(t^2 - \sin t)}\{t(1 + e^{-t} \cos 1) - e^{-t} \sin 1\},$

$\alpha_i = 1,\ i = 0, 1$

$\beta(t) = t^2,\ \gamma(t) = t,$

$E(t) = e^{(t^2 - \sin t)}\{1 + e^{-t} \sin 1\},$

$s(t) = 1.$

The exact solution of the problem is $u(x,\ t) = e^{(t^2 - \sin t)}\{1 + e^{-t} \cos x\}$ and $p(t) = 2t - \cos t,$ see [20].

This problem can be solved by the method described in Section 3. In Tables 1, 2 the absolute error between the exact solution and the approximate solution shows a new method when $m = n = 3, 5, 7$ is given and the absolute error of the new method and the method given in [20] are also compared. In addition, Figure 1 shows the absolute error function $|\ u_{5,5}(x,\ 0.5) - u(x,\ 0.5)\ |$ at the interval $0 < x < 1$ and the absolute error function $|\ p_5(t) - p(t)\ |$ in the interval $0 < t < 1$ of the new method.

Table 1. Results for $u(x, 0.5)$ and the absolute error $|u_{m,n}(x, 0.5) - (u, 0.5)|$ from Example 1.

x	exact $u(x, 0.5)$	error			
		$m = n = 3$	$m = n = 5$		$m = n = 7$
			Our method	Sinc-collocation method [20]	
0.1	1.27477	2.99×10^{-4}	7.94×10^{-5}	6.2×10^{-3}	9.56×10^{-8}
0.2	1.26756	2.21×10^{-4}	3.44×10^{-8}	3.3×10^{-3}	1.32×10^{-9}
0.3	1.25564	1.66×10^{-4}	2.08×10^{-6}	5.4×10^{-3}	8.31×10^{-9}
0.4	1.23911	2.42×10^{-4}	5.34×10^{-6}	2.8×10^{-3}	6.44×10^{-8}
0.5	1.21815	4.99×10^{-4}	3.44×10^{-5}	4.1×10^{-3}	5.82×10^{-6}
0.6	1.19296	4.88×10^{-5}	8.48×10^{-7}	5.2×10^{-4}	7.47×10^{-9}
0.7	1.16379	1.97×10^{-4}	2.07×10^{-6}	8.4×10^{-4}	2.24×10^{-8}
0.8	1.13093	1.89×10^{-5}	6.17×10^{-8}	2.0×10^{-3}	1.89×10^{-9}
0.9	1.09472	2.08×10^{-4}	4.99×10^{-6}	4.9×10^{-3}	1.74×10^{-7}

Table 2. Results for $p(t)$ and absolute error $|p_m(t) - p(t)|$ from Example 1.

t	Exact $p(t)$	error			
		$m = 3$	$m = 5$		$m = 7$
			Our method	Sinc-collocation method [20]	
0.1	– 0.79500	1.66×10^{-3}	3.38×10^{-4}	4.6×10^{-3}	6.68×10^{-7}
0.2	– 0.58006	1.38×10^{-3}	1.68×10^{-7}	1.3×10^{-3}	6.74×10^{-9}
0.3	– 0.35533	8.10×10^{-4}	1.01×10^{-5}	2.3×10^{-3}	4.03×10^{-8}
0.4	– 0.12106	1.62×10^{-3}	2.69×10^{-5}	1.3×10^{-3}	7.25×10^{-7}
0.5	0.12241	1.43×10^{-3}	1.67×10^{-5}	1.5×10^{-3}	2.83×10^{-7}
0.6	0.37466	2.38×10^{-4}	4.06×10^{-5}	3.4×10^{-4}	3.57×10^{-8}
0.7	0.63515	1.69×10^{-4}	1.01×10^{-5}	6.7×10^{-4}	1.09×10^{-7}
0.8	0.90329	9.63×10^{-5}	3.01×10^{-7}	1.4×10^{-3}	9.24×10^{-9}
0.9	1.17839	1.72×10^{-3}	2.45×10^{-5}	4.7×10^{-3}	8.54×10^{-7}

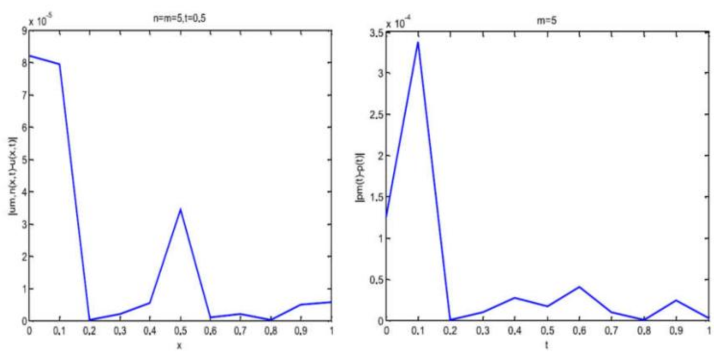

Figure 1. Plot of error function $|u_{5,5}(x, 0.5) - u(x, 0.5)|$ at the interval $0 < x < 1$ (left) furthermore error function $|p_5(t) - p(t)|$ in the interval $0 < t < 1$ (right) from example 1.

Example 2. Next, let us consider another inverse problem (1)-(5) with the following conditions:

$\tau = 0.5,\ L = 1,$

$q(x, t) = (1 - t^3)\sin x - x^2 (t - 1)^2 \exp(t^2) - 2\exp(t^2) - t^2 (\pi\cos x + t^3 + t - 3),$

$f(x) = x^2 + \pi\cos x,$

$g_1(t) = \pi + t^3,$

$g_2(t) = t\sin 1 + \exp(t^2) + \pi\cos 1 + t^3,$

$\alpha_i = 0,\ i = 0, 1$

$\beta(t) = 1,\ \gamma(t) = 1,$

$E(t) = (\pi\sin t - t\cos t)\cos(\sin t) + (t\sin t + \pi\cos t)\sin(\sin t)$

$+ \frac{1}{3}\exp(t^2)(\sin^3 t + t^3) + (t^2\sin t + t\sin^2 t)\exp(t^2) + (1 + t^3 + t^2\sin t)t,$

$s(t) = t + \sin t.$

The exact solution of the problem is $u(x, t) = t\sin x + x^2\exp(t^2) + \pi\cos x + t^3$ and $p(t) = 1 + t^2$, see [24].

Similarly, this problem can be solved by the present method like Example 1. Tables 3, 4 the absolute error between the exact solution and the approximate solution shows a new method when $m = n = 3, 5, 7$ is given, respectively. Moreover, Figure 2 also shows the absolute error function $|u_{5,5}(x, 0.25) - u(x, 0.25)|$ at the interval $0 < x < 1$ and the absolute error function $|p_5(t) - p(t)|$ in the interval $0 < t < 1$ of the new method.

Table 3. Results for $u(x, 0.25)$ and the absolute error $|u_{m,n}(x, 0.25) - u(x, 0.25)|$ from Example 2.

x	exact	error		
	$u(x,\ 0.25)$	$m = n = 3$	$m = n = 5$	$m = n = 7$
0.1	3.17713	1.47×10^{-4}	3.91×10^{-5}	4.70×10^{-8}
0.2	3.18684	1.09×10^{-4}	1.69×10^{-8}	6.48×10^{-10}
0.3	3.18659	2.59×10^{-5}	3.24×10^{-7}	1.27×10^{-9}
0.4	3.17690	1.58×10^{-4}	3.49×10^{-6}	4.21×10^{-8}
0.5	3.15861	1.29×10^{-3}	8.89×10^{-5}	1.50×10^{-5}
0.6	3.13287	5.19×10^{-4}	9.03×10^{-6}	7.96×10^{-8}
0.7	3.10110	4.24×10^{-4}	4.47×10^{-6}	4.84×10^{-8}
0.8	3.06501	1.23×10^{-4}	4.01×10^{-7}	1.23×10^{-8}
0.9	3.02654	6.61×10^{-4}	1.58×10^{-5}	5.51×10^{-7}

Table 4. Results for $p(t)$ and absolute error $|p_m(t) - p(t)|$ from Example 2.

t	exact	error		
	$p(t)$	$m = 3$	$m = 5$	$m = 7$
0	1	4.41×10^{-3}	1.62×10^{-3}	1.66×10^{-6}
0.05	1.0025	4.59×10^{-3}	1.22×10^{-3}	1.26×10^{-6}
0.10	1.01	5.85×10^{-3}	9.06×10^{-7}	3.47×10^{-8}
0.15	1.0225	5.24×10^{-3}	9.04×10^{-5}	4.54×10^{-7}
0.20	1.04104	1.83×10^{-3}	4.04×10^{-5}	3.88×10^{-7}
0.25	1.0625	7.58×10^{-2}	5.22×10^{-3}	8.81×10^{-6}

0.30	1.09	5.44×10^{-2}	4.26×10^{-6}	3.75×10^{-8}
0.35	1.1225	1.15×10^{-2}	2.04×10^{-4}	2.20×10^{-6}
0.40	1.16	6.82×10^{-3}	2.22×10^{-5}	6.81×10^{-7}
0.45	1.2025	8.18×10^{-2}	2.19×10^{-3}	7.03×10^{-5}
0.5	1.25	9.91×10^{-2}	2.13×10^{-3}	7.71×10^{-5}

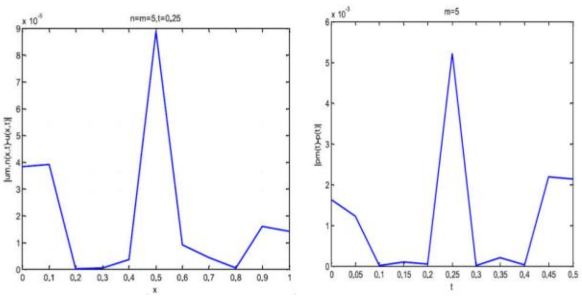

Figure 2. Plot of error function $|u_{5,5}(x, 0.25) - u(x, 0.25)|$ at the interval $0 < x < 1$ (left) furthermore error function $|p_5(t) - p(t)|$ in the interval $0 < t < 0.5$ (right) from example 2.

From the above examples, we can observe that:

First, according to Tables 1, 2 the shifted Chebyshev-Tau method has higher accuracy than the Sinc-collocation method when they have the same number.

Second, experimental data in Tables 1, 2, 3, 4 shows that the approximation accuracy of the shifted Chebyshev-Tau method is gradually increased with a rise in terms of the truncated series.

5. Conclusion

Determination of an unknown time-dependent control parameter in parabolic partial differential equations plays a very important role in many branches of science and engineering. In this article, the inverse problem of finding the time-dependent heat source together with the temperature in the heat equation, under the boundary condition and integral over determination condition has been investigated. An efficient direct solver method is developed for solving such problems using the shifted Cheyshev-Tau method. The construction of the proposed algorithm is based on the Tau approximation in addition to the shifted Chebyshev operational matrix. Illustrative numerical examples with satisfactory approximate solutions are achieved to demonstrate the accuracy of the present method. The obtained approximations of the exact solutions for the test problems make this method very attractive and contributed to the good agreement between approximate and exact values in numerical examples.

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